INDUCED MODULES: FIRST PROPERTIES OF DEFECT GROUPS

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Let $k$ be a field of characteristic $p > 0$. If $G$ is a finite group, then the group algebra $kG$ has a block decomposition

$$kG = B_1 \oplus B_2 \oplus \cdots \oplus B_r,$$

where each block $B_i \subseteq kG$ is an indecomposable two-sided ideal. Equivalently, each block $B \subseteq kG$ is an indecomposable $kG \otimes_k kG^{op}$-module. Since the map

$$(g, h) \mapsto g \otimes h^{-1}$$

induces an isomorphism $k(G \times G) \longrightarrow kG \otimes_k kG^{op}$ of associative $k$-algebras, the latter condition amounts to $B$ being an indecomposable submodule of the $(G \times G)$-module $kG$, relative to the action

$$(g, h), x := gxh^{-1} \quad \forall \ g, h \in G, \ x \in kG.$$

One can thus speak of the vertex of the $(G \times G)$-module $B$, see [4] for the definition.

Let $\Delta : G \longrightarrow G \times G : g \mapsto (g, g)$ be the diagonal embedding, whose induced algebra homomorphism $kG \longrightarrow k(G \times G)$ will also be denoted $\Delta$.

**Definition.** Let $B \subseteq kG$ be a block. A $p$-subgroup $D \subseteq G$ is called a defect group of $B$ if $\Delta(D)$ is a vertex of the $(G \times G)$-module $B$. If $\text{ord}(D) = p^d$, then $d$ is called the defect of $B$.

The name defect derives from an early result of the theory, which states that a block $B \subseteq kG$ is semi-simple (and hence simple) if and only if $d = 0$. Thus, $d$ may be viewed as a measure for the deviation of $B$ from being semi-simple.

Defects were first defined by Brauer [1], with the definition of a defect group following shortly thereafter [2]. In his seminal articles [1, 2, 3] Brauer established important properties of defect groups that were later reformulated by Green [6, 7], whose approach is the basis of our exposition.

Recall that $G$ acts on $k$ via

$$g.\alpha = \alpha \quad \forall \ g \in G, \ \alpha \in k.$$

Our first result establishes the existence of defect groups and shows that the defect of a block is well-defined.

**Theorem 1.** Let $B \subseteq kG$ be a block of $kG$.

1. $B$ possesses a defect group $D \subseteq G$.

2. If $D, D' \subseteq G$ are defect groups of $B$, then there exists $g \in G$ with $D' = gDg^{-1}$.

**Proof.** (1) We consider $k(G \times G)$ as a left and right $G$-module via $\Delta$. The bilinear map

$$\varphi : k(G \times G) \times k \longrightarrow kG ; \ ((g, h), \alpha) \mapsto \alpha gh^{-1}$$

is $kG$-balanced: Given $x \in G$, we have

$$\varphi((g, h), x, \alpha) = \varphi((gx, hx), \alpha) = \alpha gh^{-1} = \varphi((g, h), x.\alpha).$$
Hence there exists a surjective, $k$-linear map
\[ \psi : k(G \times G) \otimes_{kG} k \rightarrow kG \quad ; \quad (g, h) \otimes \alpha \mapsto \alpha gh^{-1}, \]
which is readily seen to be $k(G \times G)$-linear. Since both spaces involved have dimension $\text{ord}(G)$, $\psi$ is in fact an isomorphism, so that $kG$ is a relatively $\Delta(G)$-projective $k(G \times G)$-module. Being a direct summand of $kG$, the block $B$ enjoys the same property. According to [4, Prop.4] there exists a $p$-subgroup $D \subseteq G$ such that $\Delta(D)$ is a vertex of $B$.

(2) Let $D, D'$ be defect groups of $B$. Owing to [4, Prop.4], there exists an element $(g, h) \in G \times G$ such that
\[ \Delta(D') = (g, h)\Delta(D)(g, h)^{-1}, \]
whence $D' = gDg^{-1}$.

We would like to relate the defect group of a block to the vertices of its indecomposable modules. This necessitates the following subsidiary result, which shows that induction commutes with taking tensor products over $k$. Recall that the tensor product $M \otimes_k N$ of $G$-modules obtains the structure of a $G$-module via
\[ g.(m \otimes n) := g.m \otimes g.n \]
for all $g \in G$, $m \in M$ and $n \in N$.

**Lemma 2** (Tensor Identity). Let $H \subseteq G$ be a subgroup of the finite group $G$. If $V$ is a finite-dimensional $G$-module and $M$ is a finite-dimensional $H$-module, then we have an isomorphism
\[ kG \otimes_{kH} (M \otimes_k V) \cong (kG \otimes_{kH} M) \otimes_k V \]
of $G$-modules.

**Proof.** Given $g \in G$, we consider the $k$-linear map
\[ \lambda_g : M \otimes_k V \rightarrow (kG \otimes_{kH} M) \otimes_k V \quad ; \quad m \otimes v \mapsto (g \otimes m) \otimes g.v \]
If $a = \sum_{g \in G} \alpha_g g$ is an element of $kG$, we define $\lambda_a := \sum_{g \in G} \alpha_g \lambda_g$. There results a bilinear map
\[ \psi : kG \times (M \otimes_k V) \rightarrow (kG \otimes_{kH} M) \otimes_k V \quad ; \quad (a, x) \mapsto \lambda_a(x). \]
Since $\lambda_a(hx) = \lambda_a(h x)$ for all $a \in kG$, $h \in H$ and $x \in M \otimes_k V$, the map $\psi$ is $kH$-balanced and there exists a $k$-linear map
\[ \omega : kG \otimes_{kH} (M \otimes_k V) \rightarrow (kG \otimes_{kH} M) \otimes_k V \quad ; \quad a \otimes x \mapsto \lambda_a(x). \]
This map is actually $kG$-linear: Let $g, g' \in G$, $m \in M$ and $v \in V$. Then we have
\[
\begin{align*}
\omega(g'.(g \otimes (m \otimes v))) &= \omega(g'g \otimes (m \otimes v)) = (g'g \otimes m) \otimes g'.g.v = g'.((g \otimes m) \otimes g.v)
\end{align*}
\]
Directly from the definition, we obtain the surjectivity of $\omega$. Since both $G$-modules involved have dimension $|G/H|(\dim_k M)(\dim_k V)$, the map $\omega$ is bijective.

