1 Introduction

This talk is based on an article by C. M. Ringel (see [4]).

Let $A$ be an Artin algebra and $\text{mod } A$ the category of finitely generated $A$-modules. We fix simple modules $S(i), i = 1, \ldots, n$. Let $P(i)$ be the projective cover of $S(i)$, $Q(i)$ the injective envelope of $S(i), i = 1, \ldots, n$.

Denote by $\Delta(i)$ resp. $\nabla(i)$ the following modules:

$$\Delta(i) := P(i)/U(i), \text{ where } U(i) := \sum_{j > i} \text{Im}(f : P(j) \to P(i)) \quad \text{(standard modules)}$$

and

$$\nabla(i) := \bigcap_{j > i} \text{Ker}(f : Q(i) \to Q(j)) \quad \text{(costandard modules)}.$$ 

We get

$$\text{Ext}_A^1(\Delta(j), \Delta(i)) = 0, \ j \geq i,$$

and

$$\text{Ext}_A^1(\nabla(j), \nabla(i)) = 0, \ j \leq i.$$ 

Let $\Delta := \{\Delta(i) \mid i = 1, \ldots, n\}$ and $\nabla := \{\nabla(i) \mid i = 1, \ldots, n\}$.

By $\mathcal{F}(\Delta)$ we denote the full subcategory of finitely generated $A$-modules with filtration factors from $\Delta$ and by $\mathcal{F}(\nabla)$ the full subcategory of finitely generated $A$-modules with filtration factors from $\nabla$.

**Definition 1.** $A$ is called quasi hereditary if $AA \in \mathcal{F}(\Delta)$ and $[\Delta(i) : S(i)] = 1$.

Let $\omega := \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. The following theorem appeared in an article by M. Auslander and I. Reiten (see [1]), but was also proved in C. M. Ringel’s article (see [4]):

**Theorem 2** (Auslander-Reiten; Ringel). There is a uniquely defined basic module with $\omega = \text{add } T$ and $T$ both a tilting and cotilting module. Furthermore,

$$\mathcal{F}(\Delta) = \{X \in \text{mod } A \mid \text{Ext}_A^i(X, T) = 0 \ \forall i \geq 1\}$$

and

$$\mathcal{F}(\nabla) = \{Y \in \text{mod } A \mid \text{Ext}_A^i(T, Y) = 0 \ \forall i \geq 1\}.$$ 

**Definition 3.** A module $T$ as in the previous theorem is called characteristic tilting module.

**Corollary 4** (Ringel). The categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ determine $\Delta$ and $\nabla$. 
Proof. We construct $\Delta$ from $\mathcal{F}(\Delta)$.

$\Delta(i) = P(i)/U(i)$ as above. But we can describe $U(i)$ in a different way:

Claim:

$$U(i) = \sum_{0 \neq g \text{ surjective } X \in F(\Delta)} \ker(g : P(i) \to X).$$

Let $g : P(i) \to X$ be a non-zero surjective map and $X \in F(\Delta)$. Since $0 \neq X \in F(\Delta)$, there exists a submodule $X' \subseteq X$ with $X/X' \in \Delta$. Since there is $P(i) \to X/X'$ surjective, it follows that $X/X' \cong \Delta(i)$.

This implies that $\text{Hom}_A(P(j), X/X') = 0$ for all $j > i$. Therefore, $U(i) \subseteq \ker \pi \circ g$, where $\pi : X \to X/X'$ denotes the projection map.

Now we show that $U(i) = \ker \pi \circ g$ by a comparison of their lengths. We have an extension

$$0 \to U(i) \to P(i) \to \Delta(i) \to 0$$

and, since $\pi \circ g$ is also surjective, an extension

$$0 \to \ker \pi \circ g \to P(i) \xrightarrow{\pi \circ g} X/X' \to 0.$$

Therefore, $|U(i)| = |P(i)| - |\Delta(i)| = |P(i)| - |X/X'| = |\ker \pi \circ g|$. (The second equality holds because $X/X' \cong \Delta(i)$.)

The equality of the modules implies that $\ker g \subseteq \ker \pi \circ g = U(i)$, and also

$$\sum_{0 \neq g \text{ surjective } X \in F(\Delta)} \ker(g : P(i) \to X) \subseteq U(i).$$

But the projection map $P(i) \to \Delta(i)$ is just a (non zero) surjective map of the form $g : P(i) \to X$ with $X \in \Delta \subseteq F(\Delta)$, which implies that

$$U(i) \subseteq \sum_{0 \neq g \text{ surjective } X \in F(\Delta)} \ker(g : P(i) \to X).$$

Similarly, we get $\nabla$ from $\mathcal{F}(\nabla)$.

