Selected topics in representation theory

– Properties and use of the characteristic tilting module –

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1 Introduction

This talk is based on an article by C. M. Ringel (see [4]).

Let A be an Artin algebra and $\operatorname{mod} A$ the category of finitely generated A-modules. We fix simple modules $S(i)$, $i = 1, \ldots, n$. Let $P(i)$ be the projective cover of $S(i)$, $Q(i)$ the injective envelope of $S(i)$, $i = 1, \ldots, n$.

Denote by $\Delta(i)$ resp. $\nabla(i)$ the following modules:

$$\Delta(i) := P(i)/U(i), \text{ where } U(i) := \sum_{j>i} \operatorname{Im}(f : P(j) \to P(i)) \quad (\textit{standard modules})$$

and

$$\nabla(i) := \bigcap_{j>i} \operatorname{Ker}(f : Q(i) \to Q(j)) \quad (\textit{costandard modules}).$$

We get

$$\operatorname{Ext}^1_A(\Delta(j), \Delta(i)) = 0, \; j \geq i,$$

and

$$\operatorname{Ext}^1_A(\nabla(j), \nabla(i)) = 0, \; j \leq i.$$

Let $\Delta := \{\Delta(i) \mid i = 1, \ldots, n\}$ and $\nabla := \{\nabla(i) \mid i = 1, \ldots, n\}$.

By $\mathcal{F}(\Delta)$ we denote the full subcategory of finitely generated A-modules with filtration factors from Δ and by $\mathcal{F}(\nabla)$ the full subcategory of finitely generated A-modules with filtration factors from ∇.

Definition 1. A is called *quasi hereditary* if ${}_AA \in \mathcal{F}(\Delta)$ and $[\Delta(i) : S(i)] = 1$.

Let $\omega := \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. The following theorem appeared in an article by M. Auslander and I. Reiten (see [1]), but was also proved in C. M. Ringel's article (see [4]):

Theorem 2 (Auslander-Reiten; Ringel)**.** *There is a uniquely defined basic module with $\omega = \operatorname{add} T$ and T both a tilting and cotilting module. Furthermore,*

$$\mathcal{F}(\Delta) = \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(X, T) = 0 \; \forall i \geq 1\}$$

and

$$\mathcal{F}(\nabla) = \{Y \in \operatorname{mod} A \mid \operatorname{Ext}^i_A(T, Y) = 0 \; \forall i \geq 1\}.$$

Definition 3. A module T as in the previous theorem is called *characteristic tilting module.*

Corollary 4 (Ringel)**.** *The categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ determine Δ and ∇.*

Proof. We construct Δ from $\mathcal{F}(\Delta)$.

$\Delta(i) = P(i)/U(i)$ as above. But we can describe $U(i)$ in a different way:

Claim:

$$U(i) = \sum_{\substack{0 \neq g \text{ surjective} \\ X \in \mathcal{F}(\Delta)}} \operatorname{Ker}(g : P(i) \to X).$$

Let $g : P(i) \to X$ be a non-zero surjective map and $X \in \mathcal{F}(\Delta)$. Since $0 \neq X \in \mathcal{F}(\Delta)$, there exists a submodule $X' \subseteq X$ with $X/X' \in \Delta$. Since there is $P(i) \to X/X'$ surjective, it follows that $X/X' \cong \Delta(i)$.

This implies that $\operatorname{Hom}_A(P(j), X/X') = 0$ for all $j > i$. Therefore, $U(i) \subseteq \operatorname{Ker} \pi \circ g$, where $\pi : X \to X/X'$ denotes the projection map.

Now we show that $U(i) = \operatorname{Ker} \pi \circ g$ by a comparison of their lengths. We have an extension

$$0 \to U(i) \to P(i) \to \Delta(i) \to 0$$

and, since $\pi \circ g$ is also surjective, an extension

$$0 \to \operatorname{Ker} \pi \circ g \to P(i) \stackrel{\pi \circ g}{\to} X/X' \to 0.$$

Therefore, $|U(i)| = |P(i)| - |\Delta(i)| = |P(i)| - |X/X'| = |\operatorname{Ker} \pi \circ g|$. (The second equality holds because $X/X' \cong \Delta(i)$.)

The equality of the modules implies that $\operatorname{Ker} g \subseteq \operatorname{Ker} \pi \circ g = U(i)$, and also

$$\sum_{\substack{0 \neq g \text{ surjective} \\ X \in \mathcal{F}(\Delta)}} \operatorname{Ker}(g : P(i) \to X) \subseteq U(i).$$

But the projection map $P(i) \to \Delta(i)$ is just a (non zero) surjective map of the form $g : P(i) \to X$ with $X \in \Delta \subseteq \mathcal{F}(\Delta)$, which implies that

$$U(i) \subseteq \sum_{\substack{0 \neq g \text{ surjective} \\ X \in \mathcal{F}(\Delta)}} \operatorname{Ker}(g : P(i) \to X).$$

Similarly, we get ∇ from $\mathcal{F}(\nabla)$. □

A description of the indecomposable modules in ω is given by the following proposition:

Proposition 5 (Ringel)**.** *The characteristic tilting module T with $\operatorname{add} T = \omega$ can be decomposed as $T = \bigoplus_{i=1}^{n} T(i)$ into indecomposable modules $T(i)$, $i = 1, \ldots, n$, such that there exist extensions*

$$0 \to \Delta(i) \stackrel{f_i}{\to} T(i) \to X(i) \to 0$$

and

$$0 \to Y(i) \to T(i) \stackrel{g_i}{\to} \nabla(i) \to 0$$

where f_i is a left $\mathcal{F}(\nabla)$-approximation and $X(i) \in \mathcal{F}(\{\Delta(j) \mid j < i\})$ and g_i is a right $\mathcal{F}(\Delta)$-approximation and $Y(i) \in \mathcal{F}(\{\nabla(j) \mid j < i\})$.

2 Construction of quasi hereditary algebras using the characteristic tilting module

Let A be an Artin algebra with costandard modules $\nabla := \{\nabla(i) \mid i = n, \ldots, 1\}$, $T = \bigoplus_{i=1}^{n} T(i)$ be the characteristic tilting module for A, $A' := \mathrm{End}_A({}_AT)$, and denote the functor $\mathrm{Hom}_A(T, -) : \mathrm{mod}\, A \to \mathrm{mod}\, A'$ by F.

Theorem 6 (Ringel). *A' is quasi hereditary where $\Delta' := \{F(\nabla(i)) \mid i = 1, \ldots, n\}$ is the set of standard modules. The functor F induces an equivalence between $\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta')$.*

Proof. The module ${}_AT$ is a tilting module. Therefore (and because of the description of $\mathcal{F}(\nabla)$ due to Auslander-Reiten), F is a full exact embedding of $\mathcal{F}(\nabla)$ onto an extension closed subcategory of $\mathrm{mod}\, A'$ (see for example Happel [2] or Miyashita [3]) which contains the projective A'-modules.

Let $i' := n - i + 1$ for $i = 1, \ldots, n$ and $\Delta'(i) = F(\nabla(i'))$.

The image of $\mathcal{F}(\nabla)$ under the functor F is $\mathcal{F}(\Delta')$.

Now we show that the modules in Δ' are defined in such a way that A' becomes a quasi hereditary algebra w.r.t. Δ'.

The indecomposable projective A'-modules are just $F(T(i')) =: P'(i)$, $i = 1, \ldots, n$. Let $S'(i) := \mathrm{top}(P'(i))$, $i = 1, \ldots, n$, be the corresponding simple A'-modules.

We have to show that the $\Delta'(i)$ are standard modules, $[\Delta'(i) : S'(i)] = 1$ and that $A' \in \mathcal{F}(\Delta)$.

First we show that $\mathrm{Hom}_{A'}(P'(j), \Delta'(i)) = 0$ for $j > i$. We have that $\mathrm{Hom}_A(T(j'), \nabla(i')) = 0$ because $S(i')$ is not a composition factor of $T(j')$, but $S(i') = \mathrm{soc}\, \nabla(i')$.

By the proposition above, we get an extension in $\mathrm{mod}\, A$:

$$0 \to Y(i') \to T(i') \to \nabla(i') \to 0,$$

where $Y(i') \in \mathcal{F}(\{\nabla(j') \mid j' < i'\})$ and $Y(i'), T(i'), \nabla(i') \in \mathcal{F}(\nabla)$.

Applying F to the extension, we get an extension in $\mathrm{mod}\, A'$:

$$0 \to F(Y(i')) \to P'(i) \to \Delta'(i) \to 0,$$

where $F(Y(i')) \in \mathcal{F}(\{\Delta'(j) \mid j > i\})$.

Since $F(Y(i')) \in \mathcal{F}(\{\Delta'(j) \mid j > i\})$ and $S'(i) = \mathrm{top}\, P'(i)$, it follows that $\mathrm{top}\, \Delta'(i) = S'(i)$. We get that the top composition factors of $F(Y(i'))$ can only be $S'(j)$ with $j > i$.

This implies that $\Delta'(i)$ is the largest factor module of $P'(i)$ with composition factors $S'(j)$ with $j \leq i$. It is the indecomposable projective A'/I_{n-i}-module with top $S'(i)$. Here, I_t denotes the ideal in $\mathrm{End}_A({}_AT)$ of morphisms factoring through $\mathrm{add}(T(1) \oplus \ldots \oplus T(t))$.

$\mathrm{End}_{A'}(\Delta'(i)) \cong \mathrm{End}_A(\nabla(i'))$ is a division ring. Therefore, $[\Delta(i) : S'(i)] = 1$.

Clearly, $A' \in \mathcal{F}(\Delta)$.

So A' is quasi hereditary with standard modules Δ'. □

References

[1] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math. **86** (1991), no. 1, 111–152.

[2] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, Lond. Math. Lect. Note Ser., vol. 119, Cambridge University Press, Cambridge, 1988.

[3] Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. **193** (1986), 113–146.

[4] C.M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. **208** (1991), 209–223.