Recall that any block $B \subseteq kG$ is of the form $B = kGe$, where $e \in kG$ is a central, primitive idempotent of $kG$. Given an indecomposable $kG$-module $M$, we thus have $e.M = (0)$ or $e.M = M$. In the latter case, we say that $M$ belongs to $B$.

**Theorem 3.** Let $B \subseteq kG$ be a block with defect group $D$. Then every indecomposable $kG$-module $M$ belonging to $B$ has a vertex $D_M \subseteq D$. 


Proof. We let $G$ act on $kG$ via conjugation, i.e.,
$$g.a := gag^{-1} \quad \forall a \in kG, \: g \in G.$$ 
Note that this amounts to pulling back the $(G \times G)$-action on $kG$ along $\Delta$. Since $B \subseteq kG$ is a two-sided ideal, $B \subseteq kG$ is a $G$-submodule relative to this operation. The multiplication
$$\mu : B \otimes_k M \rightarrow M \mid b \otimes m \mapsto bm$$
is a homomorphism of $G$-modules: Given $g \in G$, $b \in B$ and $m \in M$, we have
$$\mu(g(b \otimes m)) = \mu(gbg^{-1} \otimes gm) = gbg^{-1}gm = g(bm) = g\mu(b \otimes m).$$
Let $e \in kG$ be the central primitive idempotent of $B$, so that $B = ke$. Then
$$\iota : M \rightarrow B \otimes_k M \mid m \mapsto e \otimes m$$
is a homomorphism of $G$-modules. Since $M$ belongs to $B$, we obtain $\mu \circ \iota = \text{id}_M$, so that $M$ is a direct summand of $B \otimes_k M$.

As $D \subseteq G$ is a defect group of $B$, the $G$-module $B$ is relatively $D$-projective. Consequently, $B$ is a direct summand of $kG \otimes_{kD} B|D$. In view of Lemma 2, the tensor product $B \otimes_k M$ is a direct summand of $(kG \otimes_{kD} B|D) \otimes_k M \cong kG \otimes_{kD} (B|D \otimes_k M|D)$. By the above, this implies that $M$ is relatively $D$-projective, so that $D$ contains a vertex of $M$, cf. [4, Prop.4].

There exists exactly one block $B_0(G) \subseteq kG$ to which the trivial $G$-module $k$ belongs. The block $B_0(G)$ is customarily referred to as the principal block. The following result shows why $B_0(G)$ is thought of as being the “most complicated” block of $kG$:

**Corollary 4.** Every defect group $D \subseteq G$ of the principal block $B_0(G)$ is a Sylow-$p$-subgroup of $G$.

Proof. Owing to Theorem 3, $D$ contains a vertex $D'$ of the trivial module $k$. Being a $p$-group, $D'$ is contained in a Sylow-$p$-subgroup $P \subseteq G$. As $k$ is relatively $D'$-projective, $k$ is a summand of $kG \otimes_{kD'} k$. By Mackey’s Theorem [4], the trivial $P$-module $k|P$ is a summand of
$$\bigoplus_{P \subseteq D'} kP \otimes_{k(P \cap D')} k = \bigoplus_{P \subseteq D'} kP \otimes_{k(P \cap D')} k,$$
where $D' := gD'g^{-1}$. Repeated application of Green’s Indecomposability Theorem [5] (to a chain of normalizers in $P$ starting with $\text{Nor}_P(P \cap D')$) implies that each summand is an indecomposable $kP$-module. Theorem of Krull-Remak-Schmidt now ensures that $k|P$ is isomorphic to one of these summands. Hence there exists an element $g$ with $P = D'g$, so that $P = D'$.

**Corollary 5.** Let $B \subseteq kG$ be a block with defect group $D$.

1. If $D$ is cyclic, then $B$ has finite representation type.
2. If $D = \{1\}$, then $B$ is simple.

Proof. Suppose that $\text{ord}(D) = p^r$. As $D$ is cyclic, the group algebra $kD \cong k[X]/(X^{p^r})$ has finite representation type, with indecomposable modules $N_1, \ldots, N_{p^r}$. In view of Theorem 3, every indecomposable $B$-module is relatively $D$-projective, and hence a direct summand of some $kG \otimes_{kD} N_i$. Consequently, there are only finitely many isomorphism classes of such modules. If $D = \{1\}$, then each indecomposable $B$-module $M$ is a direct summand of $kG \otimes_k k \cong kG$ and is thus projective. This implies that $B$ is simple.

\textsuperscript{1}This argument actually shows that induction functors of $p$-groups preserve indecomposables. In our situation, Frobenius reciprocity gives $\text{Hom}_{kP}(kP \otimes_{k(P \cap D')} k, k) \cong \text{Hom}_{k(kP \cap D')} k(k, k)$, which, in view of $kP$ being local, implies that the top of the induced module is simple.
Remark. The converse statements of (1) and (2) of Corollary 5 also hold, but their proofs necessitate the so-called Brauer correspondence of blocks.

References