Selected topics in representation theory

– General representations of quivers and canonical decompositions I –

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We consider representations of a quiver Q and would like to find out general properties of the representations for a fixed dimension vector. The aim of these talks is to investigate how such "general representations" look like and how they can be obtained. The main reference for this is [6] (and [4]).

1 Definitions/Preliminaries

We consider representations over a fixed algebraically closed field K.

The vector space $\mathrm{rep}(Q, \mathbf{d})$ of representations of a quiver $Q = (Q_0, Q_1, s, t)$ with dimension vector $\mathbf{d}$ comes along with an algebraic group action of the group $\mathrm{GL}(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{GL}(d_i)/C$, where $C = \{(\lambda \cdot \mathrm{id}_{K^{d_i}})_{i \in Q_0} \mid \lambda \in K^*\}$, such that the isomorphism classes of representations are in $1-1$ correspondence with the orbits of $\mathrm{GL}(\mathbf{d})$.

Kac has shown in [3] that for a fixed dimension vector $\mathbf{d}$, there is an open dense subset $\mathrm{rep}_{\mathrm{can}}(Q, \mathbf{d}) \subseteq \mathrm{rep}(Q, \mathbf{d})$ such that all representations $R_p \in \mathrm{rep}_{\mathrm{can}}(Q, \mathbf{d})$ decompose as a direct sum of indecomposable representations $V_{a,p}$, $a \in I$, and the family of dimension vectors $\{\underline{\dim}(V_{a,p}) \mid a \in I\}$ is independent of the chosen representation R_p.

Definition. A *general representation of dimension vector* $\mathbf{d}$ is a representation $R \in \mathrm{rep}_{\mathrm{can}}(Q, \mathbf{d})$. The decomposition $\mathbf{d} = \sum \underline{\dim}(V_{a,p})$ (as above) is called the *canonical decomposition of* $\mathbf{d}$.

Furthermore, we define another open subset of $\mathrm{rep}(Q, \mathbf{d})$, namely

$$\mathrm{rep}^0(Q, \mathbf{d}) = \{V_j \in \mathrm{GL}(\mathbf{d})V_i \mid V_i \in \mathrm{rep}(Q, \mathbf{d}),\ \dim(\mathrm{GL}(\mathbf{d})V_i) \text{ is maximal}\}.$$

For a quiver Q, let us define a quadratic form $\langle -, - \rangle$ on $\mathbb{Z}^{Q_0}$ in the following way:

$$\langle -, - \rangle : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

$$\langle \mathbf{d_1}, \mathbf{d_2} \rangle = \sum_{i \in Q_0} d_{1i} d_{2i} - \sum_{\alpha \in Q_1} d_{1,s(\alpha)} d_{2,t(\alpha)}.$$

By Ringel [5], we know that

$$\langle \mathbf{d_1}, \mathbf{d_2} \rangle = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}^1(V, W)$$

for all representations V and W with dimension vectors $\underline{\dim}(V) = \mathbf{d_1}$ and $\underline{\dim}(W) = \mathbf{d_2}$.

The functions $\dim \mathrm{Hom}(V, W)$ and $\dim \mathrm{Ext}(V, W)$ are upper semicontinuous functions on $\mathrm{rep}(Q, \mathbf{d_1}) \times \mathrm{rep}(Q, \mathbf{d_2})$, and their minimal values will be denoted by $\hom(\mathbf{d_1}, \mathbf{d_2})$ and $\mathrm{ext}(\mathbf{d_1}, \mathbf{d_2})$, resp.

(Similarly, for a fixed representation V, $\hom(\mathbf{d}, V)$, $\hom(V, \mathbf{d})$, $\mathrm{ext}(\mathbf{d}, V)$, and $\mathrm{ext}(V, \mathbf{d})$ are defined. And it should also be clear what $\mathrm{end}(\mathbf{d})$ means.)

Definition. A root $\mathbf{d}$ is called a *Schur root* if $\mathrm{end}(\mathbf{d}) = 1$, i. e. there is an indecomposable representation V with $\dim \mathrm{End}(V) = 1$.

