SIMPLE MODULES AND p-REGULAR CLASSES

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Let Λ be a finite dimensional algebra over an algebraically closed field k. One fundamental problem is to determine the number s_Λ of isomorphism classes of simple Λ-modules. If Λ is semi-simple, then Wedderburn's Theorem yields an isomorphism

$$\Lambda \cong \mathrm{Mat}_{n_1}(k) \oplus \cdots \oplus \mathrm{Mat}_{n_{s_\Lambda}}(k),$$

so that $s_\Lambda = \dim_k \mathfrak{Z}(\Lambda)$ is the dimension of the center $\mathfrak{Z}(\Lambda)$ of Λ.

If $\Lambda = kG$ is the group algebra of a finite group G, then $\dim_k \mathfrak{Z}(kG)$ is the number c_G of conjugacy classes of kG, and Maschke's Theorem implies $c_G = s_{kG}$ whenever $\mathrm{char}(k) \nmid \mathrm{ord}(G)$.

The examples of local group algebras show that $s_{kG} \neq c_G$ for not necessarily semi-simple group algebras. Suppose that $\mathrm{char}(k) = p > 0$, and consider an abelian group G. Then

$$G = P \times Q$$

is a direct product of its Sylow-p-subgroup P and a group Q of order prime to p. Every simple kG-module is given by an algebra homomorphism $\lambda : kG \longrightarrow k$, which corresponds a group homomorphism $\lambda : G \longrightarrow k^\times$ from G to the multiplicative group $k^\times = k \setminus \{0\}$ of the field k. Since $k^\times$ has no elements of order a proper p-power, it follows that $s_{kG} = s_{kQ} = \mathrm{ord}(Q)$ is the number of *p-regular elements* of G. This is the content of Dickson's early result [3] concerning this problem.

About thirty years later, Brauer [1] provided a solution for arbitary finite groups. He returned to the subject again in his article [2].

We henceforth assume that k is an algebraically closed field of characteristic $p > 0$.

Definition. Let G be a finite group. A conjugacy class $C \subset G$ is called *p-regular* if it contains an element whose order is not divisible by p.

Theorem. *Let G be a finite group. Then s_{kG} coincides with the number of p-regular classes of G.*

We begin by giving a characterization of s_Λ for an arbitrary k-algebra Λ. In the sequel, J denotes the (Jacobson) radical of Λ. We consider Λ as a Lie algebra via the commutator product

$$[x, y] := xy - yx \qquad \forall\ x, y \in \Lambda.$$

Let $\Lambda^{(1)} = [\Lambda, \Lambda]$ be the derived algebra, and define

$$\mathcal{N}_p(\Lambda) := \{x \in \Lambda\ ;\ \exists\ n \in \mathbb{N}_0 \text{ with } x^{p^n} \in \Lambda^{(1)}\}.$$

We record the following basic properties:

(1) If $\Lambda = \Lambda_1 \times \Lambda_2$ is a product of algebras, then $\mathcal{N}_p(\Lambda) = \mathcal{N}_p(\Lambda_1) \times \mathcal{N}_p(\Lambda_2)$.
(2) $(x + y)^p \equiv x^p + y^p \mod(\Lambda^{(1)})$.
(3) $(xy - yx)^p \equiv (xy)^p - (yx)^p = [x, y(xy)^{p-1}] \equiv 0 \mod(\Lambda^{(1)}) \quad \forall\ x, y \in \Lambda$.
(4) Let $\pi : \Lambda \longrightarrow \Lambda/J$ be the canonical projection. Then $\mathcal{N}_p(\Lambda) = \pi^{-1}(\mathcal{N}_p(\Lambda/J))$.

Date: June 14, 2006.

Lemma 1. *There exist linear maps $\omega_i : \Lambda \longrightarrow k$ for $1 \le i \le s_\Lambda$ such that*
(a) $\omega_i(xy) = \omega_i(yx) \quad \forall\, x, y \in \Lambda$, *and*
(b) $\omega_i(x^p) = \omega_i(x)^p \quad \forall\, x \in \Lambda$, *and*
(c) $\mathcal{N}_p(\Lambda) = \bigcap_{i=1}^{s_\Lambda} \ker \omega_i$.

Proof. We write

$$\Lambda/J \cong \bigoplus_{i=1}^{s_\Lambda} \mathrm{Mat}_{n_i}(k)$$

and let

$$\omega_i := \mathrm{tr}_i \circ \mathrm{pr}_i \circ \pi$$

be the composition of the projections $\pi : \Lambda \longrightarrow \Lambda/J$, $\mathrm{pr}_i : \Lambda/J \longrightarrow \mathrm{Mat}_{n_i}(k)$ and the trace function $\mathrm{tr}_i : \mathrm{Mat}_{n_i}(k) \longrightarrow k$. Since tr_i satisfies (a) and (b) and $\mathrm{pr}_i \circ \pi$ is a homomorphism of k-algebras, properties (a) and (b) hold.

