Selected topics in representation theory

– Resolutions of simple modules over the polynomial ring $k[X_1, \ldots, X_n]$ –

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1 The exterior algebra

Definition. Let R be a commutative ring, $E \in \operatorname{mod} R$ a finitely generated R-module.

Set $T^r(E) = \bigotimes_{i=1}^r E$, $T^0 = R$. Then

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E)$$

is called the *tensor algebra* of E.[1]

Denote by $\mathfrak{a}_r \subseteq T^r(E)$ the ideal generated by $\{x_1 \otimes \ldots \otimes x_r \mid x_i = x_j \text{ for some } i \neq j\}$.

Set $\bigwedge^r(E) = T^r(E)/\mathfrak{a}_r$. Then

$$\bigwedge(E) = \bigoplus_{r=0}^{\infty} \bigwedge^r(E)$$

is called the *exterior algebra* of E.

The image of an element $x_1 \otimes \ldots \otimes x_r \in T^r(E)$ under the projection to $\bigwedge^r(E)$ is denoted by $x_1 \wedge \ldots \wedge x_r$.

Remark. Let E be a free R-module of dimension n over R. Then $\bigwedge(E)$ can be described as a quotient of the free algebra in n variables:

$$\bigwedge(E) = R\langle X_1, \ldots X_n \rangle / (X_i^2, X_i X_j + X_j X_i).$$

- If $r > n$, then $\bigwedge^r(E) = 0$.
- Let $\{v_1, \ldots, v_n\}$ be a basis of E over R. For $1 \leq r \leq n$, $\bigwedge^r(E)$ is free over R, the elements $v_{i_1} \wedge \ldots \wedge v_{i_r}$, $i_1 < \ldots < i_r$, form a basis of $\bigwedge^r(E)$ over R, and $\dim_R \bigwedge^r(E) = \binom{n}{r}$.

2 Tensor products of complexes

Let $K^\bullet$ and $L^\bullet$ be two complexes of R-modules, R a commutative ring, such that $K_m = 0$, $L_m = 0$ for all $m < 0$, i.e. $K^\bullet : \cdots \to K_n \to K_{n-1} \to \cdots \to K_0 \to 0 \to \cdots$, and similarly for $L^\bullet$.

We define a new complex $K^\bullet \otimes L^\bullet$ as follows: The module in degree n is

$$(K^\bullet \otimes L^\bullet)_n = \sum_{p+q=n} K_p \otimes L_q,$$

and the differentials are defined as

$$d(u \otimes v) = du \otimes v + (-1)^p u \otimes dv.$$

One can check that this gives really a complex.

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[1] Tensors products are always taken over the ring R.

3 The Koszul complex

Definition. Let R be a commutative ring, $M \in \operatorname{mod} R$. A sequence $\mathbf{x} = (x_1, \ldots, x_r)$ in R is called *M-regular* if

- x_1 is not a zero divisor for M, and
- x_i is not a zero divisor for $M/(x_1, \ldots, x_{i-1})M$ for all $2 \leq i \leq r$.

(Here, $(x_1, \ldots, x_{i-1})M$ denotes the ideal in M generated by $\{x_1, \ldots, x_{i-1}\}$.)

It is called *regular* if it is an R-regular sequence.

Definition. Let $\mathbf{x} = (x_1, \ldots., x_r)$ be a regular sequence in R. We define the *Koszul complex* $K(\mathbf{x})$ for R by

- $K_0(\mathbf{x}) = R$
- $K_1(\mathbf{x}) =$ the free R-module with basis $\{e_1, \ldots e_r\}$
- $K_p(\mathbf{x}) =$ the free R-module with basis $\{e_{i_1} \wedge \ldots \wedge e_{i_p} \mid i_1 < \ldots < i_p\}$
- $K_r(\mathbf{x}) =$ the free R-module with basis $\{e_1 \wedge \ldots \wedge e_r\}$

(Note that $e_{i_1} \wedge \ldots \wedge e_{i_p}$ is just a notion for the basis element and has (so far) nothing to do with the exterior algebra.)

The boundary maps are defined as follows —and this is where the sequence $\mathbf{x}$ has an influence. . . —:

- $d : K_1(\mathbf{x}) \to K_0(\mathbf{x}), d(e_i) = x_i \ \forall i = 1, \ldots, r$
- $d : K_p(\mathbf{x}) \to K_{p-1}(\mathbf{x}), d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_p}$.

This defines really a complex.

Definition. We define the *Koszul complex* for $M \in \operatorname{mod} R$ (w.r.t. the regular sequence $\mathbf{x}$) by tensoring the complex for R with M: $K(\mathbf{x}) \otimes M$.

In order to calculate the Koszul complex for a regular sequence $\mathbf{x} = (x_1, \ldots, x_r)$, we can calculate the Koszul complexes for non zero divisors in R, and we obtain the Koszul complex also inductively, since there is a natural isomorphism

$$K(\mathbf{x}) \cong K(x_1) \otimes \ldots \otimes K(x_r).$$

Proof. Homework. □

Notation. For $M \in \operatorname{mod} R$ and a regular sequence $\mathbf{x}$ we denote the homology groups of the Koszul complex for M by

$$H_p(\mathbf{x}, M) = H_p(K(\mathbf{x}) \otimes M).$$

The following construction is very useful.

Let $C^\bullet$ be a complex of R-modules and $x \in R$ a non zero divisor.

We have a short exact sequence of complexes

$$0 \to R \to K(x) \to R[-1] \to 0. \tag{1}$$

(Here, R denotes the stalk complex concentrated in degree zero with entry R, and $R[-1]$ the stalk complex concentrated in degree -1 with entry R.)

