Elementary Modules

(Selected Topics in
Representation Theory)

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Contents

1 Definitions

Let k be an algebraically closed field. Throughout A will denote a finite-dimensional, basic, connected, hereditary k-algebra. Recall that a module $M \in \text{mod}-A$ is called a *brick*, provided $\text{End}(M) \cong k$.

An indecomposable regular module M is called *quasi-simple*, if in the Auslander-Reiten sequence $0 \to \tau M \to X \to M \to 0$, X is indecomposable, where τ denotes the Auslander-Reiten translate. If A is a tame algebra, then quasi-simple modules lie at the mouth of the tubes in the regular components of the Auslander-Reiten quiver. If A is wild, the quasi-simple modules lie at the bottom of the $\mathbb{Z}A_\infty$ components.

Definition 1.1. *Let A be a representation-infinite, hereditary algebra. A regular module $E \neq 0$ is called* elementary, *if there exists no short exact sequence $0 \to U \to E \to V \to 0$, with U, V nonzero regular A-modules.*

W. Crawley-Boevey suggested the study of elementary modules. First results were published by F. Lukas and O. Kerner in [L2] and [KL]. The following are easy to show:

1. If E is elementary, then so is $\tau^n E$, for all $n \in \mathbb{Z}$.

2. Elementary modules are quasi-simple.

3. If A is tame and E is a quasi-simple, regular A-module, then E is elementary.

Elementary modules are of intereset, since any nonzero regular module has a filtration whose subfactors are elementary: If A is a representation-infinite, hereditary algebra, let R be a nonzero regular A-module. Let $\mathcal{U}_R = \{$all proper regular submodules $U \subset R$, such that R/U is regular$\}$. Choose a maximal submodule from $\mathcal{U}_R$, say R_1 and consider $\mathcal{U}_{R_1} = \{$all proper regular submodules $U \subset R_1$, such that R_1/U is regular$\}$. Continuing we get a descending chain of regular submodules of R :

$$R = R_0 \supset R_1 \supset \ldots \supset R_l \supset R_{l+1} = 0$$

For $1 \leq i \leq l+1$, let $X = R_{i-1}/R_i$. Then X is nonzero regular and not a middle term of a short exact sequence of regular modules, i.e., there is no exact sequence $0 \to S \to X \to T \to 0$, $S \neq 0, T \neq 0$ and both S, T regular. So X is elementary.

2 Properties

Lemma 2.1. *Let A be a wild, hereditary algebra.*

(a) *Let $X \neq 0$ be regular. Then there exists $N \in \mathbb{N}_1$, such that for all regular modules R and all $f \in \mathrm{Hom}(\tau^l X, R)$, with $l \geq N$,* $\mathrm{Ker}\, f$ *is regular.*

(a') *Let $X' \neq 0$ be regular. Then there exists $M \in \mathbb{N}_1$, such that for all regular modules S and all $f \in \mathrm{Hom}(S, \tau^{-m} X')$, with $m \geq N$,* $\mathrm{Cok}\, f$ *is regular.*

(b) *Let Y be regular. If Y has no nontrivial regular factor modules, then so has $\tau^l Y$, for all $l \geq 0$.*

Proof. (a) It is well-known that the dimensions $\dim_k \tau^{-l} P$ grow expoentially with l for P projective. So there exists an $N \in \mathbb{N}$, such that $\dim_k \tau^{-l} P > \dim_k X$, for all $l \geq N$ and for all nonzero projective modules P. For $l \geq N$ and R regular, consider a nonzero $f \in \mathrm{Hom}(\tau^l X, R)$. We have a short exact sequence $0 \to \mathrm{Ker}\, f \to \tau^l \to Imf \to 0$ with Imf regular and $\mathrm{Ker}\, f$ without nonzero preinjective direct summand. Apply τ^{-l} :

$$0 \to \tau^{-l}\,\mathrm{Ker}\, f \to X \to \tau^{-l} Imf \to 0$$

But $\tau^{-l}\,\mathrm{Ker}\, f$ being a regular submodule of X, is, by the dimension inequality, only possible, if $\mathrm{Ker}\, f$ is regular.

(a') This is dual to (a).