A description of the indecomposable modules in $\omega$ is given by the following proposition:

**Proposition 5** (Ringel). *The characteristic tilting module $T$ with $\text{add } T = \omega$ can be decomposed as $T = \bigoplus_{i=1}^n T(i)$ into indecomposable modules $T(i)$, $i = 1, \ldots, n$, such that there exist extensions*

$$0 \to \Delta(i) \xrightarrow{f_i} T(i) \to X(i) \to 0$$

*and*

$$0 \to Y(i) \to T(i) \xrightarrow{g_i} \nabla(i) \to 0$$

*where $f_i$ is a left $\mathcal{F}(\nabla)$-approximation and $X(i) \in \mathcal{F}(\{\Delta(j) \mid j < i\})$ and $g_i$ is a right $\mathcal{F}(\Delta)$-approximation and $Y(i) \in \mathcal{F}(\{\nabla(j) \mid j < i\})$.*/
2 Construction of quasi hereditary algebras using the characteristic tilting module

Let $A$ be an Artin algebra with costandard modules $\nabla := \{\nabla(i) \mid i = n, \ldots, 1\}$, $T = \bigoplus_{i=1}^{n} T(i)$ be the characteristic tilting module for $A$, $A' := \text{End}_{A}(A^{\text{op}})$, and denote the functor $\text{Hom}_{A}(T, -) : \text{mod} A \to \text{mod} A'$ by $F$.

**Theorem 6** (Ringel). $A'$ is quasi hereditary where $\Delta' := \{F(\nabla(i)) \mid i = 1, \ldots, n\}$ is the set of standard modules. The functor $F$ induces an equivalence between $\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta')$.

**Proof.** The module $A^{\text{op}}$ is a tilting module. Therefore (and because of the description of $\mathcal{F}(\nabla)$ due to Auslander-Reiten), $F$ is a full exact embedding of $\mathcal{F}(\nabla)$ onto an extension closed subcategory of $\text{mod} A'$ (see for example Happel [2] or Miyashita [3]) which contains the projective $A'$-modules.

Let $i' := n - i + 1$ for $i = 1, \ldots, n$ and $\Delta'(i) = F(\nabla(i'))$.

The image of $\mathcal{F}(\nabla)$ under the functor $F$ is $\mathcal{F}(\Delta')$.

Now we show that the modules in $\Delta'$ are defined in such a way that $A'$ becomes a quasi hereditary algebra w.r.t. $\Delta'$.

The indecomposable projective $A'$-modules are just $F(T(i')) =: P'(i)$, $i = 1, \ldots, n$. Let $S'(i) := \text{top}(P'(i))$, $i = 1, \ldots, n$, be the corresponding simple $A'$-modules.

We have to show that $\Delta'(i)$ is standard modules, $[\Delta'(i) : S'(i)] = 1$ and that $A' \in \mathcal{F}(\Delta)$.

First we show that $\text{Hom}_{A'}(P'(j), \Delta'(i)) = 0$ for $j > i$. We have that $\text{Hom}_{A'}(T(j'), \nabla(i')) = 0$ because $S(i')$ is not a composition factor of $T(j')$, but $S(i') = \text{soc} \nabla(i')$.

By the proposition above, we get an extension in mod $A$:

$$0 \to Y(i') \to T(i') \to \nabla(i') \to 0,$$

where $Y(i') \in \mathcal{F}(\{\nabla(j') \mid j' < i'\})$ and $Y(i'), T(i'), \nabla(i') \in \mathcal{F}(\nabla)$.

Applying $F$ to the extension, we get an extension in mod $A'$:

$$0 \to F(Y(i')) \to P'(i) \to \Delta'(i) \to 0,$$

where $F(Y(i')) \in \mathcal{F}(\{\Delta'(j) \mid j > i\})$.

Since $F(Y(i')) \in \mathcal{F}(\{\Delta'(j) \mid j > i\})$ and $S'(i) = \text{top} P'(i)$, it follows that $\text{top} \Delta'(i) = S'(i)$.

We get that the top composition factors of $F(Y(i'))$ can only be $S'(j)$ with $j > i$.

This implies that $\Delta'(i)$ is the largest factor module of $P'(i)$ with composition factors $S'(j)$ with $j \leq i$. It is the indecomposable projective $A'/I_{n-1}$-module with top $S'(i)$. Here, $I_i$ denotes the ideal in $\text{End}_{A}(A^{\text{op}})$ of morphisms factoring through $\text{add}(T(1) \oplus \ldots \oplus T(i))$.

$\text{End}_{A'}(\Delta'(i)) \cong \text{End}_{A}(\nabla(i'))$ is a division ring. Therefore, $[\Delta(i) : S'(i)] = 1$.

Clearly, $A' \in \mathcal{F}(\Delta)$.

So $A'$ is quasi hereditary with standard modules $\Delta'$. \hfill \qed
References


