Let $A$ be an Artin algebra, and denote the category of finitely generated left $A$-modules by $\text{mod}\ A$.

1 Approximations

Let $\mathcal{X}$ be a full subcategory of $\text{mod}\ A$, and $M \in \text{mod}\ A$.

**Definition.** A *right* $\mathcal{X}$-approximation of $M$ is a map $f : X \rightarrow M$ with $X \in \mathcal{X}$ so that for any map $f' : X' \rightarrow M$ with $X' \in \mathcal{X}$ there is a map $g : X' \rightarrow X$ such that $f' = f \circ g$.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & M \\
\downarrow{g} & & \\
X' & \xleftarrow{f'} & 
\end{array}
\]

Dually, define a *left* $\mathcal{X}$-approximation of $M$ to be a map $f : M \rightarrow X$ with $X \in \mathcal{X}$ so that for any map $f' : M \rightarrow X'$ with $X' \in \mathcal{X}$ there is a map $g : X \rightarrow X'$ such that $f' = g \circ f$.

\[
\begin{array}{ccc}
M & \xleftarrow{f} & X \\
\downarrow{f'} & & \\
X' & \xrightarrow{g} & 
\end{array}
\]

A subcategory $\mathcal{X}$ of $\text{mod}\ A$ is called *functorially finite* if every $M \in \text{mod}\ A$ has both a right and a left $\mathcal{X}$-approximation.

**Notation.** Let $\Theta = \{\Theta(1), \ldots, \Theta(n)\}$ be a sequence of $A$-modules with $\text{Ext}^1_A(\Theta(j), \Theta(i)) = 0$ for all $j \geq i$. Denote by $\mathcal{F}(\Theta)$ the full subcategory of $\text{mod}\ A$ of modules with filtration factors in $\Theta$.

2 Main Theorem

One of the theorems in [2] is the following:

**Theorem (Ringel).** The subcategory $\mathcal{F}(\Theta)$ is functorially finite in $\text{mod}\ A$.

There is also another theorem which assures then the existence of (relative) AR-sequences for a certain full subcategory of $\text{mod}\ A$ (see [1]):

**Theorem (Auslander, Smalø).** A functorially finite subcategory which is closed under extensions and direct summands has relative AR-sequences.

We denote by $\mathcal{X}(\Theta)$ the full subcategory in $\text{mod}\ A$ of all modules which are direct summands of modules in $\mathcal{F}(\Theta)$. Since $\mathcal{X}(\Theta)$ is closed under extensions and direct summands and it is also functorially finite in $\text{mod}\ A$, we obtain immediately:
Corollary. The category $\mathcal{X}(\Theta)$ has almost split sequences.

Note that $\mathcal{F}(\Theta)$ is generally not closed under direct summands.

Example. Consider the quiver $Q = \circ_1 \to \circ_2 \to \circ_3$ and its path algebra $kQ$.

Take $\Theta = \{I(2), P(2)\}$. Then $P(1), P(3) \in \mathcal{X}(\Theta)$, but $P(1), P(3) \notin \mathcal{F}(\Theta)$. (Here, $P(i)$ (resp. $I(i)$) denotes the indecomposable projective (resp. injective) $kQ$-module corresponding to the point $i$.)

3 Proof of the Theorem

Let $\mathcal{X}$ be an arbitrary full subcategory of $\text{mod } A$, and denote by $\mathcal{Y}$ the full subcategory of $\text{mod } A$ of all modules $Y$ with $\text{Ext}^1_A(X,Y) = 0$ for all $X \in \mathcal{X}$.

Lemma. Let $0 \to Y \to X \overset{f}{\to} M \to 0$ with $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ be exact. Then $f$ is a right $\mathcal{X}$-approximation of $M$.

Proof. Suppose there is a map $f' : X' \to M$ with $X' \in \mathcal{X}$. Taking the pull back, we obtain the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & Y & \overset{f}{\to} & E & \overset{X'}{\to} & 0 \\
0 & \to & Y & \overset{f}{\to} & X & \overset{f'}{\downarrow} & M & \to & 0
\end{array}
\]

The induced exact sequence splits, since $Y \in \mathcal{Y}$ and $X' \in \mathcal{X}$. So there is a map $g : X' \to X$ with $f' = f \circ g$. \hfill \qed

Lemma. Suppose that $\mathcal{X}$ is closed under extensions and for every $N \in \text{mod } A$ there is an exact sequence $0 \to N \to Y_N \to X_N \to 0$ with $Y_N \in \mathcal{Y}$ and $X_N \in \mathcal{X}$. Then every module $M \in \text{mod } A$ has a right $\mathcal{X}$-approximation.