(b) Assume, to get a contradiction, $\tau^l Y$ has a nontrivial regular factor module Z. Then we get an exact sequence:

$$0 \to U \to \tau^l Y \to Z \to 0$$

Apply τ^{-l} :

$$0 \to \tau^{-l} U \to Y \to \tau^{-l} Z \to 0$$

contradicting the fact that Y has no nontrivial regular factor module. □

Proposition 2.2. *Let A be a representation-infinite, hereditary algebra. Let E be an indecomposable regular A-module. Then the following are equivalent:*

1. *E is elementary.*
2. *There exists $N \in \mathbb{N}_1$, such that $\tau^l E$ has no nontrivial regular factor module for all $l \geq N$.*
3. *There exists $M \in \mathbb{N}_1$, such that $\tau^{-l} E$ has no nontrivial regular submodule for all $l \geq M$.*
4. *If $Y \neq 0$ is a regular submodule of E, then E/Y is preinjective.*
5. *If X is a proper submodule of E with E/X regular, then X is preprojective.*

Proof. (4) ⇒ (1) and (5) ⇒ (1) are clear by definition of elementary modules.

(1) ⇒ (2) : Assume V is a nontrivial regular factor module of $\tau^l E$ for $l \geq N$. Then in $0 \to U \to \tau^l E \to V \to 0$, U is regular by the lemma. So, by applying τ^{-l}, we get an exact sequence $0 \to \tau^{-l} U \to E \to \tau^{-l} V \to 0$ with $\tau^{-l} U$ and $\tau^{-l} V$ nonzero regular, contradicting that E is elementary.

(1) ⇒ (3) is dual to (1) ⇒ (2) and we immediately get (2) ⇒ (1) and (3) ⇒ (1).

(1) ⇒ (4) : E is elementary and suppose $E/Y = Z_1 \oplus Z_2$ with $Z_1 \neq 0$ regular and Z_2 preinjective. We get the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & & & Z_2 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & E/Y & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & Z_2 & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Since K is a submodule of E, it has no nonzero preinjective direct summand. From $\mathbf{dim}\, K = \mathbf{dim}\, Y + \mathbf{dim}\, Z_2$ we have that K is regular. But this contradicts the fact that E is elementary. So E/Y is preinjective.

(1) ⇒ (5) is the dual situation to (1) ⇒ (4). □

Corollary 2.3. *If E is elementary, Y regular with $\boldsymbol{dim}\, Y = \boldsymbol{dim}\, E$, then either $Y \cong E$ or Y and E are orthogonal, i.e.,* $\mathrm{Hom}(E, Y) = 0 = \mathrm{Hom}(Y, E)$.

If S is indecomposable and regular, such that $\mathbf{dim}\, S$ or $\dim_k S$ is minimal among all nonzero regular modules, then S is elementary. One can further show that if E is elementary, then E is a brick. Note that in contrast to the tame case, if A is wild hereditary, then there are quasi-simple modules which are not bricks, thus cannot be elementary.

3 Different lengths

Let $K(2)$ be the Kronecker quiver with path algebra $B = kK(2)$. Let $K(3)$ be the extended Kronecker quiver with three arrows (α, β, γ) in the same direction, and let $A = kK(3)$ be its path algebra. A is wild hereditary, whereas B is tame hereditary. Note that any representation over $K(2)$ can be considered as a representation over $K(3)$ by letting one arrow correspond the zero map (e.g. $\gamma = 0$). There is an embedding $\operatorname{mod}-B \hookrightarrow \operatorname{mod}-A$.

Consider the following two representations $P_2(B), R(B)$ in $\operatorname{mod}-B$:

$$\begin{array}{ccc} k & \longrightarrow & k^3 \\ \binom{1}{0}\Big\downarrow \binom{0}{1}\Big\downarrow & & \Big\downarrow I_3 \Big\downarrow C \\ k^2 & \longrightarrow & k^3 \end{array}$$

where $C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. $P_2(B)$ with $\mathbf{dim}\, P_2(B) = \binom{1}{2}$ is projective over $K(2)$, whereas $R(B)$ with $\mathbf{dim}\, R(B) = \binom{3}{3}$ is a regular B-module and we have $P_2(B) \hookrightarrow R(B)$ with cokernel $I_1(B)$, preinjective, of dimension vector $\binom{2}{1}$. Now look at the exact sequence:

$$0 \to P_2(B) \to R(B) \to I_1(B) \to 0$$

Considered as A-modules, $P_2(B)$ and $I_1(B)$ are regular and elementary. So $0 \subset P_2(B) \subset R(B)$ is a chain of regular submodules of $R(B)$, with elementary factor module $R(B)/P_2(B) \cong I_1(B)$. But in $\operatorname{mod}-B$, since $R(B)$ has quasi-length 3, there exists a chain of regular modules with elementary factor modules of greater length.

