Let $\Lambda$ be an artin algebra. As shown in our previous lecture [3], Riedtmann's Theorem sets
the stage for the determination of the regular components of the Auslander-Reiten quiver of $\Lambda$.
More precisely, given such a component $Q$, abstract considerations, which make no reference to the
category $\text{mod } \Lambda$ of finitely generated left $\Lambda$-modules, provide a directed tree $T_Q$ and an admissible
subgroup $\Pi \subset \text{Aut}(\mathbb{Z}[T_Q])$ such that
$$Q \cong \mathbb{Z}[T_Q]/\Pi.$$  
We say that $Q$ is tree-infinite if the tree class $\bar{T}_Q$ of $Q$ is an infinite tree.

In this lecture, we determine $T_Q$ and $\Pi$ of tree-infinite components subject to some hypotheses
concerning the growth numbers of the modules belonging to $Q$. The relevant conditions obtain in
classical contexts, such as group algebras of finite groups.

Here is the magic number:
$$\omega := \sqrt{\frac{1}{2}} + \sqrt{\frac{23}{108}} + \sqrt{\frac{1}{2} - \sqrt{\frac{23}{108}}}.$$  

**Theorem** ([6]). Let $Q$ be a non-periodic, tree-infinite, regular component of the Auslander-Reiten
quiver of $\Lambda$. If the growth numbers $\rho^L_Q$ and $\rho^R_Q$ of $Q$ are smaller that $\omega$, then
$$Q \cong \mathbb{Z}[T], \text{ where } T \in \{A_{\infty}, A_{\infty}^\infty, B_{\infty}, C_{\infty}, D_{\infty}\}.$$  

The result refers to valued quivers, thus the distinction between $A_{\infty}$, $B_{\infty}$ and $C_{\infty}$. To ease the
technical aspects, we will ignore valuations, thereby eliminating $B_{\infty}$ and $C_{\infty}$. This simplification
is legitimate in case $\Lambda$ is a finite dimensional algebra over an algebraically closed field.

An AR-component $Q$ is called regular if it contains neither projective nor injective vertices.
Regular components are stable representation quivers with $\tau = \text{DTr}$.

Let $Q$ be a regular AR-component. We say that $Q$ is periodic if for each isoclass $[M] \in Q$ there
exists a natural number $m \geq 1$ such that
$$\text{DTr}^m(M) \cong M.$$  

Thanks to a result due to Happel-Preiser-Ringel [4, Cor.2], periodic components are of the form
$Q \cong \mathbb{Z}[A_{\infty}] / (\tau^m)$ for some $m \geq 1$.

We let $k$ be a commutative artinian ring such that $\Lambda$ is a finitely generated $k$-module. The length
of a $k$-module module $M$ will be denoted $\ell(M)$. There exists a natural number $\ell_\Lambda$ such that
$$\max\{\ell(\text{DTr}(M)), \ell(\text{TrD}(M))\} \leq \ell_\Lambda \ell(M)$$  

for every $M \in \text{mod } \Lambda$. We may thus make the following definition, which does not depend on the
choice of $k$:

---

**Date:** January 18, 2006.
Definition. Let $M \in \mod \Lambda$. Then
\[ \rho^\ell_M := \limsup_{n \to \infty} \sqrt[n]{\ell(DTr^n(M))} \quad \text{and} \quad \rho^r_M := \limsup_{n \to \infty} \sqrt[n]{\ell(TrD^n(M))} \]
are called the left and right growth numbers of $M$, respectively.

Remark. By definition, we have
\[ \ell(DTr^n(M)) \approx (\rho^\ell_M)^n \]
for infinitely many $n$, so that our notion of growth refers to exponential growth.

Lemma 1. If $Q$ is a regular component of the AR-quiver of $\mod \Lambda$, then
\[ \rho^\ell_M := \rho^\ell_N \quad \text{and} \quad \rho^r_M := \rho^r_N \quad [M], [N] \in Q. \quad \square \]

Accordingly, we can define the left and right growth numbers
\[ \rho^\ell_Q := \rho^\ell_M \quad \text{and} \quad \rho^r_Q := \rho^r_M \quad [M] \in Q \]
of the component $Q$.

The proof of our Theorem rests on a comparison between the growth numbers of $Q$ and the spectral radius of the Coxeter transformation of $T_Q$.

