ALGEBRAS OF FINITE GLOBAL DIMENSION: THE NO LOOPS CONJECTURE

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In this talk we outline the proof of the following result, due to Igusa [2, Thm.3.2], which confirms the no-loops conjecture for algebras over algebraically closed fields:

Theorem. *Let Λ be a finite-dimensional algebra over an algebraically closed field k. If* $\operatorname{gldim}\Lambda < \infty$*, then* $\operatorname{Ext}^1_\Lambda(S,S) = (0)$ *for every simple Λ-module S.*

Our theorem turns out to be a fairly direct consequence of earlier work by Lenzing [3], who employed K-theoretic methods to obtain information on nilpotent elements in rings of finite global dimension. We thus begin in a more general setting, assuming that Λ is an arbitrary ring. Let $\operatorname{mod}\Lambda$ be the category of finitely generated Λ-modules. Given a full subcategory $\mathcal{F} \subseteq \operatorname{mod}\Lambda$, we define $K_1(\mathcal{F})$ to be the free abelian group generated by pairs (M,f), with $M \in \mathcal{F}$ and $f \in \operatorname{End}_\Lambda(M)$, subject to the following relations:

(a) $(M, f+g) = (M,f) + (M,g)$

(b) $(M',f') + (M'',f'') = (M,f)$, for every commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & (0) \\ & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\ (0) & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & (0) \end{array}$$

with exact rows.

(c) $(M, \beta\circ\alpha) = (N, \alpha\circ\beta)$, if $\alpha : M \longrightarrow N$ and $\beta : N \longrightarrow M$.

If $\mathcal{F}' \subseteq \mathcal{F}$ are full subcategories of $\operatorname{mod}\Lambda$, then the inclusion $\iota : \mathcal{F}' \longrightarrow \mathcal{F}$ induces a homomorphism

$$\iota_* : K_1(\mathcal{F}') \longrightarrow K_1(\mathcal{F}) \quad ; \quad [(M,f)] \longrightarrow [(M,f)]$$

of abelian groups. We denote by $\mathcal{P}(\Lambda)$ and $\mathcal{P}_0(\Lambda)$ be the full subcategories of $\operatorname{mod}\Lambda$, whose objects are the finitely generated projective modules and the module Λ, respectively.

Our analysis of various K_1-groups utilizes trace functions for projective modules which take values in the factor group $\Lambda/[\Lambda,\Lambda]$. Here $[a,b] := ab - ba$ is the usual Lie product on Λ. Given $P \in \mathcal{P}(\Lambda)$, we recall that

$$\psi_P : \operatorname{Hom}_\Lambda(P,\Lambda)\otimes_\Lambda P \longrightarrow \operatorname{End}_\Lambda(P) \quad ; \quad \psi(f\otimes p)(q) := f(q)p$$

is an isomorphism (cf. [1, (II.4.4)]). Using the map

$$\varphi_P : \operatorname{Hom}_\Lambda(P,\Lambda)\otimes_\Lambda P \longrightarrow \Lambda/[\Lambda,\Lambda] \quad ; \quad f\otimes p \mapsto [f(p)],$$

we define

$$\operatorname{tr}_P : \operatorname{End}_\Lambda(P) \longrightarrow \Lambda/[\Lambda,\Lambda] \quad ; \quad f \mapsto \varphi_P\circ\psi_P^{-1}(f).$$

This function enjoys the usual properties of a trace:

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- If P is free and f is represented by some matrix $A := (a_{ij}) \in \mathrm{Mat}_n(\Lambda)$, then $\mathrm{tr}_P(f) = [\sum_{i=1}^n a_{ii}] \in \Lambda/[\Lambda,\Lambda]$.
- Given free modules P and P', we have
$$(*) \qquad \mathrm{tr}_P(g\circ f) = \mathrm{tr}_{P'}(f\circ g)$$
for $f \in \mathrm{Hom}_\Lambda(P,P')$ and $g \in \mathrm{Hom}_\Lambda(P',P)$.
- If P and Q are finitely generated projective Λ-modules and $f \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}_P(f) = \mathrm{tr}_{P\oplus Q}(f\oplus 0)$. Thus, by considering projective modules as direct summands of suitable free modules, we obtain $(*)$ for all projective modules.