Proof. Let $M \in \text{mod } A$.

Case 1: There is an epimorphism $\pi : X \to M$ with $X \in \mathcal{X}$.

Let $K = \ker \pi$. We get a commutative diagram with exact rows and columns (taking the push out sequences):

\[
\begin{array}{ccccccccc}
0 & \to & 0 \\
0 & \to & K & \to & Y_K & \to & X_K & \to & 0 \\
0 & \to & X & \to & Z & \to & X_K & \to & 0 \\
\pi & \downarrow & f & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
M & \to & M \\
0 & \to & 0
\end{array}
\]
Now, \( X, X_K \in \mathcal{X} \) and \( \mathcal{X} \) is closed under extensions, so \( Z \in \mathcal{X} \). Use the previous lemma for the second row to obtain that \( f \) a right \( \mathcal{X} \)-approximation of \( M \).

**Case 2:** There is no epimorphism \( X \to M \) with \( X \in \mathcal{X} \).

Consider the submodule \( M' \subseteq M \) generated by the images of maps \( X' \to M \) with \( X' \in \mathcal{X} \). Since \( M \) is finitely generated, there exists a finite set of maps \( X_i \to M \) with \( X_i \in \mathcal{X} \) such that the images generate \( M' \).

Since \( \mathcal{X} \) is closed under extensions (and therefore under direct sums), \( \mathcal{X} = \bigoplus_i X_i \in \mathcal{X} \), and there is an epimorphism \( X \to M' \) with \( X \in \mathcal{X} \). Now the conditions in **Case 1** are fulfilled for \( X \) and \( M' \), and we get a right \( \mathcal{X} \)-approximation for \( M' \), say \( f' \). If \( i : M' \to M \) denotes the inclusion map, then \( i \circ f' \) gives a right \( \mathcal{X} \)-approximation of \( M \). (Every map \( \tilde{X} \to M \) with \( \tilde{X} \in \mathcal{X} \) factors via the inclusion \( i \).)

Let now \( \Theta = \{ \Theta(1), \ldots, \Theta(n) \} \) be a sequence of \( A \)-modules as above, \( \mathcal{X} = \mathcal{F}(\Theta) \), and \( \mathcal{Y} = \mathcal{Y}(\Theta) = \{ Y \in \text{mod} \ A \mid \text{Ext}_A^1(X, Y) = 0 \ \forall X \in \mathcal{F}(\Theta) \} = \{ Y \in \text{mod} \ A \mid \text{Ext}_A^1(\Theta(i), Y) = 0 \ \forall i = 1, \ldots, n \} \).

**Question.** How can we assure in our case that we have the exact sequences of the form above, \( 0 \to N \to Y_N \to X_N \to 0 \) with \( Y_N \in \mathcal{Y} \) and \( X_N \in \mathcal{X} \)?

**Lemma.** Let \( t \in \{1, \ldots, n\} \), and \( N \in \text{mod} \ A \) such that \( \text{Ext}_A^1(\Theta(j), N) = 0 \) for all \( j > t \). Then there is an exact sequence \( 0 \to N \to N' \to Q \to 0 \) with \( Q = \Theta(t)^{\text{res}} \) and \( \text{Ext}_A^1(\Theta(j), N') = 0 \) for all \( j \geq t \).

**Proof.** Uses universal extensions and a little homological algebra.

**Lemma.** Let \( t \in \{1, \ldots, n\} \), and \( N \in \text{mod} \ A \) such that \( \text{Ext}_A^1(\Theta(j), N) = 0 \) for all \( j > t \). Then there exists an exact sequence \( 0 \to N \to Y \to X \to 0 \) with \( X \in \mathcal{F}(\{\Theta(1), \ldots, \Theta(t)\}) \) and \( Y \in \mathcal{Y}(\Theta) \).

Note the the special case \( t = n \) gives us the “required” sequences, therefore the right \( \mathcal{F}(\Theta) \)-approximations for any module \( M \in \text{mod} \ A \).

Now the theorem follows if we just take the dual constructions to get the left \( \mathcal{F}(\Theta) \)-approximations.

**References**
