Throughout, we shall be working over a field $k$. All $k$-vector spaces are assumed to be finite-dimensional.

Let $\Lambda$ be a $k$-algebra. The objective of this lecture is to characterize non-degenerate associative forms on $\Lambda$ in terms of coalgebra structures. Such an interrelation apparently plays a rôle within topological quantum field theory, which exclusively deals with commutative Frobenius algebras.

**Definition.** A coalgebra $(C, \Delta, \varepsilon)$ is a triple, consisting of a $k$-vector space $C$ and two linear maps $\Delta : C \to C \otimes_k C$ and $\varepsilon : C \to k$ such that

1. $(\text{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_C) \circ \Delta$ (co-associativity), and
2. $(\varepsilon \otimes \text{id}_C) \circ \Delta = \text{id}_C = (\text{id}_C \otimes \varepsilon) \circ \Delta$ (counit).

Here $\otimes$ refers to the ordinary tensor product of maps followed by scalar multiplication.

We require two simple observations:

- If $(C, \Delta, \varepsilon)$ is a coalgebra and $f : V \to C$ is an isomorphism of $k$-vector spaces, then $V$ obtains the structure of a coalgebra via

  $$\Delta_V := (f^{-1} \otimes f^{-1}) \circ \Delta \circ f \quad \text{and} \quad \varepsilon_V := \varepsilon \circ f.$$ 

- If $\Lambda$ is a $k$-algebra with multiplication $m : \Lambda \otimes_k \Lambda \to \Lambda$, then the dual space $\Lambda^*$ carries a coalgebra structure, with comultiplication $m^* : \Lambda^* \to \Lambda^* \otimes_k \Lambda^*$ and counit $1 : \Lambda \to k$ given by

  $$m^*(f) = \sum_{i=1}^n f_i \otimes g_i \iff f(ab) = \sum_{i=1}^n f_i(a)g_i(b) \quad \forall \ a, b \in \Lambda$$

  and

  $$1(f) = f(1) \quad \forall \ f \in \Lambda^*.$$ 

Recall that $\Lambda$ is a Frobenius algebra if $\Lambda$ possesses a non-degenerate associative form

$$(, ) : \Lambda \times \Lambda \to k,$$

that is,

$$(ax, b) = (a, xb) \quad \forall \ a, b, x \in \Lambda$$

If $(, ) : \Lambda \times \Lambda \to k$ is such a form, then there exists an automorphism $\mu : \Lambda \to \Lambda$ such that

$$(y, x) = (\mu(x), y) \quad \forall \ x, y \in \Lambda.$$ 

This automorphism is referred to as the Nakayama automorphism of the Frobenius algebra $(\Lambda, (, ))$. As explained in [1], $\mu$ corresponds to the Nakayama permutation $\nu$ of the self-injective algebra $\Lambda$.

The associative form $(, )$ induces an isomorphism

$$\Theta : \Lambda \to \Lambda^* : \ a \mapsto (a, -)$$

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of \( k \)-vector spaces. In view of the above observations, \( \Theta \) in turn gives rise to the following coalgebra structure on \( \Lambda \):

\[
\Delta := (\Theta^{-1} \otimes \Theta^{-1}) \circ m^* \circ \Theta ; \quad \varepsilon := 1 \circ \Theta.
\]

We can re-write this in the following form:

\[
(*) \quad \Delta(a) = \sum_{i=1}^{n} a_i \otimes b_i \quad \Leftrightarrow \quad (a, xy) = \sum_{i=1}^{n} (a_i, x)(b_i, y) \quad \forall x, y \in \Lambda
\]

and

\[
(**) \quad \varepsilon(a) = (a, 1) \quad \forall a \in \Lambda.
\]

**Proposition 1.** Let \( \Lambda \) be a \( k \)-algebra. Then the following statements are equivalent:

1. \( \Lambda \) is a Frobenius algebra.
2. There exists a coalgebra structure \((\Lambda, \Delta, \varepsilon)\) such that

\[
\Delta(x) = \sum_{i=1}^{n} a_i x \otimes b_i = \sum_{i=1}^{n} a_i \otimes x b_i \quad \forall x \in \Lambda.
\]

