NAKAYAMA ALGEBRAS: KUPISCH SERIES AND MORITA TYPE

ROLF FARNSTEINER

Throughout, $\Lambda$ is assumed to be a finite dimensional $k$-algebra, defined over an algebraically closed field $k$. We let $J$ be the (Jacobson) radical of $\Lambda$. A $\Lambda$-module $M$ of length $\ell(M)$ is called \textit{uniserial} if the following equivalent conditions hold:

- $M$ possesses exactly one composition series.
- $(J^i M)_{i \geq 0}$ is a composition series of $M$.
- For every $i \in \{0, \ldots, \ell(M)\}$, $J^i M$ is the unique submodule of length $\ell(M) - i$.

The algebra $\Lambda$ is referred to as \textit{pro-uniserial} if all its projective indecomposable modules are uniserial.

Let $S(\Lambda)$ denote a complete set of representatives of the simple $\Lambda$-modules.

**Proposition 1** (Thm. 9 of [2]). The following statements are equivalent:

1. $\Lambda$ is pro-uniserial
2. $\sum_{T \in S(\Lambda)} \dim_k \text{Ext}^1_{\Lambda}(S, T) \leq 1$ for every $S \in S(\Lambda)$.

**Proof.** (1) $\Rightarrow$ (2). Let $S$ be an element of $S(\Lambda)$ with projective cover $P(S)$. There results an exact sequence

\[ (*) \quad (0) \rightarrow JP(S) \rightarrow P(S) \rightarrow S \rightarrow (0). \]

If $T \in S(\Lambda)$ is another simple $\Lambda$-module, then general theory implies that

\[ (**) \quad \text{Ext}^1_{\Lambda}(S, T) \cong \text{Hom}_{\Lambda}(JP(S)/J^2P(S), T). \]

Since $P(S)$ is uniserial, the module $JP(S)/J^2P(S)$ is either $(0)$ or simple. Schur’s Lemma then yields $\dim_k \text{Ext}^1_{\Lambda}(S, T) = 1$ for at most one $T \in S(\Lambda)$.

(2) $\Rightarrow$ (1). Let $S$ be an element of $S(\Lambda)$ and consider the exact sequence $(*)$. The module $JP(S)/J^2P(S)$ is semi-simple, and condition (2) in conjunction with $(**)$ shows that $JP(S)/J^2P(S)$ is either zero or simple.

Given $n > 1$, suppose that $J^{n-1}P(S)/J^nP(S)$ is simple. If $Q$ is a projective cover of $J^{n-1}P(S)$, then it is also a projective cover of $J^{n-1}P(S)/J^nP(S)$, and the above observation ensures that $JQ/J^2Q$ is zero or simple. The surjective map $\pi : Q \rightarrow J^{n-1}P(S)$ induces a surjection $\tilde{\pi} : JQ/J^2Q \rightarrow J^nP(S)/J^{n+1}P(S)$, so that the latter module is also either zero or simple. It now follows inductively that the Loewy series of $(J^iP(S))_{0 \leq i \leq \ell(P(S))}$ is a composition series. Consequently, $P(S)$ is uniserial. \hfill $\square$

**Corollary 2.** The algebra $\Lambda$ is pro-uniserial if and only if $\Lambda/J^2$ is pro-uniserial.

**Proof.** Setting $\Lambda' := \Lambda/J^2$, we note that the pullback functor

\[ \pi^* : \text{mod} \Lambda' \rightarrow \text{mod} \Lambda \]

induces a bijection between the simple modules. Moreover, $P(S)/J^2P(S)$ is the projective cover of the simple $\Lambda$-module $S$, considered as a $\Lambda'$-module. It readily follows from $(**)$, that

\[ \text{Ext}^1_{\Lambda}(\pi^*(S), \pi^*(T)) \cong \text{Ext}^1_{\Lambda'}(S, T) \quad \forall S, T \in S(\Lambda'). \]

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Our assertion is now a direct consequence of Proposition 1.

**Definition.** The algebra $\Lambda$ is a Nakayama algebra if every projective indecomposable and every injective indecomposable $\Lambda$-module is uniserial.

