HOPF MODULES AND INTEGRALS: MASCHKE'S THEOREM FOR LIE ALGEBRAS

ROLF FARNSTEINER

Throughout, we will be working over a field k of characteristic $p > 0$. Let $(\mathfrak{g}, [p])$ be a restricted Lie algebra over k. Recall that for any element $x \in \mathfrak{g}$, the left multiplication by x is denoted

$$\operatorname{ad} x : \mathfrak{g} \longrightarrow \mathfrak{g} \quad ; \quad y \mapsto [x, y].$$

The p-map $\mathfrak{g} \longrightarrow \mathfrak{g}$; $x \mapsto x^{[p]}$ satisfies the formal properties of an associative p-power operator. In particular, we have

- $(\operatorname{ad} x)^p = \operatorname{ad} x^{[p]} \ \ \forall\, x \in \mathfrak{g}$,
- $(\alpha x)^{[p]} = \alpha^p x^{[p]} \ \ \forall\, \alpha \in k,\ x \in \mathfrak{g}$,
- $(x + y)^{[p]} = x^{[p]} + y^{[p]}$ for $x, y \in \mathfrak{g}$ with $[x, y] = 0$.

Thus, if $\mathfrak{g}$ is an abelian Lie algebra, then the p-map is p-semilinear. If the ground field k is perfect, this implies that $[p]$ is surjective if and only if $[p]$ is injective.

Let $U(\mathfrak{g})$ be the enveloping algebra of the ordinary Lie algebra $\mathfrak{g}$. The factor algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\})$$

is called the *restricted enveloping algebra* of $\mathfrak{g}$. Up to isomorphism it is uniquely determined by the following universal property: Given any associative k-algebra Λ and any linear map $f : \mathfrak{g} \longrightarrow \Lambda$ with

(a) $f([x, y]) = f(x)f(y) - f(y)f(x)$ and
(b) $f(x^{[p]}) = f(x)^p$ for all $x, y \in \mathfrak{g}$

there exists exactly one homomorphism $\hat{f} : U_0(\mathfrak{g}) \longrightarrow \Lambda$ with $\hat{f} \circ \iota = f$. Here, $\iota : \mathfrak{g} \longrightarrow U_0(\mathfrak{g})$ is the composite of the canonical map $\mathfrak{g} \longrightarrow U(\mathfrak{g})$ with the projection $U(\mathfrak{g}) \longrightarrow U_0(\mathfrak{g})$. We take Jacobson's analog [5] of the Theorem of Poincaré-Birkhoff-Witt for granted:

Theorem 1. *Let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$. Then the monomials*

$$\iota(x_1)^{a_1} \cdots \iota(x_n)^{a_n} \quad ; \quad 0 \leq a_i \leq p - 1$$

form a basis of $U_0(\mathfrak{g})$ over k. □

Remarks. (a) In view of Theorem 1 the canonical map ι is injective and it will henceforth be suppressed.

(b) If $\mathfrak{h} \subset \mathfrak{g}$ is a p-subalgebra of $\mathfrak{g}$, then $U_0(\mathfrak{h})$ is a subalgebra of $U_0(\mathfrak{g})$ and $U_0(\mathfrak{g})$ is a free left and right $U_0(\mathfrak{h})$-module. Consequently, the canonical restriction functor

$$\operatorname{mod} U_0(\mathfrak{g}) \longrightarrow \operatorname{mod} U_0(\mathfrak{h}) \quad ; \quad M \mapsto M|_{U_0(\mathfrak{h})}$$

sends projectives to projectives.

(c) Given $x \in \mathfrak{g}$ with $x^{[p]} = 0$, we have $U_0(kx) \cong \begin{cases} k[X]/(X^p) & \text{for } x \neq 0 \\ k & \text{otherwise.} \end{cases}$

Date: May 24, 2006.

(d) Using the universal property, one endows $U_0(\mathfrak{g})$ with the structure of a Hopf algebra by defining

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad , \quad \eta(x) = -x \quad , \quad \varepsilon(x) = 0 \qquad \forall\, x \in \mathfrak{g}.$$

The following criterion for the semi-simplicity of $U_0(\mathfrak{g})$ was first established by Hochschild in [4]. His proof was based on his theory of restricted cohomology groups [3]. Subsequently, Hochschild's work was superseded by Nagata's result (cf. [1, IV,§3,3.6]), which provides a criterion for a cocommutative Hopf algebra to be semi-simple. We shall give an elementary proof by exploiting the separability of semi-simple enveloping algebras.

Theorem (Hochschild)**.** *Let $(\mathfrak{g}, [p])$ be a finite dimensional restricted Lie algebra. Then the following statements are equivalent:*

(1) *The algebra $U_0(\mathfrak{g})$ is semi-simple.*
(2) *The Lie algebra $\mathfrak{g}$ is abelian and $\langle \mathfrak{g}^{[p]} \rangle = \mathfrak{g}$.*

Proof. (1) $\Rightarrow$ (2). We proceed in several steps, assuming first k to be algebraically closed.

