HOPF MODULES AND INTEGRALS: NAKAYAMA AUTOMORPHISMS

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We adopt the notation from our previous lectures [3, 4] and assume in particular that H is a finite dimensional Hopf algebra over a field k. As noted in [4, Cor. 1], H is a Frobenuis algebra, with any non-zero left integral $\pi \in \int_{H^*}^{\ell}$ giving rise to a non-degenerate associative form $(\ ,\ )_\pi$.

Since H is a Frobenius algebra, the Nakayama functor on $\operatorname{mod} H$ is just the twist by any Nakayama automorphism H, see [2]. Recall that the Nakayama automorphism μ relative to $(\ ,\ )_\pi$ is defined via

$$\pi(yx) = \pi(\mu(x)y) \qquad \forall\ x, y \in H.$$

In this lecture, we relate μ to the so-called left modular function of H, thereby showing in particular that μ has finite order. The relevant results can be found in [7] and [5].

Given $x \in \int_H^r$, we have $hx \in \int_H^r$ for every $h \in H$. Consequently, there exists an algebra homomorphism $\zeta_r : H \longrightarrow k$ with

$$hx = \zeta_r(h)x \qquad \forall\ h \in H,\ x \in \int_H^r.$$

This homomorphism is referred to as the *right modular function* of H. Clearly, $\int_H^\ell = \int_H^r$ if and only if $\zeta_r = \varepsilon$ coincides with the counit of H. In that case H is called *unimodular*. The *left modular function* ζ_ℓ is defined analogously. By [4, Thm.], we have

$$\zeta_\ell = \zeta_r \circ \eta^{-1}.$$

If $\gamma : H \longrightarrow k$ is a homomorphism of k-algebras, then the convolution

$$\mathrm{id}_H * \gamma : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} h_{(1)}\gamma(h_{(2)})$$

is an automorphism of the algebra H, whose inverse is given by $\mathrm{id}_H * (\gamma \circ \eta)$.

Theorem. *The automorphism $\eta^{-2} \circ (\mathrm{id}_H * \zeta_\ell)$ is a Nakayama automorphism of the Frobenius algebra H.*

Proof. Let $\pi \in \int_{H^*}^\ell$ be a non-zero left integral of H^*, so that

$$(x, y)_\pi = \pi(xy)$$

endows H with the structure of a Frobenius algebra. Directly from the defining property of π, we obtain

$$(*) \qquad \pi(h)1 = \sum_{(h)} \pi(h_{(2)})h_{(1)} \qquad \forall\ h \in H.$$

Since the map $H \longrightarrow H^*$; $h \mapsto \pi . h$ is an isomorphism of right H-modules, there exists a unique element $u_\pi \in H$ such that

$$\pi . u_\pi = \varepsilon.$$

In other words, we have

$$\pi(u_\pi h) = \varepsilon(h) \qquad \forall\ h \in H.$$

Date: May 3, 2006.

In view of $\pi((u_\pi h - \varepsilon(h)u_\pi)x) = (0)$ for all $h, x \in H$, we see that $u := u_\pi$ is a non-zero right integral of H. Given $x \in H$, we now obtain, observing $(*)$,

$$\begin{aligned}\sum_{(u)} \eta(u_{(1)})\pi(u_{(2)}x) &= \sum_{(u),(x)} \eta(u_{(1)})u_{(2)}x_{(1)}\pi(u_{(3)}x_{(2)}) = \sum_{(u),(x)} \varepsilon(u_{(1)})x_{(1)}\pi(u_{(2)}x_{(2)}) \\ &= \sum_{(x)} x_{(1)}\pi(ux_{(2)}) = \sum_{(x)} x_{(1)}\varepsilon(x_{(2)}) = x.\end{aligned}$$

Thus, letting μ be the Nakayama automorphism relative to $(\ ,\ )_\pi$, we have

$$\mu^{-1}(x) = \sum_{(u)} \eta(u_{(1)})\pi(u_{(2)}\mu^{-1}(x)) = \sum_{(u)} \pi(xu_{(2)})\eta(u_{(1)}).$$

Using $(*)$ again, we compute $\eta^{-2} \circ \mu^{-1}$:

$$\begin{aligned}(\eta^{-2} \circ \mu^{-1})(x) &= \sum_{(u)} \pi(xu_{(2)})\eta^{-1}(u_{(1)}) = \sum_{(u),(x)} \pi(x_{(2)}u_{(3)})x_{(1)}u_{(2)}\eta^{-1}(u_{(1)}) \\ &= \sum_{(u),(x)} \pi(x_{(2)}u_{(2)})x_{(1)}\varepsilon(u_{(1)}) = \sum_{(x)} \pi(x_{(2)}u)x_{(1)} = \sum_{(x)} \zeta_r(x_{(2)})\pi(u)x_{(1)} \\ &= \sum_{(x)} \zeta_r(x_{(2)})x_{(1)} = (\mathrm{id}_H * \zeta_r)(x).\end{aligned}$$

It follows that

$$\mu = (\mathrm{id}_H * \zeta_r)^{-1} \circ \eta^{-2} = (\mathrm{id}_H * (\zeta_r \circ \eta)) \circ \eta^{-2} = \eta^{-2} \circ (\mathrm{id}_H * (\zeta_r \circ \eta^{-1})) = \eta^{-2} \circ (\mathrm{id}_H * \zeta_\ell),$$

as desired. □

The following result is valid for any finite dimensional Hopf algebra. For simplicity we give a proof for the case where $\eta^2 = \mathrm{id}_H$, which holds whenever H is cocommutative.