**Remarks.** (1) A self-injective algebra is a Nakayama algebra if and only if it is pro-uniserial.

(2) The algebra $\Lambda = k[1 \rightarrow 2 \leftarrow 3]$ is pro-uniserial, but the injective indecomposable $\Lambda$-module $I_2$ belonging to the vertex 2 has a top of length 2, so that $\Lambda$ is not a Nakayama algebra.

(3) Using duality, we see that $\Lambda$ is a Nakayama algebra if and only if $\Lambda$ and $\Lambda^{\text{op}}$ are pro-uniserial. Consequently, Corollary 2 also holds for Nakayama algebras.

(4) An algebra $\Lambda$ is a Nakayama algebra if and only if Proposition 1 and its dual

$$
\sum_{T \in S(\Lambda)} \dim_k \text{Ext}^1_{\Lambda}(T, S) \leq 1 \quad \forall S \in S(\Lambda)
$$

hold.

**Proposition 3.** Let $\Lambda$ be a Nakayama algebra. Then every indecomposable $\Lambda$-module is uniserial, and $\Lambda$ has finite representation type.

**Proof.** We prove the first assertion by induction on the Loewy length $\ell(\Lambda)$ of $\Lambda$, the case $\ell(\Lambda) = 1$ being trivial. Assuming $\ell := \ell(\Lambda) \geq 2$, we consider an indecomposable $\Lambda$-module $M$. If $J^{\ell-1}M = (0)$, then $M$ is an indecomposable module for the Nakayama algebra $\Lambda/J^{\ell-1}$, and the inductive hypothesis yields the assertion. Alternatively, there exists a simple left ideal $S \subset J^{\ell-1}$ with $S.M \neq (0)$. We can therefore find $m \in M \setminus \{0\}$ such that

$$
\hat{\psi}_m : S \rightarrow M \ ; \ s \mapsto s.m
$$

is injective. Hence there is a map $\hat{\psi}_m : M \rightarrow E(S)$ to the injective envelope $E(S)$ of $S$, whose composite with $\psi_m$ is the canonical inclusion $S \hookrightarrow E(S)$. As $E(S)$ is uniserial, we can find $i \geq 0$ with $\hat{\psi}_m(M) = J^iE(S)$. Consequently, $J^{\ell-1}M \subset \ker \hat{\psi}_m$, while $J^{\ell-1}M \not\subset \ker \hat{\psi}_m$. As a result $\hat{\psi}_m$ is surjective and $J^{\ell-1}E(S) \neq (0)$. Since the uniserial projective cover $\pi : P \rightarrow E(S)$ of $E(S)$ satisfies $\ell(P) = \ell(E(S)) = \ell(E(S))$, we have $P \cong E(S)$. As $M$ is indecomposable, it now follows that $\hat{\psi}_m$ is an isomorphism. Thus, $M$ is uniserial.

As an upshot of the above, every indecomposable $\Lambda$-module $M$ has a simple top and is thus of the form

$$
M \cong P(S)/J^iP(S) \quad 0 \leq i \leq \ell(\Lambda),
$$

for some simple module $S$. Consequently, $\Lambda$ has finite representation type.

**Example.** The path algebra $k[\tilde{D}_4]$ of the four subspace quiver $\tilde{D}_4$ is pro-uniserial, but not of finite representation type. The same holds of course for any subspace quiver involving at least four subspaces.

We let $Q_\Lambda$ be the Gabriel quiver of $\Lambda$ and denote by $A_n$ and $\tilde{A}_n$ the quivers with vertices $\{1, \ldots, n\}$ and $\mathbb{Z}/(n+1)$, respectively and arrows $i \rightarrow i + 1$.

An analogue of following result, which is an easy consequence of Proposition 1 and its dual, was established by Kupisch prior to the introduction of quivers.
**Theorem 4** (cf. Satz 5 of [3]). Let Λ be a connected Nakayama algebra. Then $Q_\Lambda = A_n, \tilde{A}_n$.