**Proof.** (2) \( \Rightarrow \) (1). We define \(( , ) : \Lambda \times \Lambda \longrightarrow k\) via

\[
(x, y) := \varepsilon(xy) \quad \forall x, y \in \Lambda.
\]

Then \(( , )\) is an associative form. If \( x \in \Lambda \) satisfies \((a, x) = 0\) for every \( a \in \Lambda\), then

\[
x = (\varepsilon \otimes \text{id}_\Lambda) \circ \Delta(x) = \sum_{i=1}^{n} \varepsilon(a_i x)b_i = \sum_{i=1}^{n} (a_i, x)b_i = 0,
\]

so that \(( , )\) is non-degenerate.

(1) \( \Rightarrow \) (2). Given a non-degenerate associative form \(( , )\) with Nakayama automorphism \( \mu \), we define \( \Delta \) and \( \varepsilon \) via \((*)\) and \((***)\). We write \( \Delta(1) = \sum_{i=1}^{n} a_i \otimes b_i \) and obtain for \( x, y, z \in \Lambda \)

\[
\sum_{i=1}^{n} (a_i, y)(xb_i, z) = \sum_{i=1}^{n} (a_i, y)(x, b_i z) = \sum_{i=1}^{n} (a_i, y)(b_i z, \mu^{-1}(x)) = \sum_{i=1}^{n} (a_i, y)(b_i, z \mu^{-1}(x)) = (1, yz \mu^{-1}(x)) = (yz, \mu^{-1}(x)) = (x, yz),
\]

so that \( \Delta(x) = \sum_{i=1}^{n} a_i \otimes xb_i \). The other identity follows similarly. \( \square \)

In order to construct examples, we need to identify \( \Delta(1) \). We call an ordered pair \((\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})\) of ordered bases of \( \Lambda \) a dual pair for \(( , )\) if

\[
(x_i, y_j) = \delta_{ij} \quad 1 \leq i, j \leq n.
\]

**Lemma 2.** Let \((\Lambda, ( , ))\) be a Frobenius algebra with Nakayama automorphism \( \mu \) and associated coalgebra \((\Lambda, \Delta, \varepsilon)\). If \((\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\})\) is a dual pair, then the following identities hold:

1. \( \Delta(1) = \sum_{i=1}^{n} x_i \otimes y_i \).
2. \( \sum_{i=1}^{n} \mu(a) x_i \otimes y_i = \sum_{i=1}^{n} x_i \otimes y_i a \quad \forall a \in \Lambda.\)
Proof. Writing $\Delta(1) = \sum_{j=1}^{m} a_j \otimes b_j$, Proposition 1 implies
\[
x = (\text{id}_{\Lambda} \otimes \varepsilon) \circ \Delta(x) = \sum_{j=1}^{m} a_j \varepsilon(x b_j) = \sum_{j=1}^{m} (x b_j) a_j
\]
for every $x \in \Lambda$. Consequently, $\{a_1, \ldots, a_n\}$ generates the vector space $\Lambda$. After suitable renumbering we may thus assume that $\{a_1, \ldots, a_n\}$ is a basis of $\Lambda$. If $\Delta(1) = \sum_{i=1}^{n} a_i \otimes b_i'$, the above identity now implies that $\{(a_1, \ldots, a_n), \{b_1', \ldots, b_n'\}\}$ is a dual pair.

