HOPF MODULES AND INTEGRALS: THE SPACE OF INTEGRALS

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Throughout, \( H \) denotes a finite dimensional Hopf algebra over a field \( k \). As usual, the comultiplication, the counit and the antipode of \( H \) are denoted \( \Delta, \varepsilon \) and \( \eta \), respectively. Recall that

\[
\int^l_H := \{ x \in H ; hx = \varepsilon(h)x \ \forall \ h \in H \} \quad \text{and} \quad \int^r_H := \{ x \in H ; xh = \varepsilon(h)x \ \forall \ h \in H \}
\]

are the subspaces of left and right integrals of \( H \), respectively. The object of this lecture is the ensuing

Theorem ([3]). The following statements hold:

1. \( \dim_k \int^r_H = 1 \).
2. The antipode \( \eta \) is bijective.
3. \( \eta(\int^r_H) = \int^l_H \).

The main idea of the proof is to endow \( H^* \) with the structure of a Hopf module and use the fundamental theorem [2] to show \( \dim_k \int^r_{H^*} = 1 \). Since \( H^* \) is also a Hopf algebra, the asserted result follows.

The multiplication and comultiplication on \( H^* \) are given by the following formulae:

\[
(\varphi \psi)(h) := \sum_{(h)} \varphi(h^{(1)})\psi(h^{(2)}) \quad \forall \ \varphi, \psi \in H^*, \ h \in H
\]

and

\[
\Delta(\varphi) = \sum_{(\varphi)} \varphi^{(1)} \otimes \varphi^{(2)} \Leftrightarrow \varphi(hh') = \sum_{(h)} \varphi^{(1)}(h)\varphi^{(2)}(h') \quad \forall \ h, h' \in H.
\]

These rules are obtained by dualizing those for \( H \). For instance, the multiplication \( m_{H^*} \) is the composite

\[
m_{H^*} : H^* \otimes_k H^* \longrightarrow (H \otimes_k H)^* \xrightarrow{\Delta^*} H^*.
\]

The counit and the antipode of \( H^* \) are defined via

\[
\varepsilon^*(\varphi) = \varphi(1) \quad \text{and} \quad \eta^*(\varphi) = \varphi \circ \eta \quad \forall \ \varphi \in H^*,
\]

respectively. In a similar fashion, the vector space \( H^* \) obtains the structure of a Hopf module for \( H \) by postulating

\[
(h \cdot \varphi)(x) := \varphi(\eta(h)x) \quad \forall \ h, x \in H, \ \varphi \in H^*
\]

as well as

\[
\nabla(\varphi) = \sum_{(\varphi)} \varphi^{(0)} \otimes \varphi^{(1)} \Leftrightarrow \varphi \psi = \sum_{(\varphi)} \psi(\varphi^{(0)})\varphi^{(1)} \quad \forall \ \psi \in H^*
\]

for every \( \varphi \in H^* \). Taking these structures for granted, we can prove our Theorem.

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Proof. By the fundamental theorem of Hopf modules (cf. [2]), the multiplication induces an isomorphism
\[ \Phi : H \otimes_k (H^*)^{coH} \rightarrow H^* ; \quad h \otimes \varphi \mapsto h \cdot \varphi. \]
Given \( \varphi \in (H^*)^{coH} \), we have \( \nabla(\varphi) = 1 \otimes \varphi \), so that \( \varphi \psi = \psi(1) \varphi \) for all \( \psi \in H^* \). Consequently, \( (H^*)^{coH} \subset \int_{H^*}^r \). The reverse inclusion follows analogously. Since \( \dim_k H = \dim_k H^* \), we obtain \( \dim_k \int_{H^*}^r = 1 \). Replacing \( H \) by \( H^* \), while observing \( (H^*)^* \cong H \), yields (1).

Assertion (3) now follows from direct computation, using the fact that \( \eta \) is an anti-homomorphism of associative algebras. \( \square \)

Examples. (1) Suppose that \( H = kG \) is the group algebra of a finite group. Then \( x := \sum_{g \in G} g \) is a two-sided integral of \( kG \).

(2) In general, integrals of Hopf algebras are not easy to find. Suppose that \( \text{char}(k) = p > 0 \) and let \( g = kt \oplus kx \) be the two-dimensional non-abelian restricted Lie algebra with restricted enveloping algebra \( U_0(g) \). Thus, \( U_0(g) \) is generated by \( t \) and \( x \) subject to the relations \( t^p = t, x^p = 0, tx - xt = x \). The generators are primitive elements (that is, they satisfy \( \Delta(y) = y \otimes 1 + 1 \otimes y \) and hence are annihilated by \( \varepsilon \). Moreover, \( \eta(t) = -t \) and \( \eta(x) = -x \). Then
\[ (t^{p-1} - 1)x^{p-1} \in \int_{U_0(g)}^l \]
is a non-zero (!) left integral and \( x^{p-1}(t^{p-1} - 1) \) is a right integral. Since
\[ (t^{p-1} - 1)x^{p-1}t = (t^{p-1} - 1)x^{p-1} \]
the left integral is not a right integral.

