CARTIER'S THEOREM

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Let k be a field. Recall that a k-algebra H together with k-linear maps

$$\Delta : H \longrightarrow H \otimes_k H \quad ; \quad \varepsilon : H \longrightarrow k \quad ; \quad \eta : H \longrightarrow H$$

is called a *Hopf algebra* if

- Δ and ε are homomorphisms of k-algebras,
- η is an anti-homomorphism of k-algebras,
- Δ and ε satisfy axioms dual to that of an associative multiplication and an identity element.

We shall write

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

for every $h \in H$. Among the axioms mentioned above we shall require

(1) $\sum_{(h)} h_{(1)}\varepsilon(h_{(2)}) = h = \sum_{(h)} \varepsilon(h_{(1)})h_{(2)}$ for all $h \in H$, and
(2) $\sum_{(h)} h_{(1)}\eta(h_{(2)}) = \varepsilon(h)1 = \sum_{(h)} \eta(h_{(1)})h_{(2)}$ for all $h \in H$.

The category of H-modules has tensor products. If M and N are H-modules, then $M \otimes_k N$ obtains the structure of an H-module by pulling the $H \otimes_k H$-structure on $M \otimes_k N$ back along Δ:

$$h.(m \otimes n) := \sum_{(h)} h_{(1)}m \otimes h_{(2)}n \qquad \forall\ h \in H,\ m \in M,\ n \in N.$$

A commutative k-algebra A is referred to as *reduced* if 0 is the only nilpotent element of A. The object of this lecture is the following fundamental result, which implies that locally algebraic group schemes of characteristic zero are smooth, cf. [1, (II,§6,1.1)].

Theorem (Cartier). *Let H be a finitely generated commutative Hopf algebra. If* $\mathrm{char}(k) = 0$*, then H is reduced.*

Remark. The polynomial ring $k[X]$ obtains the structure of a Hopf algebra by defining

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad ; \quad \varepsilon(X) = 0 \quad ; \quad \eta(X) = -X.$$

If $\mathrm{char}(k) = p > 0$, then the ideal $I := (X^p)$ satisfies

$$\Delta(I) \subseteq I \otimes_k H + H \otimes_k I \quad ; \quad I \subseteq \ker \varepsilon \quad ; \quad \eta(I) = I.$$

Hence the local algebra $H := k[X]/I$ inherits the Hopf algebra structure from $k[X]$, showing that Cartier's Theorem fails in positive characteristic.

The proof of Cartier's Theorem requires the following result from commutative algebra:

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Theorem 1 (Krull's Intersection Theorem,[2, 3])**.** *Let R be a commutative noetherian ring,* $\operatorname{Max}(R)$ *be the set of maximal ideals of R. Then*

$$\bigcap_{\mathfrak{M}\in\operatorname{Max}(R)}\ \bigcap_{n\in\mathbb{N}}\mathfrak{M}^n=(0).$$

Let A be a commutative k-algebra. A k-linear map, $d : A \longrightarrow M$ with values in an A-module M is called a *derivation* of A into M, provided

$$d(xy) = x.d(y) + y.d(x) \qquad \forall\ x, y \in A.$$

Given a derivation $d : A \longrightarrow M$, the map

$$\mathrm{id}_A \otimes d : A \otimes_k A \longrightarrow A \otimes_k M \quad ; \quad a \otimes b \mapsto a \otimes d(b)$$

is a derivation of the k-algebra $A\otimes_k A$ with values in the $(A\otimes_k A)$-module $A\otimes_k M$, whose structure is defined by

$$(a \otimes b).(a' \otimes m) := aa' \otimes b.m.$$

Let H be a commutative Hopf algebra with augmentation ideal $I := \ker\varepsilon$. We consider the map

$$\pi_H : H \longrightarrow I/I^2 \quad ; \quad h \mapsto h - \varepsilon(h)1 + I^2,$$

which is readily seen to be a derivation. Hence $\mathrm{id}_H \otimes\pi_H$ is a derivation of $H \otimes_k H$ with values in $H \otimes_k (I/I^2)$. Consequently,

$$d_H := (\mathrm{id}_H \otimes\pi_H) \circ \Delta$$

is a derivation of H into $H\otimes_k (I/I^2)$, where the latter space is the tensor product of the H-modules H and I/I^2.

Proof of the Theorem. Since the k-algebra H is finitely generated, the vector space I/I^2 is finite-dimensional. Hence there are elements $x_1, \dots , x_r \in I$ such that their residue classes form a basis of I/I^2. Using multi-index notation, we write

$$x^m := x_1^{m_1}\cdots x_r^{m_r} \quad\text{and}\quad |m| := \sum_{i=1}^{r} m_i$$

for $m \in \mathbb{N}_0^r$. The following claim says that the graded algebra, defined by the powers of the augmentation ideal I, is a polynomial ring in r variables.