In view of property (4), it suffices to verify

$$\mathcal{N}_p(\Lambda/J) = \bigcap_{i=1}^{s_\Lambda} \ker(\mathrm{tr}_i \circ \mathrm{pr_i}).$$

If $\Gamma = \mathrm{Mat}_n(k)$ is a matrix algebra, then $\Gamma^{(1)} = \mathfrak{sl}(n)$ is the special linear Lie algebra. Since $\mathrm{tr}(x^p) = \mathrm{tr}(x)^p$ for all $x \in \Gamma$, we obtain $\mathcal{N}_p(\Gamma) = \ker \mathrm{tr}$. It follows that

$$\bigcap_{i=1}^{s_\Lambda} \ker(\mathrm{tr}_i \circ \mathrm{pr_i}) = \ker \mathrm{tr}_1 \times \cdots \times \ker \mathrm{tr}_{s_\Lambda} = \prod_{i=1}^{s_\Lambda} \mathcal{N}_p(\mathrm{Mat}_{n_i}(k)),$$

so that property (1) yields the desired result. □

Lemma 2. *We have $s_\Lambda = \dim_k \Lambda/\mathcal{N}_p(\Lambda)$.*

Proof. Using the above notation, we let $v_j \in \mathrm{Mat}_{n_j}(k)$ be a matrix of trace 1 and put $u_j := (\delta_{ij} v_i)_{1 \le i \le s_\Lambda} \in \Lambda/J$. Picking $x_j \in \pi^{-1}(u_j)$, we obtain

$$\omega_i(x_j) = \delta_{ij}.$$

In view of (c), the map $\omega : \Lambda \longrightarrow k^{s_\Lambda}\ ; x \mapsto (\omega_1(x), \ldots, \omega_{s_\Lambda}(x))$ induces an isomorphism $\Lambda/\mathcal{N}_p(\Lambda) \cong k^{s_\Lambda}$, as desired. □

In the context of symmetric algebras, we have the following description of the center $\mathfrak{Z}(\Lambda)$ and the derived Lie algebra $\Lambda^{(1)}$:

Lemma 3. *Let Λ be a symmetric algebra. Then*

$$\mathfrak{Z}(\Lambda) = (\Lambda^{(1)})^\perp \quad \textit{and} \quad \mathfrak{Z}(\Lambda)^\perp = \Lambda^{(1)}.$$

Proof. Let $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ be a non-degenerate symmetric associative form. Given $c, x, y \in \Lambda$, we have

$$(cx - xc, y) = (c, xy) - (y, xc) = (c, xy) - (yx, c) = (c, xy - yx),$$

so that $c \in \mathfrak{Z}(\Lambda)$ if and only if $c \in (\Lambda^{(1)})^\perp$.

Since $(\ ,\ )$ is non-degenerate, we have $X = (X^\perp)^\perp$ for every subspace $X \subset \Lambda$. Consequently, the above also shows $\mathfrak{Z}(\Lambda)^\perp = ((\Lambda^{(1)})^\perp)^\perp = \Lambda^{(1)}$. □

Recall that the projection onto 1 endows kG with the structure of a symmetric algebra. Given a conjugacy class $C \subset G$, we let $z_C := \sum_{g\in G} g$ be the corresponding central element. Denoting by $\mathrm{Cl}(G)$ the set of conjugacy classes of G, Lemma 3 yields

$$(*) \qquad kG^{(1)} = \{\sum_{g\in G} \alpha_g g \ ; \ \sum_{g\in C} \alpha_g = 0 \quad \forall\ C \in \mathrm{Cl}(G)\}.$$

We now turn to the proof of the main theorem:

Proof. Given an element $g \in G$ of order n, the cyclic subgroup $\langle g\rangle \subset G$ generated by g is the direct product of its Sylow subgroups. Consequently, g uniquely decomposes as

$$g = g_p g_r$$

with $g_pg_r = g_rg_p$, $\mathrm{ord}(g_p) = p^\ell$ and g_r being p-regular. Since $((g_p - 1)g_r)^{p^\ell} = 0$, it follows that

$$g = (g_p - 1)g_r + g_r \equiv g_r \qquad \mathrm{mod}(\mathcal{N}_p(kG)).$$

Let $h \in G$. In view of $\omega_i(hgh^{-1}) = \omega_i(g)$ for all $i \in \{1, \ldots, s_{kG}\}$, Lemma 1 gives

$$hgh^{-1} \equiv g \quad \mathrm{mod}\, \mathcal{N}_p(kG).$$

Let $c_1, \ldots, c_t$ be elements of G, each belonging to exactly one of the p-regular classes of G. As an upshot of our discussion, the canonical projection map $\sigma : kG \longrightarrow kG/\mathcal{N}_p(kG)$ induces a surjection

$$\sigma : \bigoplus_{i=1}^{t} kc_i \longrightarrow kG/\mathcal{N}_p(kG).$$

It remains to be shown that σ is injective.

Let $x = \sum_{i=1}^{t} \alpha_i c_i$ be an element of $\ker \sigma$. Then we have $x^{p^n} \in kG^{(1)}$ for some $n \in \mathbb{N}_0$, so that properties (2) and (3) imply

$$\sum_{i=1}^{t} \alpha_i^{p^n} c_i^{p^n} \in kG^{(1)}.$$

Observe that the $c_i^{p^n}$ still belong to different p-regular classes of G. Identity $(*)$ now yields $\alpha_i^{p^n} = 0$ for every i, so that $x = 0$.

Consequently, $s_{kG} = \dim_k kG/\mathcal{N}_p(kG) = t$ is the number of p-regular classes of G. □

Example. Let $G = \mathrm{SL}(2, p)$ be the special linear group over $\mathbb{F}_p$. Then G has $(p - 1)p(p + 1)$ elements and is known to afford p p-regular classes. Thus, G has p simple modules, given by the first p symmetric powers of the standard module (the first power being the trivial module).

References

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