THE THEOREM OF WEDDERBURN-MALCEV: CONJUGACY OF MAXIMAL SEPARABLE SUBALGEBRAS

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Throughout, A denotes a finite dimensional algebra over a field k. We denote by $\operatorname{Rad}(A)$ the Jacobson (nilpotent) radical of A. In our previous lecture [1] we gave an account of the cohomological proof of Wedderburn's Principal Theorem. It was shown that the result follows from the vanishing of the second Hochschild cohomology $H^2(A/\operatorname{Rad}(A), M)$ with coefficients in any bimodule M. In that case, we can find a subalgebra $S \subset A$ with

$$A = S \oplus \operatorname{Rad}(A).$$

Thus, $S \cong A/\operatorname{Rad}(A)$ is uniquely determined up to isomorphism.

Our objective here is to show that this isomorphism comes from an inner automorphism of A. The proof is again cohomological and involves the first Hochschild cohomology groups

$$H^1(A, M) = \operatorname{Ext}^1_{A^e}(A, M)$$

with coefficients in an (A, A)-bimodule M. A linear map $d : A \longrightarrow M$ is referred to as a *derivation* if

$$d(ab) = a.d(b) + d(a).b \quad \forall\ a, b \in A.$$

Given $m \in M$, the linear map

$$d_m : A \longrightarrow M \ \ ; \ \ a \mapsto a.m - m.a$$

is the *inner derivation* effected by m. Using the bar resolution one sees that the group $H^1(A, M)$ is the factor space of derivations by inner derivations.

Theorem 1 (Malcev)**.** *Suppose there is a subalgebra $S \subset A$ such that $A = S \oplus \operatorname{Rad}(A)$. If $T \subset A$ is a separable subalgebra, then there exists an element $n \in \operatorname{Rad}(A)$ such that*

$$(1 + n)T(1 + n)^{-1} \subset S.$$

Proof. We first consider the case where $\operatorname{Rad}(A)^2 = (0)$. The decomposition $A = S \oplus \operatorname{Rad}(A)$ provides k-linear maps $f : T \longrightarrow S$ and $g : T \longrightarrow \operatorname{Rad}(A)$ such that

$$t = f(t) + g(t) \qquad \forall\ t \in T.$$

Direct computation shows that

- f is a homomorphism of k-algebras, and
- $g(st) = f(s)g(t) + g(s)f(t) \quad \forall\ s, t \in T.$

Thus, by endowing $M := \operatorname{Rad}(A)$ with the structure of a (T, T)-bimodule induced by f, we see that g is a derivation of T with values in M. Since T is separable, [1, Lemma 2] implies that the enveloping algebra $T^e := T \otimes_k T^{\mathrm{op}}$ is semi-simple, so that $H^1(T, M) = (0)$. Consequently, there exists an element $n \in \operatorname{Rad}(A)$ such that

$$g(t) = t.n - n.t = f(t)n - nf(t) \qquad \forall\ t \in T.$$

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As a result, $t = f(t) + g(t) = f(t)(1+n) - nf(t)$, so that, observing $\mathrm{Rad}(A)^2 = (0)$, we obtain

$$(1+n)t(1+n)^{-1} = (1+n)f(t) - nf(t)(1-n) = f(t) \in S$$

for every $t \in T$.

We now prove the theorem by induction on the nilpotency class of $\mathrm{Rad}(A)$, that is, the number $\ell \in \mathbb{N}$ satisfying

$$\mathrm{Rad}(A)^\ell = (0) = \mathrm{Rad}(A)^{\ell-1}.$$

Assuming $\ell \geq 2$, we consider the algebra $A' := A/\mathrm{Rad}(A)^{\ell-1}$, whose radical has nilpotency class $\leq \ell - 1$. Let $\pi : A \longrightarrow A'$ be the canonical projection, and consider the subalgebras $S' := \pi(S)$ and $T' := \pi(T)$ of A'. Since S and T are semi-simple, we have $(\ker \pi) \cap S = (0) = (\ker \pi) \cap T$. Upon application of the inductive hypothesis to $A' = S' \oplus \mathrm{Rad}(A')$, we find an element $m \in \mathrm{Rad}(A)$ such that

$$T_1 := (1+m)T(1+m)^{-1} \subset S \oplus \mathrm{Rad}(A)^{\ell-1} =: B.$$

Thus, T_1 is a separable subalgebra of B and $\mathrm{Rad}(B)^2 = (0)$. Our earlier observations now provide $m' \in \mathrm{Rad}(A)^{\ell-1}$ such that

$$(1+m')(1+m)T(1+m)^{-1}(1+m')^{-1} \subset S.$$

Since $(1+m')(1+m) = 1+m+m'$, the element $n := m+m'$ has the requisite property. □