Consider the isomorphism $\Omega : \Lambda \otimes_k \Lambda \rightarrow \text{Hom}_k(\Lambda, \Lambda)$, given by
\[
\Omega(a \otimes b)(z) := (a, z)b \quad \forall \ a, b, z \in \Lambda.
\]
If $\{(x_1, \ldots, x_n), \{y_1, \ldots, y_n\}\}$ is a dual pair, then $\Omega(\sum_{i=1}^{n} x_i \otimes y_i) = \text{id}_{\Lambda}$, so that
\[
\Delta(1) = \sum_{i=1}^{n} a_i \otimes b_i' = \sum_{i=1}^{n} x_i \otimes y_i.
\]
It remains to verify (2). Note that
\[
\Omega(\mu(a) x \otimes y)(z) = (\mu(a) x, z)y = (\mu(a), xz)y = (x, za)y = \Omega(x \otimes y)(za),
\]
whence
\[
\Omega(\sum_{i=1}^{n} \mu(a) x_i \otimes y_i)(z) = \Omega(\sum_{i=1}^{n} x_i \otimes y_i)(za) = za = \text{id}_{\Lambda}(za) = \sum_{i=1}^{n} (x_i, z)y_i a = \Omega(\sum_{i=1}^{n} x_i \otimes y_i a)(z),
\]
as desired. \qed

Examples. (1) Let $G$ be a finite group. The group algebra $kG$ is a Frobenius algebra with associative form given by
\[
(g, h) = \delta_{gh, 1} \quad \forall \ g, h \in G.
\]
Writing $G = \{g_1, \ldots, g_n\}$, we obtain a dual pair $\{(g_1, \ldots, g_n), \{g_1^{-1}, \ldots, g_n^{-1}\}\}$. In view of Proposition 1 and Lemma 2, the associated coalgebra structure on $kG$ satisfies
\[
\Delta(x) = \sum_{g \in G} gx \otimes g^{-1} = \sum_{i=1}^{n} g \otimes x g^{-1} \quad \forall \ x \in kG.
\]
This markedly differs from the usual coalgebra structure on $kG$, reflecting the fact that the counit of a Hopf algebra $H$ defines a non-degenerate form on $H$ if and only if $\dim_k H = 1$.

Let $m : kG \otimes_k kG \rightarrow kG$ be the multiplication. Then $m \circ \Delta(x) \in \mathcal{F}(kG)$, the center of $kG$, so that $m \circ \Delta$ is invertible only if $G$ is abelian. Moreover, $m \circ \delta(1) = \text{ord}(G)1$, rendering the separability of $kG$ another necessary condition for invertibility. Clearly, both conditions together are also sufficient.

(2) Let $\Lambda := k[X]/(X^n)$ be a truncated polynomial ring with canonical basis $\{1, \ldots, x^{n-1}\}$. Then
\[
(x^i, x^j) = \delta_{n-1, i+j}
\]
defines a non-degenerate associative form on $\Lambda$ such that $\{(1, \ldots, x^{n-1}), \{x^{n-1}, \ldots, 1\}\}$ is a dual pair. Consequently,
\[
\Delta(a) = \sum_{i=0}^{n-1} x^i a \otimes x^{n-1-i} \quad \forall \ a \in \Lambda.
\]
Since $(m \circ \delta)(a) = nx^{n-1}a$, the map $m \circ \delta$ is invertible if and only if $n = 1$.

(3) Let $\Lambda := \text{Mat}_n(k)$ be the algebra of $(n \times n)$-matrices. Then
\[
(x, y) = \text{tr}(xy) \quad \forall \ x, y \in \Lambda
\]
enows Λ with the structure of a Frobenius algebra such that the standard basis of Λ defines a dual pair \((E_{ij}, E_{ji})\). Direct computation yields
\[
m \circ \Delta(x) = (\sum_{j=1}^{n} x_{jj})I_n,
\]
so that \(m \circ \Delta\) is invertible if and only if \(n = 1\). (Here \(I_n\) denotes the identity matrix.)

The passage between non-degenerate associative forms on Λ is effected by invertible elements of Λ. Let \(u \in \Lambda^\times\) be invertible. If \(\{a, b\} = (ua, b)\) \(\forall a, b \in \Lambda\), then a dual pair \(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\}\) of Λ relative to \((, )\) induces a dual pair \(\{u^{-1}x_1, \ldots, u^{-1}x_n\}, \{y_1, \ldots, y_n\}\) relative to \(\{, \}\). Letting Λ act on \(\Lambda \otimes_k \Lambda\) via left multiplication on the first factor, we may thus write
\[
\Delta_{(,)} = u^{-1}\Delta_{(,)}.
\]
Consequently, \(m \circ \Delta_{(,)} = 1\). (Here \(\Lambda_{(,)}\) denotes the identity matrix.)

