The Harder-Narasimhan filtration

(based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007)

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1 Definitions

Let $Q$ be a finite quiver with set of vertices $I$, and let $\theta : \mathbb{Z}I \to \mathbb{Z}$ be a linear function, called stability. We also define $\dim$ on $\mathbb{Z}I$ by $\dim d = \sum_{i \in I} d_i$.

Let us recall those definitions which are needed to define the Harder-Narasimhan filtration:

Definition 1.1.

1. For a non-zero dimension vector $d \in \mathbb{N}I$, we define its slope by
   \[ \mu(d) = \frac{\theta(d)}{\dim d} \in \mathbb{Q}. \]

   We define the slope of a non-zero representation $X$ of $Q$ (over some field) as the slope of its dimension vector, thus $\mu(X) = \mu(\dim X) \in \mathbb{Q}$.

2. We call the representation $X$ semistable if $\mu(U) \leq \mu(X)$ for all non-zero subrepresentations $U$ of $X$, and we call $X$ stable if $\mu(U) < \mu(X)$ for all non-zero proper subrepresentations $U$ of $X$.

2 Harder-Narasimhan filtration

Definition 2.1. A filtration $0 = X_0 \subset X_1 \subset \ldots \subset X_s = X$ of a representation $X$ is called Harder-Narasimhan (abbreviated by HN) if the subquotients $X_i/X_{i-1}$ are semistable for $i = 1, \ldots, s$ and $\mu(X_1/X_0) > \mu(X_2/X_1) > \ldots > \mu(X_s/X_{s-1})$.

Theorem 2.1. Any representation $X$ possesses a unique Harder-Narasimhan filtration.

For the proof of this theorem, we need the concept of strongly contradicting semistability:
**Definition 2.2.** A subrepresentation $U$ of a representation $X$ is called strongly contradicting semistable (or just ss) if its slope is maximal among the slopes of subrepresentations of $X$, that is, $\mu(U) = \max \{ \mu(V) | V \subset X \}$, and it is of maximal dimension with this property.

In a previous lecture it was shown that any representation $X$ admits a unique ss/scss subrepresentation. This is crucial for the proof of the theorem:

**Proof of Theorem 2.1.** We will first prove existence, then uniqueness.

Existence is proved by induction over the dimension of $X$. Let $X_1$ be the ss/scss of $X$. By induction, we have a HN filtration

$$0 = Y_0 \subset Y_1 \subset \ldots \subset Y_{s-1} = X/X_1.$$

Via the projection $\pi : X \to X/X_1$, we pull this back to a filtration of $X$ defined by $X_i = \pi^{-1}(Y_{i-1})$ for $i = 1, \ldots, s$.

Now $X_1/X_0$ is semistable since $X_1$ is the ss/scss of $X$, and $X_{i+1}/X_i \cong Y_i/Y_{i-1}$ is semistable by the choice of the $Y_i$ for $i = 1, \ldots, s - 1$.

We also get $\mu(X_2/X_1) > \ldots > \mu(X_s/X_{s-1})$ from the corresponding property of the slopes of the subquotients in the HN filtration of $X/X_1$. Since $X_2$ is a subrepresentation of $X$ of strictly larger dimension than $X_1$, we have $\mu(X_1) > \mu(X_2)$ since $X_1$ is ss/scss in $X$, and thus $\mu(X_1/X_0) = \mu(X_1) > (X_2/X_1)$.

Existence is also proved by induction on the dimension. Assume that

$$0 = X'_0 \subset \ldots \subset X'_s = X$$

is a HN filtration of $X$. Let $t$ be minimal such that $X_1$ is contained in $X'_t$, thus the inclusion induces a non-zero map from $X_1$ to $X'_t/X'_{t-1}$. Both representations being semistable and $X_1$ being ss/scss, we have

$$\mu(X'_1) \leq \mu(X_1) \leq \mu(X'_t/X'_{t-1}) \leq \mu(X'_1),$$

thus $\mu(X_1) = \mu(X'_t)$ and $t = 1$, which means $X_1 \subset X'_1$. Again since $X_1$ is ss/scss, we conclude that $X_1 = X'_1$.

By induction, we know that the induced filtrations on the factor $X/X_1$ coincide, thus the filtrations of $X$ coincide, which proves uniqueness.

**References**