(i) *If $x \in \mathfrak{g}$ is an element with $x^{[p]} = 0$, then $x = 0$.*

Consider the p-subalgebra $\mathfrak{h} := kx$. By assumption, the trivial $U_0(\mathfrak{h})$-module $k = k|_{U_0(\mathfrak{h})}$ is projective, so that Remark (c) implies $U_0(\mathfrak{h}) = k$ and $x = 0$. ⋄

(ii) *$\mathfrak{g}$ is abelian and $[p]$ is bijective.*

Given $x \in \mathfrak{g}$, we consider the abelian p-subalgebra $\mathfrak{h} := \sum_{i \geq 0} kx^{[p]^i}$. In view of (i), the p-map is injective on $\mathfrak{h}$ and hence bijective. Consequently, there exist $\alpha_1, \ldots, \alpha_n \in k$ with $x = \sum_{i=1}^n \alpha_i x^{[p]^i}$. It follows $\operatorname{ad} x$ satisfies the polynomial $\sum_{i=1}^n \alpha_i X^{p^i} - X$, so that $\operatorname{ad} x$ is diagonalizable.

Now let α be an eigenvalue of $\operatorname{ad} x$. Then there exists $y \in \mathfrak{g}$ with $[x, y] = \alpha y$. Since the linear map $\operatorname{ad} y$ is diagonalizable, its restriction to the $(\operatorname{ad} y)$-invariant subspace $V := kx + ky$ enjoys the same property. As $(\operatorname{ad} y)^2|_V = 0$, we obtain $(\operatorname{ad} y)|_V = 0$, whence $\alpha = 0$. It follows that $\operatorname{ad} x = 0$ for every $x \in \mathfrak{g}$, so that $\mathfrak{g}$ is abelian. Thus, $[p]$ is p-semilinear and (i) yields the bijectivity of $[p]$. ⋄

(iii) *If k is an arbitrary field, then $\mathfrak{g}$ is abelian and $\mathfrak{g} = \langle \mathfrak{g}^{[p]} \rangle$.*

We let K be an algebraic closure of k and consider the restricted Lie algebra $\mathfrak{g}_K := \mathfrak{g} \otimes_k K$, whose Lie bracket and p-map are defined via

$$[x \otimes \alpha, y \otimes \beta] := [x, y] \otimes \alpha\beta \quad ; \quad (x \otimes \alpha)^{[p]} := x^{[p]} \otimes \alpha^p \qquad \forall\, x, y \in \mathfrak{g},\ \alpha, \beta \in K.$$

The universal property provides an isomorphism

$$U_0(\mathfrak{g}_K) \cong U_0(\mathfrak{g}) \otimes_k K$$

of Hopf algebras. Thanks to [2, Corollary 3] the algebra $U_0(\mathfrak{g}_K)$ is semi-simple, and (ii) ensures that $\mathfrak{g}_K$ is abelian with surjective p-map. Hence $\mathfrak{g}$ is abelian and there are elements $x_1, \ldots, x_n$ of $\mathfrak{g}$, such that $\{x_1^{[p]} \otimes 1, \ldots, x_n^{[p]} \otimes 1\}$ is K-basis of $\mathfrak{g}_K$. Accordingly, the set $\{x_1^{[p]}, \ldots, x_n^{[p]}\} \subset \mathfrak{g}$ is linearly independent, so that $\mathfrak{g} = \sum_{i=1}^n kx_i^{[p]} = \langle \mathfrak{g}^{[p]} \rangle$. ⋄

(2) $\Rightarrow$ (1). Setting $n := \dim_k \mathfrak{g}$, we pick elements $x_1, \ldots, x_n$ in $\mathfrak{g}$ such that $\{x_1^{[p]}, \ldots, x_n^{[p]}\}$ is a basis of $\mathfrak{g}$. Since $\mathfrak{g}$ is abelian, $\{x_1, \ldots, x_n\}$ is also a basis of $\mathfrak{g}$ over k. Let $u \in U_0(\mathfrak{g})$ be an element with $u^p = 0$. Using Theorem 1 we write

$$u = \sum \alpha_{a_1, \ldots, a_n} x_1^{a_1} \cdots x_n^{a_n}.$$

Since $U_0(\mathfrak{g})$ is abelian, we obtain

$$0 = u^p = \sum \alpha^p_{a_1, \ldots, a_n} (x_1^p)^{a_1} \cdots (x_n^p)^{a_n} = \sum \alpha^p_{a_1, \ldots, a_n} (x_1^{[p]})^{a_1} \cdots (x_n^{[p]})^{a_n}.$$

Another application of Theorem 1 gives $\alpha^p_{a_1,\dots,a_n} = 0$, so that $u = 0$. Consequently, the radical of $U_0(\mathfrak{g})$ is trivial, and the algebra $U_0(\mathfrak{g})$ is semi-simple. □

By combining Hochschild's result with the theory of finite group schemes one can show the following result.

Theorem 2 (Nagata). *Let H be a finite dimensional cocommutative Hopf algebra. Then H is semi-simple if and only if $H \cong \Lambda[G]$ is a skew group algebra, where G is a finite group with $p \nmid \mathrm{ord}(G)$, and $\Lambda \cong K_1 \times \cdots \times K_n$ is a product of finite extension fields of k.* □

References

[1] M. Demazure and P. Gabriel. *Groupes Algébriques.* Masson & Cie, Paris 1970

[2] R. Farnsteiner. *Hopf modules and integrals: The space of integrals.* Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`

[3] G. Hochschild. *Cohomology of restricted Lie algebras.* Amer. J. Math. **76** (1954), 555-580

[4] ———. *Representations of restricted Lie algebras of characteristic p.* Proc. Amer. Math. Soc. **5** (1954), 603-605

[5] N. Jacobson. *Restricted Lie algebras of characteristic p.* Trans. Amer. Math. Soc. **50** (1941), 15-25