A root is called *real* if $q(\mathbf{d}) = 1$, where q denotes the Tits form of the underlying graph.

2 Two theorems of Kac on general representations

The following theorem was proved in [4].

Theorem. *Let* $\mathbf{d} \in \mathbb{N}^{Q_0}$. $\mathbf{d} = \sum_{a\in I} \mathbf{d_a}$ *is the canonical decomposition of* $\mathbf{d}$ *if and only if all* $\mathbf{d_a}$, $a \in I$, *are Schur roots and there exist representations* $U_a \in \mathrm{rep}^0(Q, \mathbf{d_a})$ *such that* $\mathrm{Ext}(U_a, U_b) = 0$ *for* $a \neq b$. *Moreover,* $\bigoplus_{a\in I} U_a \in \mathrm{rep}^0(Q, \mathbf{d}) \cap \mathrm{rep}_{\mathrm{can}}(Q, \mathbf{d})$.

If $\mathbf{d} = \sum_{a\in I} \mathbf{d_a}$ *is the canonical decomposition of* $\mathbf{d}$*, then* $\langle \mathbf{d_a}, \mathbf{d_b} \rangle \geq 0$ *for all* $a \neq b$.

3 A first necessary condition for canonical decompositions

In [2], Happel and Ringel proved a lemma which has the following consequence:

Lemma. *Let* V *and* W *be indecomposable representations of a quiver such that* $\mathrm{Ext}(W, V) = 0$. *Then any non zero homomorphism* $f : V \to W$ *is injective or surjective.*

As a consequence, we get the following:

Theorem. *Let* $\{V_a \mid a \in I\}$ *be a set of non isomorphic indecomposable representations with* $\dim \mathrm{End}(V_a) = 1$ *and* $\mathrm{Ext}(V_a, V_b) = 0$ *for* $a \neq b$. *We define a relation* $a \to b$ *if and only if there exists a non-zero homomorphism* $V_a \to V_b$. *The transitive relation generated by* $\to$ *is a partial order. In particular,* $\mathrm{Hom}(V_a, V_b) = 0$ *or* $\mathrm{Hom}(V_b, V_a) = 0$ *for* $a \neq b$.

Therefore, if $\mathbf{d} = \sum \mathbf{d_a}$ *is the canonical decomposition of* $\mathbf{d}$ *and* $a \neq b$*, then* $\mathbf{d_a} = \mathbf{d_b}$ *is a real Schur root or* $\langle \mathbf{d_a}, \mathbf{d_b} \rangle \langle \mathbf{d_b}, \mathbf{d_a} \rangle = 0$.

Proof.

Part 1.

- [Reflexivity] Clearly, there is a map $V_a \to V_a$, namely the identity map.

- [Antisymmetry] Every map $V_a \to V_b$ is injective or surjective (by the above Lemma).

  Now we show that in a sequence of maps $V_{a_1} \xrightarrow{f_1} V_{a_2} \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} V_{a_k}$, it is not possible to have two consecutive maps $f_i : V_{a_i} \to V_{a_{i+1}}$ and $f_{i+1} : V_{a_{i+1}} \to V_{a_{i+2}}$ where the first one is surjective, but not injective, and the second one is injective, but not surjective.

  The composition would give us a non zero map $V_{a_i} \to V_{a_{i+2}}$ which was neither surjective nor injective, which is not possible (contradiction to the assumption that $\mathrm{Ext}(V_a, V_b) = 0$ for all $a \neq b$ and the Lemma by Happel and Ringel).

  Let $0 \neq v_{i+1} \in V_{a_{i+1}}$. Since f_i is surjective, there is a $v_i \in V_{a_i}$ such that $f(v_i) = v_{i+1}$. Since f_{i+1} is injective and $v_{i+1} \neq 0$, $f_{i+1}(v_{i+1}) \neq 0$, and so there is a $v_i \in V_{a_i}$ such that $f_{i+1} \circ f_i(v_i) = f_{i+1}(v_{i+1}) \neq 0$.