4 Finiteness condition

If A is tame hereditary, the set of dimension vectors of elementary (i.e. quasi-simple modules) is finite. If A is wild hereditary, then we have $\mathbf{dim}\, \tau^i E \neq \mathbf{dim}\, \tau^j E$, for $i \neq j$. Let Φ be the Coxeter transformation (corresponding to τ). Then $\Phi^j(\mathbf{dim}\, E) = \mathbf{dim}\, \tau^j E$ for all $j \in \mathbb{Z}$.

For $x \in \mathbb{Z}^n$, $(\Phi^j(x))_{j \in \mathbb{Z}}$ is called the *Coxeter orbit* of x.

Theorem 4.1 (Lukas, 1991). *If A is hereditary, then there exists only finitely many Coxeter orbits of dimension vectors of elementary modules.*

Proof. We want to show that the set $\{(\mathbf{dim}\,\tau^j E)_{j\in\mathbb{Z}},\ E \text{ elementary}\}$ is finite.

If A is tame hereditary, this is clear. So let A be wild herediary. The idea consists of constructing a vector $c \in \mathbb{N}^n$, such that each τ-orbit $(\tau^i E)$ of any elementary module E contains some $\tau^j E$ with $\mathbf{dim}\,\tau^j E < c$. c can be chosen depending only on the quiver, not on the base field.

For the proof first note that each regular component conatins only finitely many non-sincere modules. So choose an indecomposable regular module R, such that $\tau^{-n}R$ is sincere for all $n \geq 0$. If X is elementary, then using the lemma, one can show that there exists $E = \tau^j X$, such that $\mathrm{Hom}(R, E) = 0$, but $\mathrm{Hom}(\tau^- R, E) \neq 0$. Take a nonzero $f \in \mathrm{Hom}(\tau^- R, E)$ and let $U = Imf, K = \mathrm{Ker}\, f, C = \mathrm{Cok}\, f$. Then we get two exact sequences:

$$0 \to K \to \tau^- R \to U \to 0$$

$$0 \to U \to E \to C \to 0$$

Applying $\mathrm{Hom}(R, \_)$ we get:

$$\ldots \to \mathrm{Ext}(R, \tau^- R) \to \mathrm{Ext}(R, U) \to 0$$

$$\ldots \to \mathrm{Hom}(R, E) \to \mathrm{Hom}(R, C) \to \mathrm{Ext}(R, U) \to 0$$

But $\mathrm{Hom}(R, E) = 0$, so

$$\dim_k \mathrm{Hom}(R, C) \leq \dim_k \mathrm{Ext}(R, U) \leq \dim_k \mathrm{Ext}(R, \tau^- R) =: s.$$

Since E is elementary, C is preinjective by the lemma. So C can be written as

$$C = \bigoplus_{i\in\mathbb{N}_0} \bigoplus_{j=1}^{n} \tau^i I(j)^{l_{i,j}},$$

where $I(1), \ldots, I(n)$ are indecomposable injective and almost all $l_{i,j} = 0$. By above inequality one can show:

$$\sum_{i\in\mathbb{N}_0} \sum_{j=1}^{n} l_{i,j} \cdot \dim_k \mathrm{Hom}(\tau^{-i} R, I(j)) \leq s$$

Since the components of the dimension vectors grow exponentially, there exists i_0 with $\dim \mathrm{Hom}(\tau^{-i} R, I(j)) \geq s$, for all $i \geq i_0$ and all $j = 1, \ldots, n$. So $l_{i,j} = 0$ for all $i \geq i_0$ and for all j. Since $\mathrm{Hom}(\tau^{-i} R, I(j)) \neq 0$ for all $i \geq 0$ and for all j, only finitely many $l_{i,j}$ satisfy the condition of the second inequality. Therefore we get an upper bound c for $\mathbf{dim}\, C = \mathbf{dim}\,\mathrm{Cok}\, f$, only depending on R. In particular, $\mathbf{dim}\, E \leq \mathbf{dim}\, R + c$, and there are only finitely many roots smaller or equal to $\mathbf{dim}\, R + c$. □

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