In this talk we outline the proof of the following result, due to Igusa [2, Thm.3.2], which confirms the no-loops conjecture for algebras over algebraically closed fields:

**Theorem.** Let $\Lambda$ be a finite-dimensional algebra over an algebraically closed field $k$. If $\text{gldim } \Lambda < \infty$, then $\text{Ext}^1_{\Lambda}(S,S) = (0)$ for every simple $\Lambda$-module $S$.

Our theorem turns out to be a fairly direct consequence of earlier work by Lenzing [3], who employed $K$-theoretic methods to obtain information on nilpotent elements in rings of finite global dimension. We thus begin in a more general setting, assuming that $\Lambda$ is an arbitrary ring. Let $\text{mod } \Lambda$ be the category of finitely generated $\Lambda$-modules. Given a full subcategory $F \subseteq \text{mod } \Lambda$, we define $K^1(F)$ to be the free abelian group generated by pairs $(M,f)$, with $M \in F$ and $f \in \text{End}_{\Lambda}(M)$, subject to the following relations:

(a) $(M,f + g) = (M,f) + (M,g)$
(b) $(M',f') + (M'',f'') = (M,f)$, for every commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
& & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0
\end{array}
\]

with exact rows.

(c) $(M,\beta \circ \alpha) = (N,\alpha \circ \beta)$, if $\alpha : M \rightarrow N$ and $\beta : N \rightarrow M$.

If $F' \subseteq F$ are full subcategories of $\text{mod } \Lambda$, then the inclusion $\iota : F' \rightarrow F$ induces a homomorphism $\iota_* : K_1(F') \rightarrow K_1(F) ; ([M,f]) \mapsto ([M,f])$ of abelian groups. We denote by $\mathcal{P}(\Lambda)$ and $\mathcal{P}_0(\Lambda)$ be the full subcategories of $\text{mod } \Lambda$, whose objects are the finitely generated projective modules and the module $\Lambda$, respectively.

Our analysis of various $K_1$-groups utilizes trace functions for projective modules which take values in the factor group $\Lambda/[[\Lambda,\Lambda]]$. Here $[a,b] := ab - ba$ is the usual Lie product on $\Lambda$. Given $P \in \mathcal{P}(\Lambda)$, we recall that

$$\psi_P : \text{Hom}_{\Lambda}(P,\Lambda) \otimes_{\Lambda} P \rightarrow \text{End}_{\Lambda}(P) ; \psi(f \otimes p)(q) := f(q)p$$

is an isomorphism (cf. [1, (II.4.4)]). Using the map

$$\varphi_P : \text{Hom}_{\Lambda}(P,\Lambda) \otimes_{\Lambda} P \rightarrow \Lambda/[[\Lambda,\Lambda]] ; f \otimes p \mapsto [f(p)],$$

we define

$$\text{tr}_P : \text{End}_{\Lambda}(P) \rightarrow \Lambda/[[\Lambda,\Lambda]] ; f \mapsto \varphi_P \circ \psi_P^{-1}(f).$$

This function enjoys the usual properties of a trace:
If $P$ is free and $f$ is represented by some matrix $A := (a_{ij}) \in \text{Mat}_n(\Lambda)$, then $\text{tr}_P(f) = \sum_{i=1}^n a_{ii} \in \Lambda/[\Lambda, \Lambda]$.

Given free modules $P$ and $P'$, we have

\begin{equation}
\text{tr}_P(g \circ f) = \text{tr}_P'(f \circ g) \tag{\ast} \end{equation}

for $f \in \text{Hom}_\Lambda(P, P')$ and $g \in \text{Hom}_\Lambda(P', P)$.

If $P$ and $Q$ are finitely generated projective $\Lambda$-modules and $f \in \text{End}_\Lambda(P)$, then $\text{tr}_P(f) = \text{tr}_{P \oplus Q}(f \oplus 0)$. Thus, by considering projective modules as direct summands of suitable free modules, we obtain (\ast) for all projective modules.

**Lemma 1.** The map $\iota_* : K_1(P_0(\Lambda)) \to K_1(\mathcal{P}(\Lambda))$ is surjective.

**Proof.** Let $[(P, f)]$ be an element of $K_1(\mathcal{P}(\Lambda))$. Then there exists a finitely generated projective $\Lambda$-module $Q$ such that $P \oplus Q = \Lambda^n$, so that (b) implies

$$[\Lambda^n, f \oplus 0] = [(P, f)] + [(Q, 0)].$$

In view of (a), the second summand vanishes, so that $[(P, f)] = [(\Lambda^n, g)]$ for some $g \in \text{End}_\Lambda(\Lambda^n)$. Writing $\Lambda^n = \Lambda^{n-1} \oplus \Lambda$, we obtain

$$g = \begin{pmatrix}
g_{n-1} & \gamma \\
0 & g_1
\end{pmatrix},$$

with $g_i \in \text{End}_\Lambda(\Lambda^i)$. Setting

$$h := \begin{pmatrix}
g_{n-1} & \gamma \\
0 & g_1
\end{pmatrix} \text{ and } h' := \begin{pmatrix}
0 & 0 \\
\omega & 0
\end{pmatrix},$$

relation (a) gives

$$[(\Lambda^n, g)] = [(\Lambda^n, h)] + [(\Lambda, h')].$$

Relation (b) then yields

$$[(\Lambda^n, h)] = [(\Lambda^{n-1}, g_{n-1})] + [(\Lambda, g_1)] \text{ as well as } [(\Lambda, h')] = 0.$$ 

This readily implies the surjectivity of $\iota_*$. \qed

**Proposition 2.** The map

$$\text{Tr} : K_1(\mathcal{P}(\Lambda)) \to \Lambda/[\Lambda, \Lambda] : [(P, f)] \mapsto \text{tr}_P(f)$$

is an isomorphism.

