INDUCED MODULES: GRADED ALGEBRAS AND GREEN’S INDECOMPOSABILITY THEOREM

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Throughout, \( k \) is assumed to be an algebraically closed field of characteristic \( \text{char}(k) = p > 0 \). Let \( N \trianglelefteq G \) be a normal subgroup of a finite group \( G \). As usual, \( kG \) denotes the group algebra of \( G \).

**Theorem** (Green’s Indecomposability Theorem, [3]). Suppose that \( G/N \) is a \( p \)-group. If \( M \) is an indecomposable \( kN \)-module, then the \( kG \)-module \( kG \otimes_{kN} M \) is indecomposable.

We shall prove this result by establishing a general statement on group-graded algebras, that has also been useful in the representation theory of infinitesimal group schemes (cf. [2]). In the sequel, all algebras and modules are finite-dimensional.

**Definition.** Let \( G \) be a finite group. A \( k \)-algebra

\[
R = \bigoplus_{g \in G} R_g
\]

is said to be \( G \)-graded if \( R_g R_h \subseteq R_{gh} \) for all \( g, h \in G \). We call \( R \) strongly \( G \)-graded if \( R_g R_h = R_{gh} \) for all \( g, h \in G \).

For instance, if \( N \trianglelefteq G \) is a normal subgroup, then \( kG \) is a strongly \( G/N \)-graded \( k \)-algebra. In fact, if \( R = \bigoplus_{g \in G} R_g \) is \( G \)-graded and \( \pi : G \rightarrow H \) is a surjective homomorphism of finite groups, then \( R \) obtains the structure of an \( H \)-graded algebra via

\[
R_h := \bigoplus_{g \in \pi^{-1}(h)} R_g.
\]

Indecomposability corresponds to endomorphism rings being local, so we are interested in the question when a graded algebra is local. By way of motivation we record a few necessary conditions.

**Lemma 1.** Suppose that \( R = \bigoplus_{g \in G} R_g \) is a local algebra. Then the following statements hold:

1. The algebra \( R_1 \) is local.
2. If \( R_g \not\subseteq \text{Rad}(R) \) for every \( g \in G \), then \( G \) is a \( p \)-group.

**Proof.** By assumption, there exists an algebra homomorphism \( \varepsilon : R \rightarrow k \) such that \( \ker \varepsilon = \text{Rad}(R) \).

1. Since \( \ker \varepsilon |_{R_1} \) is a nilpotent ideal of codimension 1, it follows that \( R_1 \) is local.
2. Consider \( N := \bigoplus_{g \in G} (\ker \varepsilon) \cap R_g \). Then \( N \) is a nilpotent ideal of \( R \). Thus, \( \varepsilon \) induces an algebra homomorphism \( \gamma : S \rightarrow k \) of the local, \( G \)-graded algebra \( S := R/N \). By virtue of our current assumption, we have \( \text{dim}_k S_g = 1 \) for every \( g \in G \), and for every \( g \in G \) there exists a unique element \( s_g \in S_g \) such that \( \gamma(s_g) = 1 \). Consequently, we have \( s_gh = s_g \) for all \( g, h \in G \).
so that the map $G \to S : g \mapsto s_g$ induces a surjective algebra homomorphism $\zeta : kG \to S$. By equality of dimensions, this map is bijective. As a result, the group algebra $kG$ is local, forcing $G$ to be a $p$-group.

We turn to algebras that are graded by some $p$-group $G$, beginning with the case where $G$ is abelian.

**Lemma 2.** Let $R = \bigoplus_{g \in G} R_g$ be a group-graded $k$-algebra. Suppose that

(a) $G$ is an abelian $p$-group, and

(b) $\dim_k R_g < 1$ for every $g \in G$, and

(c) the elements of $R_g \setminus \{0\}$ are invertible for every $g \in G$.

Then there exists a subgroup $H \subseteq G$ with $R \cong kH$.

**Proof.** In view of (c), $H := \{h \in G : R_h \neq (0)\}$ is a subgroup of $G$, and $R = \bigoplus_{h \in H} R_h$. By general theory, the group $H$ is a direct sum of cyclic groups with generators $h_1, \ldots, h_{\ell}$ of orders $p^{\nu_1}, \ldots, p^{\nu_{\ell}}$, say. Pick $r_i \in R_{h_i}$ with $r_i^{p^{\nu_i}} = 1$. Given $i, j \in \{1, \ldots, \ell\}$, there exists $\alpha_{ij} \in k$ such that

$$r_i r_j r_i^{-1} = \alpha_{ij} r_j,$$

Thus,

$$r_j = r_i^{p^{\nu_i}} r_j r_i^{-p^{\nu_i}} = \alpha_{ij}^{p^{\nu_i}} r_j,$$

so that $\alpha_{ij} = 1$. Consequently, the elements $r_1, \ldots, r_{\ell}$ commute with each other. Since the subalgebra generated by these elements contains all homogeneous parts of $R$, we see that $R$ is commutative. By the same token, the map $T_i \mapsto r_i$ defines an isomorphism

$k[T_1, \ldots, T_{\ell}]/(T_i^{p^{\nu_i}} - 1, \ldots, T_{\ell}^{p^{\nu_{\ell}}} - 1) \sim R,$

with the truncated polynomial ring being isomorphic to $kH$.

The proof of our main result necessitates information on nilpotent elements. A subset $W \subseteq R$ of a $k$-algebra $R$ is nil if every element $w \in W$ is nilpotent. We say that $W$ is nilpotent if $W^n = (0)$ for some $n \in \mathbb{N}$. The set $W$ is referred to as weakly closed if there exists a function $\gamma : W \times W \to k$ such that $vw + \gamma(v, w)vw \in W$ for all $v, w \in W$. Here is the relevant result, which we shall take for granted (see [4, (II.2)]).