It still remains to be seen whether separability is essential for the validity of the Theorem of Wedderburn-Malcev. In his papers [3, 4] Hochschild studies these questions in detail. In [1, Lemma 2] we showed that the vanishing of the first Hochschild cohomology groups follows from the separability of A. The following result gives the converse implication (cf. [3, (4.1)]):

Lemma 2. *Suppose that $H^1(A, M) = (0)$ for every (A, A)-bimodule M. Then A is separable.*

Proof. The multiplication $\mu : A \otimes_k A \longrightarrow A$ defines an exact sequence

$$(0) \longrightarrow M \longrightarrow A \otimes_k A \xrightarrow{\mu} A \longrightarrow (0)$$

of A^e-modules. Our assumption entails the splitting of this sequence, so that

$$(*) \qquad A \otimes_k A \cong A \oplus M.$$

Let K be an extension field of k. Since we have an isomorphism $(A \otimes_k K) \otimes_K (A \otimes_k K) \cong (A \otimes_k A) \otimes_k K$ of modules over $(A \otimes_k K)^e \cong A^e \otimes_k K$, tensoring $(*)$ with K yields an isomorphism

$$(A \otimes_k K) \otimes_K (A \otimes_k K) \cong (A \otimes_k K) \oplus (M \otimes_k K)$$

of $(A \otimes_k K)^e$-modules. Consequently, property $(*)$ is invariant under base field extension, and we only have to show that it implies the semi-simplicity of A.

Let N be a finite dimensional left A-module. Upon tensoring $(*)$ with N over A, we arrive at a decomposition

$$(A \otimes_k A) \otimes_A N \cong (A \otimes_A N) \oplus (M \otimes_A N)$$

of left A-modules, with $a \in A$ acting via $a \otimes 1$. Note that the canonical isomorphisms $A \otimes_A N \cong N$ and $(A \otimes_k A) \otimes_A N \cong A \otimes_k N$ are A-linear. Since the latter A-module is free, we conclude that N is projective. Consequently, A is semi-simple. □

Wedderburn's Principal Theorem only requires the vanishing of the second cohomology groups. One might therefore hope that the result also obtains for inseparable algebras. However, we have the following result:

Theorem 3 ([4]). *If A is semi-simple and inseparable, then there exists an (A,A)-bimodule M such that $H^2(A,M) \neq (0)$.* □

Examples. Suppose that $\mathrm{char}(k) = p > 0$, and let $E{:}k$ be a purely inseparable extension of exponent one, where $E = k(\alpha)$, $a := \alpha^p \in k$, $\alpha \notin k$.

(1) We define a derivation $d : E \longrightarrow E$ via

$$d(\alpha^i) = i\alpha^{i-1} \qquad 1 \leq i \leq p-1.$$

Then d gives rise to a cocycle

$$f : E \times E \longrightarrow E \;\; ; \;\; (a,b) \mapsto \sum_{r=1}^{p-1} \frac{1}{p}\binom{p}{r} d^r(a)d^{p-r}(b),$$

which is not a coboundary. This implies that the extension

$$(0) \longrightarrow E \longrightarrow E \ltimes_f E \longrightarrow E \longrightarrow (0)$$

does not split, so that Wedderburn's Principal Theorem does not obtain.

(2) Consider the k-algebra $A := E \otimes_k E$. The multiplication

$$\mu : A \longrightarrow E \;\; ; \;\; a \otimes b \mapsto ab$$

is a homomorphism of k-algebras, whose kernel N is the (left) ideal generated by the elements $\{x \otimes 1 - 1 \otimes x \;,\; x \in E\}$. Since $x^p \in k$ for every $x \in E$, the ideal N is nilpotent and thus coincides with $\mathrm{Rad}(A)$.

Observing $a - \mu(a) \otimes 1$, $a - 1 \otimes \mu(a) \in N$ for every $a \in A$, we obtain

$$A = (E \otimes 1) \oplus N = (1 \otimes E) \oplus N.$$

Since A is commutative, the isomorphic subalgebras $E \otimes 1$ and $1 \otimes E$ are not conjugate in A.

Now that we understand the importance of separability, we need to know when a semi-simple algebra is separable. We have already seen that this holds for group algebras of finite groups and, in fact, it is true for Hopf algebras. Given a simple k-algebra A, it turns out that separability of the field extension $\mathcal{Z}(A){:}k$, given by the center $\mathcal{Z}(A)$ is decisive. Since extension fields of perfect fields are separable, one obtains in particular:

Theorem 4. *Let k be a perfect field. Then every semi-simple k-algebra A is separable.* □

References

[1] R. Farnsteiner. *The Theorem of Wedderburn-Malcev: $H^2(A,N)$ and extensions*. Lecture Notes, available at `http://www.mathematik.uni-bielefeld.de/~sek/selected.html`
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