**Proof.** Let $p$ be a directed path of maximal length in $Q_\Lambda$ subject to every vertex of $Q_\Lambda$ occurring at most once. We denote by $V(p)$ the set of vertices of $p$ and claim that $V(p) = (Q_\Lambda)_0$.

Writing $V(p) = \{p_1, \ldots, p_n\}$ with arrows $p_i \rightarrow p_{i+1}$, we suppose there is a vertex $x \in (Q_\Lambda)_0 \setminus V(p)$ which is connected to some vertex $p_i \in V(p)$. If $x \rightarrow p_i$, then the dual of Proposition 1 implies $i = 1$, and the maximality of $p$ gives a contradiction. Alternatively, we have $p_i \rightarrow x$, and the above reasoning first shows $i = n$ and then yields a contradiction. Since $Q_\Lambda$ is connected, our claim follows.

Let $\alpha \in (Q_\Lambda)_1$ be an arrow. If the starting point of $\alpha$ is $p_i$, then Proposition 1 shows that $\alpha$ belongs to the path whenever $i < n$. For $i = n$, the dual of Proposition 1 implies that $\alpha$ is the unique arrow $p_n \rightarrow p_1$. As an upshot of our discussion, we conclude that $Q_\Lambda = A_n$ in case there is no arrow originating in $p_n$, and $Q_\Lambda = \tilde{A}_{n-1}$ otherwise. \hfill $\Box$

In view of our Theorem there exists an ordering $S_1, \ldots, S_n$ of the simple Λ-modules such that their projective covers $P_i := P(S_i)$ satisfy

$$JP_i/J^2P_i \cong S_{i+1}, \quad 1 \leq i \leq n-1,$$

with $JP_n/J^2P_n \cong S_1$ if $JP_n \neq (0)$. This ordering is often called the Kupisch series of Λ. Note that the foregoing isomorphism also implies

$$\ell(P_{i+1}) \geq \ell(P_i) - 1.$$

It follows from the above, that the Morita equivalence class of Λ is determined by the $n$-tuple $(\ell(P_1), \ldots, \ell(P_n))$.

**Example.** Suppose that Λ is a connected hereditary Nakayama algebra. Then Λ is Morita equivalent to $k[A_n]$, so that $\ell(P_i) = n + 1 - i$. Note that $k[A_n]$ is isomorphic to the algebra of lower triangular $(n \times n)$-matrices.

We let $k[\tilde{A}_n]^{\dagger}$ be the space generated by all paths of length $\geq 1$.

**Corollary 5.** Let Λ be a connected Nakayama algebra. Then Λ is self-injective if and only if Λ is Morita equivalent to $k[\tilde{A}_n]/(k[\tilde{A}_n]^{\dagger})^m$ for $n = |S(\Lambda)| - 1$ and $m = \ell(\Lambda)$.

**Proof.** If Λ is Morita equivalent to $k[\tilde{A}_n]/(k[\tilde{A}_n]^{\dagger})^m$, then we have $\text{Soc}(P_i) \cong S_{i+m-1}$, where the indices are to be taken mod$(n + 1)$. In view of [1, Theorem], the algebra Λ is self-injective.

For the reverse direction, we pick $r$ such that $\ell(P_r)$ is maximal. If $n \neq 0$, then no simple Λ-module is projective and there is a surjection

$$P_{r+1} \twoheadrightarrow JP_r.$$ 

Since $\ell(P_r) \geq \ell(P_{r+1}) \geq \ell(P_r) - 1$, the assumption $\ell(P_r) \neq \ell(P_{r+1})$ implies that the above map is in fact an isomorphism. Thus, $JP_r$ is injective and hence a direct summand of $P_r$. Consequently, $JP_r = (0)$, so that $S_r$ is projective, a contradiction. We obtain $\ell(P_{r+1}) = \ell(P_r)$, and repeat the argument to see that $\ell(P_i) = \ell(\Lambda)$ for $i \in \{1, \ldots, n + 1\}$.

Since Λ has Loewy length $m = \ell(P_r)$, it follows that Λ is Morita equivalent to $k[\tilde{A}_n]/(k[\tilde{A}_n]^{\dagger})^m$. \hfill $\Box$
References