  The composition $f_{i+1} \circ f_i$ can be neither surjective nor injective.

  This means that, once we have a sequence of maps $V_{a_1} \to \cdots \to V_{a_k} \to \cdots \to V_{a_\ell} = V_{a_1}$, in which each single map is non zero, — i. e. $a_1 \to a_k$ and $a_k \to a_1$ —, all the maps have to be surjective or all the maps have to be injective. Therefore, $V_{a_1} \cong \cdots \cong V_{a_k} \cong \cdots \cong V_{a_\ell}$, and (by the assumption that all the different V_a be non isomorphic) $V_{a_1} = V_{a_k}$.

- [Transitivity] The relation is transitive by its definition.

Part 2. Let V be a general representation of dimension vector $\mathbf{d}$. By Kac's theorem above, all dimension vectors of the indecomposable direct summands V_a of V are Schur roots, and $\operatorname{Ext}(V_a, V_b) = 0$ for all $a \neq b$.

Suppose there are two isomorphic representations V_a and V_b with $a \neq b$. Then $\operatorname{Ext}(V_a, V_a) \cong \operatorname{Ext}(V_a, V_b) = 0$. Therefore, the corresponding dimension vector $\mathbf{d_a} = \mathbf{d_b}$ is a real Schur root. ($q(\mathbf{d_a}) = \langle \mathbf{d_a}, \mathbf{d_b} \rangle = \underbrace{\dim \operatorname{Hom}(V_a, V_b)}_{\geq 1} - \underbrace{\dim \operatorname{Ext}(V_a, V_b)}_{=0} \geq 1$, so $\mathbf{d_a}$ is real.)

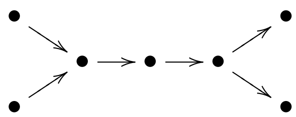
Now take from each isomorphism class of the Schur representations exactly one. Then the conditions of the first part of the theorem are fulfilled for this set of representations, and hence $\langle \mathbf{d_a}, \mathbf{d_b} \rangle = \dim \operatorname{Hom}(V_a, V_b) = 0$ or $\langle \mathbf{d_b}, \mathbf{d_a} \rangle = \dim \operatorname{Hom}(V_b, V_a) = 0$, which implies that $\langle \mathbf{d_a}, \mathbf{d_b} \rangle \langle \mathbf{d_b}, \mathbf{d_a} \rangle = 0$.

□

Warning!

The combinatorial part of the theorem of Kac stated above ($\langle \mathbf{d_a}, \mathbf{d_b} \rangle \geq 0$ for all $a \neq b$) does **not** imply that the decomposition is the canonical decomposition.

Counter example. Let

$Q =$  .

Take the dimension vector $\begin{smallmatrix}1&&&&1\\&3&2&3&\\1&&&&1\end{smallmatrix}$.