Let $H$ be a hereditary algebra with Grothendieck group $K_0(H) \cong \mathbb{Z}^m$. The Coxeter transformation $\Phi : \mathbb{Z}^m \longrightarrow \mathbb{Z}^m$ is defined via
\[ \Phi([P(S)]) = -[I(S)], \]
where $P(S)$ and $I(S)$ are the projective cover and the injective hull of the simple $H$-module $S$, respectively. If $M$ is an indecomposable, non-projective $H$-module, then, letting $\dim M \in \mathbb{Z}^m$ be the dimension vector of $M$, we have
\[ \dim DTr(M) = \Phi(\dim M). \]

Viewing $\Phi$ as a linear map of $\mathbb{C}^m$, we let
\[ \rho_H := \max \{ |\lambda| ; \lambda \in \text{Spec}(\Phi) \} \]
be the maximal modulus of all eigenvalues of $\Phi$. By the Perron-Frobenius Theorem $\rho_A$ is a simple eigenvalue of $\Phi$ and
\[ |\lambda| < \rho_A \quad \forall \lambda \neq \rho_A \in \text{Spec}(\Phi). \]

To get a feeling for the connection between the spectral radius and growth numbers, let us consider the following result:

Proposition 2 ([5]). Let $H$ be a wild connected hereditary algebra. Then there exists $x^+ \in \mathbb{R}^m$ such that for every regular $H$-module $M$ there is $\alpha_M > 0$ with
\[ \lim_{n \to \infty} \rho_H^{-n} \dim DTr^n(M) = \alpha_M x^+. \]

Upon application of the continuous function $x \mapsto \sum_{i=1}^m x_i$, we obtain
\[ \lim_{n \to \infty} \sqrt[n]{\dim DTr^n(M)} = \rho_H. \]

Our goal is to establish a similar result for arbitrary artin algebras. Given a finite quiver $\Gamma$ without oriented cycles, we put $\rho_{\Gamma} := \rho_{\mathbb{C}[\Gamma]}$. 
Lemma 3. If $T$ and $T'$ are finite directed trees with the same underlying graph, then $\rho_T = \rho_{T'}$. □

Definition. Let $T$ be a tree. The spectral radius $\varrho(T)$ is defined via

$$\varrho(T) = \sup_{T' \subset T \text{ finite}} \rho(T'),$$

where $\rho(T') := \rho_{T'}$.

Proposition 4. There exist infinite trees $T_1, \ldots, T_5$ with the following properties:

(1) $\rho(T_1) = \min\{\rho(T_i) : 1 \leq i \leq 5\} = \omega$.

(2) If $T$ be an infinite tree not belonging to $\{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}$, then $T_i \subset T$ for some $i \in \{1, \ldots, 5\}$ and $\rho(T) \geq \omega$. □

The graph $T_1$

\[\cdots \quad \cdots \quad \cdots \]

Proof of the Theorem. Let $Q$ be an AR-component as given in the Theorem. By Riedtmann’s Theorem, there exists a directed tree $T_Q$ and an admissible subgroup $\Pi \subset \text{Aut}(\mathbb{Z}[T_Q])$ such that

$$\mathbb{Z}[T_Q]/\Pi \cong Q.$$ 

By assumption, the tree class $\bar{T}_Q$ is an infinite tree.

If $\bar{T}_Q \notin \{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}$, then $T_i \subset \bar{T}_Q$ for some $i \in \{1, \ldots, 5\}$. Let us assume that $T^i \subset \bar{T}_Q$ and observe that $T^1$ can be approximated by wild trees $T_j$. We can apply Proposition 2 to each $T_j$ and combine it with a theorem by Bautista [1] to arrive at the estimate

$$\min\{\rho_Q^\ell, \rho_Q^r\} \geq \rho(\bar{T}_Q).$$

Our current assumption in conjunction with Proposition 4 now gives a contradiction, so that $T_Q \in \{A_\infty, A_\infty^\infty, B_\infty, C_\infty, D_\infty\}$. To complete the proof, we require the following facts concerning automorphisms of $\mathbb{Z}[T_Q]$:

- If $\bar{T}_Q = A_\infty, B_\infty, C_\infty$, then $\text{Aut}(\mathbb{Z}[T_Q]) = \langle \tau \rangle$.
- If $\{1\} \neq \Pi \subset \text{Aut}(\mathbb{Z}[D_\infty])$ is admissible, then there exists $n \in \mathbb{N}$ with $\tau^n \in \Pi$.
- If $T_Q \equiv A_\infty^\infty$ and $Q$ is regular, then $\Pi = \{1\}$, cf. [2].

Since the component $Q$ is not periodic, the group $\Pi$ does not contain a positive power of $\tau$ and is therefore trivial. □
References