Corollary 1. *Suppose that $\eta^2 = \mathrm{id}_H$. Then H possesses a Nakayama automorphism of finite order.*

Proof. In view of our Theorem, $\mu = \mathrm{id}_H * \zeta_\ell$ is a Nakayama automorphism of H. Next, we note that the set $\mathrm{Alg}_k(H, k)$ of algebra homomorphisms from H to k is just the group of group like elements

$$G(H^*) = \{\varphi \in H^* \setminus \{0\}\ ;\ \Delta(\varphi) = \varphi \otimes \varphi\}$$

of the Hopf algebra H^*. Since group-like elements are linearly independent, $G(H^*)$ is a finite group. In particular, there exists $n \in \mathbb{N}$ with $\zeta_\ell^n = \varepsilon$. Hence

$$\mu^n = (\mathrm{id}_H * \zeta_\ell)^n = (\mathrm{id}_H * \zeta_\ell^n) = \mathrm{id}_H,$$

as desired. □

Our last result concerns symmetry of Hopf algebras. Recall that group algebras of finite groups are always symmetric. The question of symmetry for restricted enveloping algebras and Hopf algebras was discussed by several authors [8, 6]. Given a simple H-module S, we denote by $P(S)$ its projective cover. The unique block $\mathcal{B}_0(H) \subset H$ satisfying $\varepsilon(\mathcal{B}_0(H)) \neq (0)$ is called the *principal block* of H. The following result shows in particular that weakly symmetric cocommutative Hopf algebras are symmetric.

Corollary 2. *Suppose that η^2 is an inner automorphism of H. Then the following statements are equivalent:*

(1) *H is symmetric.*
(2) *$\mathcal{B}_0(H)$ is symmetric.*
(3) *$\operatorname{Soc}(P(k)) \cong k$.*

Proof. It suffices to verify (3) ⇒ (1). Let μ be a Nakayama automorphism of H. By general theory (cf. [2]), we know that $\operatorname{Soc}(P(k)) \cong k_{\varepsilon\circ\mu}$ is the one-dimensional H-module defined by $\varepsilon \circ \mu$. In view of our Theorem, we may take $\mu = \eta^{-2} \circ (\mathrm{id}_H * \zeta_\ell)$. Observing $\varepsilon \circ \eta = \varepsilon$, we obtain

$$\varepsilon = \varepsilon \circ \mu = \varepsilon \circ (\mathrm{id}_H * \zeta_\ell) = \zeta_\ell,$$

where the last identity follows from

$$(\varepsilon \circ (\mathrm{id}_H * \zeta_\ell))(h) = \sum_{(h)} \varepsilon(h_{(1)})\zeta_\ell(h_{(2)}) = \zeta_\ell(\sum_{(h)} \varepsilon(h_{(1)})h_{(2)}) = \zeta_\ell(h) \quad \forall\ h \in H.$$

Consequently,

$$\mu = \eta^{-2} \circ (\mathrm{id}_H * \varepsilon) = \eta^{-2},$$

is an inner automorphism, implying that H is symmetric. □

References

[1] A. Berkson. *The u-algebra of a restricted Lie algebra is Frobenius.* Proc. Amer. Math. Soc. **15** (1964), 14-15
[2] R. Farnsteiner. *Self-injective algebras: Comparison with Frobenius algebras.* Lecture Notes, available at http://www.mathematik.uni-bielefeld.de/~sek/selected.html
[3] ———. *Hopf modules and integrals: The fundamental theorem.* Lecture Notes, available at http://www.mathematik.uni-bielefeld.de/~sek/selected.html
[4] ———. *Hopf modules and integrals: The space of integrals.* Lecture Notes, available at http://www.mathematik.uni-bielefeld.de/~sek/selected.html
[5] D. Fischman, S. Montgomery and H. Schneider. *Frobenius extensions of subalgebras of Hopf algebras.* Trans. Amer. Math. Soc. **349** (1996), 4857-4895
[6] J. Humphreys. *Symmetry for finite dimensional Hopf algebras.* Proc. Amer. Math. Soc. **68** (1978), 143-146.
[7] U. Oberst and H. Schneider. *Über Untergruppen endlicher algebraischer Gruppen.* Manuscripta math. **8** (1973), 217-241
[8] J. Schue. *Symmetry for the enveloping algebra of a restricted Lie algebra.* Proc. Amer. Math. Soc. **16** (1965), 1123-1124