**Theorem 3** (Jacobson’s Theorem on nil weakly closed sets). Let $W \subseteq R$ be a nil, weakly closed subset of an associative $k$-algebra $R$. Then the associative subalgebra $\text{alg}_k(W) \subseteq R$ without identity that is generated by $W$ is nilpotent.

**Theorem 4** ([2]). Let $G$ be a $p$-group, $R = \bigoplus_{g \in G} R_g$ be a $G$-graded algebra. If $R_1$ is local, then $R$ is local.

**Proof.** We first assume that $G$ is abelian and write $G$ additively. Since $R_0$ is local, there exists a linear map $\alpha : R_0 \to k$ such that

$$\ker \alpha = \{r \in R_0 : r \text{ is nilpotent}\}.$$

Given $g \in G$, we set

$$N_g := \{r \in R_g : r \text{ is nilpotent}\}.$$

Suppose that $\text{ord}(G) = p^m$. For $g \in G$ and $r \in R_g$, we have $r^{p^m} \in R_{p^m g} = R_0$. By the above, we can write

$$(*) \quad r^{p^m} = \alpha(r^{p^m})1 + x.$$
for some nilpotent element $x \in N_0$. It follows that
\[ \psi_g : R_g \to k : r \mapsto \alpha(r^{p^n}) \]
is a homogeneous polynomial function of degree $p^n$, whose zero locus $Z(\psi_g)$ is $N_g$. Since $R_g$ and $k$ are irreducible varieties and $\psi_g$ is a morphism, it follows from standard results on morphisms that $\dim N_g \geq \dim_k R_g - 1$.

By (*), a homogeneous element $r \in R$ is either nilpotent or invertible. Given $r \in N_g$ and $s \in R_h$, we have $rs \in R_{g+h}$. If $rs$ is invertible, then left multiplication by $r$ is surjective, which contradicts the nilpotence of $r$. Hence $rs \in N_{g+h}$, and a similar argument shows that $sr \in N_{g}$. Consequently, $N := \bigcup_{g \in G} N_g$ is a nil weakly closed subset of $R$. Theorem 3 now implies that the associative algebra $\text{alg}_k(N)$ without identity generated by $N$ is nilpotent. In particular, $N_g$ is a subspace of $R_g$, which, by our earlier observation, has codimension $\leq 1$. By the above, $J = \bigoplus_{g \in G} N_g$ is a nilpotent ideal of $R$, such that the factor algebra $S := R/J$ is $G$-graded with the following properties:

(a) $\dim_k S_g \leq 1$ for every $g \in G$, and
(b) every element of $S_g \setminus \{0\}$ is invertible.

Consequently, Lemma 2 provides a subgroup $H \subseteq G$ such that $S \cong kH$. In particular, $S$ is local and the algebra $R$ thus enjoys the same property.

In the general case, that is, when $G$ is not necessarily abelian, we proceed by induction on the order of $G$. The $p$-group $G$ has a non-trivial center $C(G)$. We set $G' := G/C(G)$ and denote by $\pi : G \to G'$ the canonical projection. By our introductory remarks, this map endows $R$ with a $G'$ grading such that $R_1 = \bigoplus_{g \in C(G)} R_g$ is graded with respect to the abelian $p$-group $C(G)$. By the first part of the proof $R_1$ is local, so that induction ensures that the algebra $R$ is also local. 

\[ \square \]

**Corollary 5.** Let $R = \bigoplus_{g \in G} R_g$ be a group-graded algebra. If $N \unlhd G$ is a normal subgroup of index a $p$-power such that the subalgebra $\bigoplus_{h \in N} R_h$ is local, then $R$ is local. 

\[ \square \]

If $G$ is a finite group that acts on a $k$-algebra $\Lambda$ via automorphisms
\[ (g, \lambda) \mapsto g \cdot \lambda, \]
then $\Lambda * G$ denotes the skew group algebra of $G$ with coefficients in $\Lambda$. By definition, $\Lambda * G$ is the free $\Lambda$-module with basis $G$, whose multiplication is given by
\[ (\lambda_g)(\lambda_h h) := \lambda_g(g, \lambda_h)gh \quad \forall g, h \in G, \lambda_g, \lambda_h \in \Lambda. \]

We now obtain the following generalization of Green’s theorem:

**Corollary 6.** Let $G$ be a finite group that operates on an algebra $\Lambda$ via automorphisms, and suppose that $N \unlhd G$ is a normal subgroup of index a power of $p$. If $M$ is an indecomposable $\Lambda * N$-module, then the induced module $\Lambda * G \otimes_{\Lambda * N} M$ is indecomposable.

**Proof.** The skew group algebra $\Lambda * G$ is strongly graded relative to the $p$-group $G/N$, with one-component $(\Lambda * G)_1 = \Lambda * N$. Moreover, the induced module $\Lambda * G \otimes_{\Lambda * N} M$ and its endomorphism ring are also $G/N$-graded, and $[1, (4.8)]$ provides an isomorphism
\[ \text{End}_{\Lambda G}(\Lambda * G \otimes_{\Lambda * N} M) \cong \text{End}_{\Lambda N}(M) \]
of rings. Our assumption on $M$ in conjunction with Theorem 4 now guarantees that the $k$-algebra $\text{End}_{\Lambda G}(\Lambda * G \otimes_{\Lambda * N} M)$ is local. Consequently, the module $\Lambda * G \otimes_{\Lambda * N} M$ is indecomposable. 

\[ \square \]
References