**Proof.** Let $\mathcal{H}$ be the free abelian group generated by the pairs $(P, f)$ with $P \in \mathcal{P}(\Lambda)$ and $f \in \text{End}_\Lambda(P)$. Then there exists a unique homomorphism

$$\text{tr} : \mathcal{H} \to \Lambda/[\Lambda, \Lambda] : (P, f) \mapsto \text{tr}_P(f).$$

Evidently, $\text{tr}$ is surjective and direct computation shows that $\text{tr}$ annihilates the defining relations of $K_1(\mathcal{P}(\Lambda))$. This implies the existence and surjectivity of $\text{Tr}$.

Thanks to Lemma 1, every element of $K_1(\mathcal{P}(\Lambda))$ is of the form $[(\Lambda, f)]$. If $[(\Lambda, f)] \in \ker \text{Tr}$, then $f(1) \in [\Lambda, \Lambda]$. This implies $f \in [\text{End}_\Lambda(\Lambda), \text{End}_\Lambda(\Lambda)]$, and relations (a) and (c) give $[(\Lambda, f)] = 0$. \qed

We now specialize to the case where $\Lambda$ is noetherian. Then every $M \in \text{mod } \Lambda$ affords a projective resolution, whose constituents belong to $\mathcal{P}(\Lambda)$.
Proposition 3. Suppose that $\text{gldim} \Lambda < \infty$. Then

$$\iota_* : K_1(\mathcal{P}(\Lambda)) \to K_1(\text{mod} \Lambda)$$

is an isomorphism.

Proof. Let $M \in \text{mod} \Lambda$. Since $\Lambda$ is noetherian, there is a finite projective resolution

$$P_\bullet(M) : (0) \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to (0),$$

with $P_i \in \mathcal{P}(\Lambda)$. If $f : M \to M$ is a linear map, then there exists a chain map $\varphi : P_\bullet(M) \to P_\bullet(M)$ with $\varphi_{-1} = f$. We then define

$$\kappa(M, f) := \sum_{i=0}^n (-1)^i [(P_i, \varphi_i)].$$

This definition neither depends on the choice of $\varphi$, nor on that of $P_\bullet(M)$. Moreover, $\kappa$ defines a surjective homomorphism

$$\kappa : K_1(\text{mod} \Lambda) \to K_1(\mathcal{P}(\Lambda)).$$

Owing to relation (b), we have $\iota_* \circ \kappa = \text{id}_{K_1(\text{mod} \Lambda)}$. Hence $\kappa$ is also injective, and $\iota_*$ is in fact an isomorphism. \qed

The following theorem is Lenzing’s aforementioned result, see [3, Satz 5].

Theorem 4. Suppose that $\Lambda$ has finite global dimension. If $a \in \Lambda$ is a nilpotent element, then $a \in [\Lambda, \Lambda]$.

Proof. Given any pair $(M, f)$, the commutative diagram

$$
\begin{array}{cccccc}
(0) & \to & \ker f & \xrightarrow{i} & M & \xrightarrow{f} & \text{im } f & \to & (0) \\
\downarrow 0 & & \downarrow f & & \downarrow f_{\text{im } f} & \\
(0) & \to & \ker f & \xrightarrow{i} & M & \xrightarrow{f} & \text{im } f & \to & (0)
\end{array}
$$

yields $[(M, f)] = [(\text{im } f, f_{\text{im } f})] + [(\ker f, 0)] = [(\text{im } f, f_{\text{im } f})]$. Thus, if $f$ is nilpotent, then $[(M, f)] = 0$.

Let $a \in \Lambda$ be nilpotent, and consider the right multiplication $r_a : \Lambda \to \Lambda$. Then $r_a$ is nilpotent, so that $[(\Lambda, r_a)] = 0$ in $K_1(\text{mod} \Lambda)$. Owing to Proposition 3, $[(\Lambda, r_a)]$ is the zero element in $K_1(\mathcal{P}(\Lambda))$, whence

$$0 = \text{Tr}([(\Lambda, r_a)]) = a + [\Lambda, \Lambda],$$

as asserted. \qed

Proof of the Theorem. Using Morita equivalence, we may assume that $\Lambda$ is basic (at this point $k$ being algebraically closed enters). Let $J$ be the Jacobson radical of $\Lambda$. Then $\Lambda' := \Lambda/J^2$ is also basic with Jacobson radical $J' := J/J^2$. The algebra $\Lambda'$ is graded

$$\Lambda' = \Lambda'_0 \oplus \Lambda'_1,$$

with $\Lambda'_1 = J'$ and $\Lambda'_0 = \bigoplus_{i=1}^n ke_i$ being defined by a complete set of orthogonal primitive idempotents $e_i$ of $\Lambda'$. In particular, $\Lambda'_0$ is commutative, so that $[\Lambda', \Lambda'] = [\Lambda'_0, \Lambda'_1]$. Owing to Theorem 4, we have $J \subseteq [\Lambda, \Lambda]$, whence

$$\Lambda'_1 \subseteq [\Lambda', \Lambda'] = [\Lambda'_0, \Lambda'_1].$$

Given $x \in \Lambda'_1$, we can therefore find $y_i \in \Lambda'_1$ with $x = \sum_{i=1}^n e_i y_i - y_i e_i$. Consequently,

$$e_j x e_j = e_j y_j e_j - e_j y_j e_j = 0.$$
Let $S_j$ be the simple $\Lambda$-module corresponding to $e_j$. Then we have

$$\text{Ext}_1^\Lambda(S_j, S_j) \cong \text{Ext}_1^\Lambda(J, e_j) = e_jJ'e_j = (0).$$

This concludes the proof of our Theorem. \hfill \Box

**References**