Lemma 1. *The map* $\iota_* : K_1(\mathcal{P}_0(\Lambda)) \longrightarrow K_1(\mathcal{P}(\Lambda))$ *is surjective.*

Proof. Let $[(P,f)]$ be an element of $K_1(\mathcal{P}(\Lambda))$. Then there exists a finitely generated projective Λ-module Q such that $P\oplus Q = \Lambda^n$, so that (b) implies
$$[\Lambda^n, f\oplus 0)] = [(P,f)] + [(Q,0)].$$
In view of (a), the second summand vanishes, so that $[(P,f)] = [(\Lambda^n,g)]$ for some $g \in \mathrm{End}_\Lambda(\Lambda^n)$. Writing $\Lambda^n = \Lambda^{n-1}\oplus\Lambda$, we obtain
$$g = \begin{pmatrix} g_{n-1} & \gamma \\ \omega & g_1 \end{pmatrix},$$
with $g_i \in \mathrm{End}_\Lambda(\Lambda^i)$. Setting
$$h := \begin{pmatrix} g_{n-1} & \gamma \\ 0 & g_1 \end{pmatrix} \text{ and } h' := \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix},$$
relation (a) gives
$$[(\Lambda^n,g)] = [(\Lambda^n,h)] + [(\Lambda,h')].$$
Relation (b) then yields
$$[(\Lambda^n,h)] = [(\Lambda^{n-1},g_{n-1})] + [(\Lambda,g_1)] \text{ as well as } [(\Lambda,h')] = 0.$$
This readily implies the surjectivity of ι_*. □

Proposition 2. *The map*
$$\mathrm{Tr} : K_1(\mathcal{P}(\Lambda)) \longrightarrow \Lambda/[\Lambda,\Lambda] \quad ; \quad [(P,f)] \mapsto \mathrm{tr}_P(f)$$
is an isomorphism

Proof. Let $\mathcal{H}$ be the free abelian group generated by the pairs (P,f) with $P \in \mathcal{P}(\Lambda)$ and $f \in \mathrm{End}_\Lambda(P)$. Then there exists a unique homomorphism
$$\mathrm{tr} : \mathcal{H} \longrightarrow \Lambda/[\Lambda,\Lambda] \quad ; \quad (P,f) \mapsto \mathrm{tr}_P(f).$$
Evidently, tr is surjective and direct computation shows that tr annihilates the defining relations of $K_1(\mathcal{P}(\Lambda))$. This implies the existence and surjectivity of Tr.

Thanks to Lemma 1, every element of $K_1(\mathcal{P}(\Lambda))$ is of the form $[(\Lambda,f)]$. If $[(\Lambda,f)] \in \ker \mathrm{Tr}$, then $f(1) \in [\Lambda,\Lambda]$. This implies $f \in [\mathrm{End}_\Lambda(\Lambda), \mathrm{End}_\Lambda(\Lambda)]$, and relations (a) and (c) give $[(\Lambda,f)] = 0$. □

We now specialize to the case where Λ is noetherian. Then every $M \in \mathrm{mod}\,\Lambda$ affords a projective resolution, whose constituents belong to $\mathcal{P}(\Lambda)$.

Proposition 3. *Suppose that* $\operatorname{gldim}\Lambda < \infty$*. Then*

$$\iota_* : K_1(\mathcal{P}(\Lambda)) \longrightarrow K_1(\operatorname{mod}\Lambda)$$

is an isomorphism.