By construction, the complex $K(x)$ is concentrated in degrees 1 and 0:

$$K(x): \dots \to 0 \to \underbrace{K_1(x)}_{\cong R} \xrightarrow{x} \underbrace{K_0(x)}_{\cong R} \to 0 \to \dots,$$

where x denotes the multiplication by x.

We apply $- \otimes C^\bullet$ to the sequence (1), which gives us

$$(K(x) \otimes C^\bullet)_p = (K_0(x) \otimes C_p) \oplus (K_1(x) \otimes C_{p-1}) \cong C_p \oplus C_{p-1}.$$

The boundary maps are, by definition, given by

$$d(v, w) = (dv + (-1)^{p-1} xw, dw)$$

for $(v, w) \in C_p \oplus C_{p-1}$.

Now take its homology. This leads to the long exact sequence

$$\begin{array}{l} \dots \qquad\qquad \longrightarrow \underbrace{H_{p+1}(C^\bullet[-1])}_{\cong H_p(C^\bullet)} \\ \xrightarrow{\delta_p} H_p(C^\bullet) \longrightarrow H_p(K(x) \otimes C^\bullet) \longrightarrow \underbrace{H_p(C^\bullet[-1])}_{\cong H_{p-1}(C^\bullet)} \\ \xrightarrow{\delta_{p-1}} H_{p-1}(C^\bullet) \longrightarrow H_{p-1}(K(x) \otimes C^\bullet) \longrightarrow \underbrace{H_{p-1}(C^\bullet[-1])}_{\cong H_{p-2}(C^\bullet)} \\ \qquad\qquad \dots \\ \xrightarrow{\delta_1} H_1(C^\bullet) \longrightarrow H_1(K(x) \otimes C^\bullet) \longrightarrow \underbrace{H_1(C^\bullet[-1])}_{\cong H_0(C^\bullet)} \\ \xrightarrow{\delta_0} H_0(C^\bullet), \end{array}$$

where each δ_p is induced by the multiplication by $(-1)^{p-1}x$.

Lemma. *If $\mathbf{x} = (x_1, \dots, x_r)$ is an M-regular sequence in R for $M \in \operatorname{mod} R$, then $H_p(\mathbf{x}, M) = 0$ for all $p \neq 0$, and $H_0(\mathbf{x}, M) = M/(x_1, \dots, x_r)M$.*

Proof. If $r = 1$, then we can choose in the above construction $C^\bullet = M$, the stalk complex with M concentrated in degree 0. Then $H_p(C^\bullet) = 0$ for all $p \neq 0$ and $H_0(C^\bullet) = M$.

The complex $K(x)$ is given by $0 \to R \xrightarrow{x} R \to 0$, and $K(x) \otimes M$ is $0 \to M \xrightarrow{x} M \to 0$.

So all $H_p(K(x) \otimes M) = 0$ for all $p \geq 2$ (by the long exact homology sequence above). Since x is a non zero divisor on M, the multiplication by x is injective. Furthermore, $H_1(C^\bullet) = 0$, so $H_1(K(x) \otimes M) = 0$. And $H_0(K(x) \otimes M) = M/(x)M$.

Let now $r \geq 2$.

Denote by $\mathbf{y}$ the M-regular sequence $(x_1, \dots, x_{r-1})$, and let $C^\bullet = K(\mathbf{y}) \otimes M$. As stated above, we have an isomorphism $K(\mathbf{x}) \cong K(x_r) \otimes K(\mathbf{y})$.

By induction, $H_p(K(\mathbf{y}) \otimes M) = 0$ for all $p \neq 0$. Therefore, all $H_p(K(\mathbf{x}) \otimes M) \cong H_p(K(x_r) \otimes C^\bullet) = 0$ for all $p \geq 2$ (by the long exact homology sequence above). And x_r is a non zero divisor on $M/(x_1, \dots, x_{r-1})M \cong H_0(C^\bullet)$, therefore, the multiplication by x_r is injective. Furthermore, $H_1(C^\bullet) = 0$, and hence $H_1(K(\mathbf{x}) \otimes M) \cong H_1(K(x_r) \otimes C^\bullet) = 0$. By definition, $H_0(K(\mathbf{x}) \otimes M) \cong H_0(K(x_r) \otimes C^\bullet) = M/(x_1, \dots, x_r)M$. □

Corollary. *If* $\mathbf{x} = (x_1, \dots, x_r)$ *is an* M*-regular sequence in* R *for* $M \in \operatorname{mod} R$*, then* $K(\mathbf{x}) \otimes M$ *is a free resolution of* $M/(x_1, \dots, x_r)M$*, i.e. the Koszul complex is exact.*

Now apply the construction to $M = R = k[X_1, \dots, X_n]$, with regular sequence $\mathbf{x} = (X_1 - a_1, \dots, X_n - a_n)$. We have $k[X_1, \dots, X_n]/(X_1 - a_1, \dots, X_n - a_n)k[X_1, \dots, X_n] \cong k_{a_1, \dots, a_n}$, the simple module for $k[X_1, \dots, X_n]$ defined by $(a_1, \dots, a_n) \in R^n$.

References

[1] S. Lang, **Algebra**. Revised third edition. Graduate Texts in Mathematics, 211. *Springer-Verlag, New York*, 2002. xvi+914 pp.

[2] D. G. Northcott, **Lessons on rings, modules and multiplicities**. *Cambridge University Press, London*, 1968. xiv+444 pp.

[3] C. A. Weibel, **An introduction to homological algebra**. Cambridge Studies in Advanced Mathematics, 38. *Cambridge University Press, Cambridge*, 1994. xiv+450 pp.