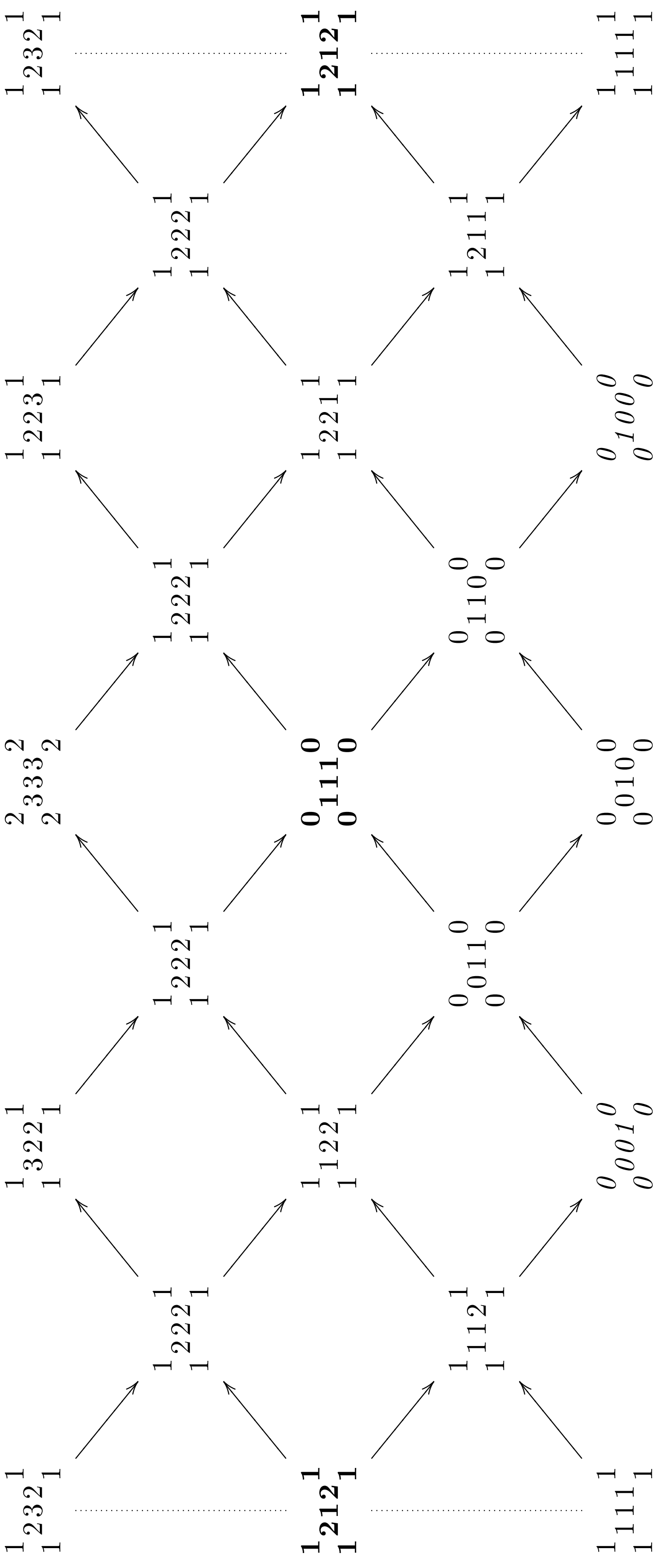
We can decompose it in two ways:

$$\begin{smallmatrix}1&&&&1\\&3&2&3&\\1&&&&1\end{smallmatrix} = \underset{\text{regular}}{\begin{smallmatrix}1&&&&1\\&2&2&2&\\1&&&&1\end{smallmatrix}} + \underset{\substack{\text{simple}\\\text{regular}}}{\begin{smallmatrix}0&&&&0\\&1&0&0&\\0&&&&0\end{smallmatrix}} + \underset{\substack{\text{simple}\\\text{regular}}}{\begin{smallmatrix}0&&&&0\\&0&0&1&\\0&&&&0\end{smallmatrix}} = \underset{\text{regular}}{\begin{smallmatrix}1&&&&1\\&2&1&2&\\1&&&&1\end{smallmatrix}} + \underset{\text{regular}}{\begin{smallmatrix}0&&&&0\\&1&1&1&\\0&&&&0\end{smallmatrix}}$$

The first decomposition is the canonical decomposition, but the second one also satisfies the combinatorial requirements.

In the first case, all dimension vectors are Schur roots and we can choose representations V_1, V_2, and V_3 such that $\operatorname{Ext}(V_i, V_j) = 0$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$.

In the second case, the dimension vectors are also Schur roots, but there are non zero homomorphisms between the corresponding representations in both ways. (The Euler form $\langle \mathbf{d_a}, \mathbf{d_b} \rangle \geq 0$ just measures the *difference* of the dimensions of the homomorphism spaces and the Ext-spaces.) Thus, this cannot be the canonical decomposition.



Mouth of the tube of rank 4 for $\tilde{\mathbb{D}}_6$ (with the orientation above)

[1]

References

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