Proof. Let $M \in \operatorname{mod}\Lambda$. Since Λ is noetherian, there is a finite projective resolution

$$P_\bullet(M): \qquad (0) \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow (0),$$

with $P_i \in \mathcal{P}(\Lambda)$. If $f : M \longrightarrow M$ is a linear map, then there exists a chain map $\varphi : P_\bullet(M) \longrightarrow P_\bullet(M)$ with $\varphi_{-1} = f$. We then define

$$\kappa(M, f) := \sum_{i=0}^{n} (-1)^i [(P_i, \varphi_i)].$$

This definition neither depends on the choice of φ, nor on that of $P_\bullet(M)$. Moreover, κ defines a surjective homomorphism

$$\kappa : K_1(\operatorname{mod}\Lambda) \longrightarrow K_1(\mathcal{P}(\Lambda)).$$

Owing to relation (b), we have $\iota_* \circ \kappa = \mathrm{id}_{K_1(\operatorname{mod}\Lambda)}$. Hence κ is also injective, and ι_* is in fact an isomorphism. □

The following theorem is Lenzing's aforementioned result, see [3, Satz 5].

Theorem 4. *Suppose that* Λ *has finite global dimension. If* $a \in \Lambda$ *is a nilpotent element, then* $a \in [\Lambda, \Lambda]$.

Proof. Given any pair (M, f), the commutative diagram

$$\begin{array}{ccccccccc} (0) & \longrightarrow & \ker f & \xrightarrow{\iota} & M & \xrightarrow{f} & \operatorname{im} f & \longrightarrow & (0) \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f|_{\operatorname{im} f}} & & \\ (0) & \longrightarrow & \ker f & \xrightarrow{\iota} & M & \xrightarrow{f} & \operatorname{im} f & \longrightarrow & (0) \end{array}$$

yields $[(M, f)] = [(\operatorname{im} f, f|_{\operatorname{im} f})] + [(\ker f, 0)] = [(\operatorname{im} f, f|_{\operatorname{im} f})]$. Thus, if f is nilpotent, then $[(M, f)] = 0$.

Let $a \in \Lambda$ be nilpotent, and consider the right multiplication $r_a : \Lambda \longrightarrow \Lambda$. Then r_a is nilpotent, so that $[(\Lambda, r_a)] = 0$ in $K_1(\operatorname{mod}\Lambda)$. Owing to Proposition 3, $[(\Lambda, r_a)]$ is the zero element in $K_1(\mathcal{P}(\Lambda))$, whence

$$0 = \operatorname{Tr}([(\Lambda, r_a)]) = a + [\Lambda, \Lambda],$$

as asserted. □

Proof of the Theorem. Using Morita equivalence, we may assume that Λ is basic (at this point k being algebraically closed enters). Let J be the Jacobson radical of Λ. Then $\Lambda' := \Lambda/J^2$ is also basic with Jacobson radical $J' := J/J^2$. The algebra Λ' is graded

$$\Lambda' = \Lambda'_0 \oplus \Lambda'_1,$$

with $\Lambda'_1 = J'$ and $\Lambda'_0 = \bigoplus_{i=1}^n ke_i$ being defined by a complete set of orthogonal primitive idempotents e_i of Λ'. In particular, Λ'_0 is commutative, so that $[\Lambda', \Lambda'] = [\Lambda'_0, \Lambda'_1]$. Owing to Theorem 4, we have $J \subseteq [\Lambda, \Lambda]$, whence

$$\Lambda'_1 \subseteq [\Lambda', \Lambda'] = [\Lambda'_0, \Lambda'_1].$$

Given $x \in \Lambda'_1$, we can therefore find $y_i \in \Lambda'_1$ with $x = \sum_{i=1}^n e_i y_i - y_i e_i$. Consequently,

$$e_j x e_j = e_j y_j e_j - e_j y_j e_j = 0.$$

Let S_j be the simple Λ-module corresponding to e_j. Then we have

$$\mathrm{Ext}^1_\Lambda(S_j, S_j) \cong \mathrm{Ext}^1_{\Lambda'}(S_j, S_j) \cong e_j J' e_j = e_j \Lambda'_1 e_j = (0).$$

This concludes the proof of our Theorem. □

References

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