We record an important consequence of the main theorem, namely \( H \) being a Frobenius algebra. Despite the title of their article [3], the authors were apparently not aware of this fact at the time of writing.\(^1\)

Corollary 1. Let \( \pi \in \int_{H^*}^r \) be non-zero left integral of \( H^* \). Then
\[ (x, y) := \pi(xy) \quad \forall \ x, y \in H \]
defines a non-degenerate associative form on \( H \). In particular, \( H \) is a Frobenius algebra.

Proof. Writing \((h \ast \varphi)(x) := \varphi(xh)\) for \( h, x \in H \) and \( \varphi \in H^* \), we consider the canonical homomorphism
\[ \Psi : H \rightarrow H^* ; \quad h \mapsto h \ast \pi. \]
In view of our theorem, \( \varphi_0 := \pi \circ \eta \) is a non-zero right integral of \( H^* \) and the map
\[ \Phi : H \rightarrow H^* ; \quad h \mapsto h \ast \varphi_0 \]
is an isomorphism. Direct computation shows that \( \eta^{-2}(\ker \Psi) \subset \ker \Phi = \{0\} \). Consequently, \( \Psi \) is an isomorphism, and [1, Lemma 1] implies the result. \( \square \)

\(^1\)On page 85 of [3] they note: “The referee has pointed out to us that our main theorem implies that every finite dimensional Hopf algebra with antipode is a Frobenius algebra.”
Our next application is often referred to as “Maschke’s Theorem for Hopf algebras”. Given two $H$-modules $M, N$, we recall that $\text{Hom}_k(M, N)$ obtains the structure of an $H$-module via
\[(h, \varphi)(m) = \sum_{(h)} h(1) \varphi(h(2)) m\]
for all $h \in H$, $m \in M$, $\varphi \in \text{Hom}_k(M, N)$.

**Corollary 2.** The following statements are equivalent:
1. $H$ is semi-simple.
2. $\varepsilon(\int H) \neq (0)$.

**Proof.** (1) $\Rightarrow$ (2). By assumption, the exact sequence
\[(0) \longrightarrow \ker \varepsilon \longrightarrow H \longrightarrow k \longrightarrow (0)\]
splits, so that $H = \ker \varepsilon \oplus \int H$.

(2) $\Rightarrow$ (1). The assumption entails the splitting of the above exact sequence. As a result, the trivial $H$-module $k$ is projective. Let $P$ be a projective $H$-module, $M$ be any $H$-module. The adjoint isomorphism
\[\text{Hom}_k(P \otimes_k M, N) \cong \text{Hom}_k(P, \text{Hom}_k(M, N))\]
induces an isomorphism
\[\text{Hom}_H(P \otimes_k M, N) \cong \text{Hom}_H(P, \text{Hom}_k(M, N)).\]
Consequently, $\text{Hom}_H(P \otimes_k M, -)$ is exact, so that $P \otimes_k M$ is projective. Setting $P = k$, we see that $k \otimes_k M \cong M$ is projective. This shows that $H$ is semi-simple. \hfill \Box

**Examples.** (1) Let $G$ be a finite group and consider the integral $x := \sum_{g \in G} g \in kG$. Then $\varepsilon(x) = \text{ord}(G) \cdot 1$, so that $kG$ is semi-simple if and only if $\text{char}(k) \nmid \text{ord}(G)$.

(2) Let $g = kt \oplus kx$ be as above. Then $\varepsilon((t^{p-1} - 1)x^{p-1}) = (0)$, so that $U_0(g)$ is not semi-simple. In fact, $\text{Rad}(U_0(g)) = U_0(g)x$.

**Corollary 3.** If $H$ is semi-simple, then $H$ is separable.

**Proof.** Let $K$ be an extension field of $k$. Then $H' := H \otimes_k K$ obtains the structure of a Hopf algebra by defining $\Delta' = \Delta \otimes \text{id}_K$. Here we use the identification $(H \otimes_k K) \otimes_K (H \otimes_k K) \cong (H \otimes_k H) \otimes_k K$. Since the counit $\varepsilon'$ of $H'$ is given by $\varepsilon \otimes \text{id}_K$, we get
\[\int_{H'}^{\ell} = \int_H^{\ell} \otimes_k K.\]
Thus, if $H$ is semi-simple, then
\[\varepsilon'\left(\int_{H'}^{\ell}\right) = \varepsilon\left(\int_H^{\ell}\right) K \neq (0),\]
so that $H'$ is also semi-simple. Consequently, $H$ is separable. \hfill \Box
References