Recall that Λ is referred to as separable if the algebra \(\Lambda K := \Lambda \otimes_k K\) is semi-simple for every field extension \(K : k\).

**Proposition 3.** Let \((\Lambda, (, ))\) be a Frobenius algebra with multiplication \(m : \Lambda \otimes_k \Lambda \rightarrow \Lambda\) and associated coalgebra \((\Lambda, \Delta, \varepsilon)\). Then the following statements are equivalent.

1. The map \(m \circ \Delta\) is an invertible homomorphism of right \(\Lambda\)-modules.
2. The algebra \(\Lambda\) is commutative and separable.

**Proof.** (1) \(\Rightarrow\) (2). By assumption, there exists an invertible element \(u \in \Lambda\) such that \(m \circ \Delta = \ell_u\), the left multiplication effected by \(u\). In view of the above, passage to another associative form allows us to assume that \(m \circ \Delta = 1\).

Let \(\{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\}\) be a dual pair of \((, )\). Given \(a \in \Lambda\), Lemma 2(2) yields
\[
\mu(a) = \mu(a)(m \circ \Delta)(1) = \mu(a)(\sum_{i=1}^{n} x_i y_i) = m(\sum_{i=1}^{n} \mu(a) x_i \otimes y_i) = m(\sum_{i=1}^{n} x_i \otimes y_i a) = a.
\]
In view of Lemma 2, the map \(a \mapsto \sum_{i=1}^{n} a x_i \otimes y_i\) is a \((\Lambda, \Lambda)\)-linear splitting of the multiplication of \(\Lambda\), and Lemma 2 of [2] ensures that \(\Lambda\) is separable.

Let \(K\) be an algebraic closure of \(k\). Then
\[
(x \otimes \alpha, y \otimes \beta)_K := (x, y)\alpha\beta \quad \forall x, y \in \Lambda, \quad \alpha, \beta \in K
\]
defines an associative form on \(\Lambda_K\) with dual pair \(\{x_1 \otimes 1, \ldots, x_n \otimes 1\}, \{y_1 \otimes 1, \ldots, y_n \otimes 1\}\). Consequently, the associated comultiplication \(\Delta_K\) satisfies
\[
m_K \circ \Delta_K = 1_{\Lambda_K}.
\]
Since \(\Lambda_K\) is semi-simple, Wedderburn’s theorem provides a decomposition
\[
\Lambda = \bigoplus_{i=1}^{r} \text{Mat}_{n_i}(K),
\]
with the summands being mutually orthogonal. Consequently, each summand satisfies (1), and the trace form of our example (3) also enjoys this property. This, however, implies \(n_i = 1\), so that \(\Lambda\) is commutative.
(2) ⇒ (1). Let $K$ be an algebraic closure of $k$. Then $\Lambda_K \cong K^n$ for some $n \in \mathbb{N}$. As before, we consider the extended form and coalgebra structure on $\Lambda_K$. Let $\{e_1, \ldots, e_n\}$ be the primitive idempotents of $\Lambda_K$. Then $(e_i, e_j)_K = 0$ for $i \neq j$, so that $\alpha_i := (e_i, e_i)_K \neq 0$. Setting $\beta_i := \alpha_i^{-1}$, we obtain a dual pair $(\{e_1, \ldots, e_n\}, \{\beta_1 e_1, \ldots, \beta_n e_n\})$. The element $u' := \sum_{i=1}^n \beta_i e_i \in \Lambda_K$ is invertible and

$$(m_K \circ \Delta_K)(x) = \sum_{i=1}^n \beta_i xe_i = u'x \quad \forall \ x \in \Lambda_K.$$ 

Since $(m_K \circ \Delta_K)(\Lambda \otimes 1) \subseteq \Lambda \otimes 1$, it follows that $u' \in \Lambda \otimes 1$. Hence there exists an invertible element $u \in \Lambda$ such that

$$(m \circ \Delta)(a) = \ell_u(a)$$

for every element $a \in \Lambda$. □

References