(i) *Let $n \geq 1$. The residue classes $\{x^m + I^{n+1}\ ;\ |m| = n\}$ form a basis of I^n/I^{n+1}.*

Given $i \in \{1, \dots , r\}$, we consider the k-linear map

$$f_i : I/I^2 \longrightarrow k \quad ; \quad \bar{x}_j \mapsto \delta_{ij}$$

as well as

$$\tilde{f}_i : H \otimes_k I/I^2 \longrightarrow H \quad ; \quad a \otimes v \mapsto f_i(v)a.$$

For $a, b \in H$ and $v \in I/I^2$, we have, observing (1),

$$a.(b \otimes v) = \sum_{(a)} a_{(1)}b \otimes a_{(2)}v = \sum_{(a)} a_{(1)}b \otimes \varepsilon(a_{(2)})v = (\sum_{(a)} a_{(1)}\varepsilon(a_{(2)}))b \otimes v = ab \otimes v,$$

so that the map $\tilde{f}_i$ is H-linear.

Consequently, the map

$$d_i : H \longrightarrow H \quad ; \quad h \mapsto \tilde{f}_i \circ d_H(h)$$

is a derivation of H with $d_i(h) = \sum_{(h)} f_i(\pi_H(h_{(2)}))h_{(1)}$ for all $h \in H$. We thus obtain

$$(\varepsilon \circ d_i)(h) = \sum_{(h)} \varepsilon(h_{(1)}) f_i(\pi_H(h_{(2)})) = f_i(\sum_{(h)} \varepsilon(h_{(1)})\pi_H(h_{(2)})) = (f_i \circ \pi_H)(h),$$

whence

$$(*) \qquad d_i(x_j) \equiv \delta_{ij} \mod I$$

for $1 \leq i \leq r$. Let $\underline{\mathbf{1}}_i \in \mathbb{N}_0^r$ be the element with coordinates δ_{ij}. Since d_i is a derivation, $(*)$ implies

$$d_i(x^m) \equiv m_i x^{m-\underline{\mathbf{1}}_i} \mod I^{|m|}.$$

In particular, $d_i(I^n) \subseteq I^{n-1}$ for all $n \geq 1$. Thus, if

$$\sum_{|m|=n} \alpha_m x^m \equiv 0 \mod I^{n+1},$$

then, applying d_i, we obtain

$$0 = \sum_{|m|=n} m_i \alpha_m x^{m-\underline{\mathbf{1}}_i} \mod I^n,$$

so that induction implies $m_i\alpha_m = 0$. Since $\mathrm{char}(k) = 0$, we conclude that $\alpha_m = 0$. ◇

(ii) *If* $h^2 = 0$*, then* $h \in \bigcap_{n\in\mathbb{N}} I^n$.

We have $h \in I$, and if $h \notin \bigcap_{n\in\mathbb{N}} I^n$, then there exists $n \in \mathbb{N}$ with $h \in I^n \setminus I^{n+1}$. We write

$$h = \sum_{|m|=n} \alpha_m x^m + z,$$

with $\alpha_m \in k$ and $z \in I^{n+1}$. Our assumption in conjunction with (i) implies

$$\sum_{m+m'=t} \alpha_m \alpha_{m'} = 0$$

for all $t \in \mathbb{N}_0^r$ with $|t| = 2n$. Upon ordering the elements of $\mathbb{N}_0^r$ lexicographically we obtain $0 = \alpha_{\tilde{m}}$, where $\tilde{m} = \max_{|m|=n}\{\alpha_m \neq 0\}$, a contradiction. ◇

It suffices to verify our theorem under the assumption that k is algebraically closed. If $\mathfrak{M} \trianglelefteq H$ is a maximal ideal of H, then Hilbert's Nullstellensatz provides an algebra homomorphism $\lambda : H \longrightarrow k$ such that $\mathfrak{M} = \ker \lambda$. Direct computation shows that λ induces an automorphism

$$\psi_\lambda : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}$$

of H, whose inverse is $\psi_{\lambda\circ\eta}$. (At this stage, we need property (2).) Since

$$(\varepsilon \circ \psi_\lambda)(h) = \sum_{(h)} \lambda(h_{(1)})\varepsilon(h_{(2)}) = \lambda(\sum_{(h)} h_{(1)}\varepsilon(h_{(2)})) = \lambda(h)$$

for every $h \in H$, it follows that $\Psi_\lambda(\mathfrak{M}) = I$.

Let $h \in H$ be nilpotent. Without loss of generality, we may assume that $h^2 = 0$. Given a maximal ideal $\mathfrak{M} = \ker \lambda$, (ii) implies that $\psi_\lambda(h) \in \bigcap_{n\in\mathbb{N}} I^n$, whence $h \in \bigcap_{n\in\mathbb{N}} \mathfrak{M}^n$. Krull's Intersection Theorem now yields $h = 0$, as desired. □

References

[1] M. Demazure and P. Gabriel, *Groupes Algébriques I.* Masson/North Holland, 1970.
[2] D. Eisenbud, *Commutative Algebra*. Graduate Texts in Mathematics **150**. Springer-Verlag, 1996.
[3] W. Krull, *Dimensionstheorie in Stellenringen.* J. reine angew. Math. **179** (1938), 204–226.