Torsion pairs induced from Harder-Narasimhan filtration

(based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007)

21 November 2007

In this lecture we want to present how to define torsion pairs from the Harder-Narasimhan filtration of a representation of a quiver. Therefore we start with a quick revision of torsion theory.

1 Torsion theory

Let $A$ be a finite-dimensional, basic, connected algebra over a fixed algebraically closed field $k$. Denote by $\text{mod} \ A$ the category of all finite-dimensional left $A$-modules.

A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of a module category is called a torsion pair (or torsion theory) if the following conditions are satisfied:

(i) $\text{Hom}(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(ii) $\text{Hom}(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.

(iii) $\text{Hom}(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

That is, there is no non-zero homomorphism from an object in $\mathcal{T}$ to an object in $\mathcal{F}$ and the two subcategories are maximal with respect to this property. $\mathcal{T}$ is called the torsion class, $\mathcal{F}$ the torsion-free class.

Each torsion pair induces an idempotent radical, called torsion radical, and conversely: $\mathcal{T}$ is a torsion class of some $(\mathcal{T}, \mathcal{F})$ if and only if there exists an idempotent radical $t$ such that $\mathcal{T} = \{ M \mid tM = M \}$. So for $M \in \text{Mod} - A$, $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. Also there is always the canonical short exact sequence $0 \to tM \to M \to M/tM \to 0$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ is called splitting if each indecomposable module $M$ either lies in $\mathcal{T}$ or in $\mathcal{F}$. Then the canonical sequence above splits. One can also show:

**Proposition 1.1.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\text{mod} \ A$. Then $(\mathcal{T}, \mathcal{F})$ is splitting if and only if $\text{Ext}_A^1(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

Of course, not every torsion pair is splitting.
2 Harder-Narasimhan filtration

Let $Q$ to be a finite quiver with set of vertices $I$, and let $\theta : ZI \rightarrow Z$ be a linear function, called stability. We also define dim on $ZI$ by $\dim d = \sum_{i \in I} d_i$. For a non-zero dimension vector $d \in \mathbb{N}I$, we define its slope by $\mu(d) = \frac{\theta(d)}{\dim d} \in \mathbb{Q}$. We define the slope of a non-zero representation $X$ of $Q$ (over some field) as the slope of its dimension vector, thus $\mu(X) = \mu(\dim X) \in \mathbb{Q}$.

We call the representation $X$ semistable if $\mu(U) \leq \mu(X)$ for all non-zero subrepresentations $U$ of $X$, and we call $X$ stable if $\mu(U) < \mu(X)$ for all non-zero proper subrepresentations $U$ of $X$.

**Definition 2.1.** A filtration $0 = X_0 \subset X_1 \subset \ldots \subset X_s = X$ of a representation $X$ is called Harder-Narasimhan (abbreviated by HN) if the subquotients $X_i/X_{i-1}$ are semistable for $i = 1, \ldots, s$ and $\mu(X_1/X_0) \geq \mu(X_2/X_1) \geq \ldots \geq \mu(X_s/X_{s-1})$.

It was shown in a previous lecture that any non-zero representation $X$ possesses a unique Harder-Narasimhan filtration, which was done with the help of the following concept:

**Definition 2.2.** A subrepresentation $U$ of a representation $X$ is called strongly contradicting semistable (or just ss) if its slope is maximal among the slopes of subrepresentations of $X$, that is, $\mu(U) = \max \{ \mu(V) \mid V \subset X \}$, and it is of maximal dimension with this property.

3 Functorial properties of the HN-filtration

The Harder-Narasimhan filtration can be interpreted functorially. Introduce for a given slope $\mu$ and each representation $X$ a family of representations $\{X^{(a)}\}$, for $a \in \mathbb{Q}$ from the Harder-Narasimhan filtration as follows: Define

$$X^{(a)} = X_k \text{ if } \mu(X_k/X_{k-1}) \geq a > \mu(X_{k+1}/X_k),$$

$$X^{(a)} = X \text{ if } a \leq \mu(X_i/X_{i-1}), \ i = 1, \ldots, s,$$

$$X^{(a)} = 0 \text{ if } a > \mu(X_i/X_{i-1}), \ i = 1, \ldots, s.$$

Recall the following results on maps between semistable representations: Let $X, Y$ be semistable and let $f : X \rightarrow Y$ a non-zero homomorphism. Then $\mu(X) \leq \mu(Y)$. Also, each homomorphism $f : X \rightarrow Y$ with $\mu(X) > \mu(Y)$ is zero.

**Lemma 3.1.** Any morphism $f : X \rightarrow Y$ respects the HN-filtration, in the sense that $f(X^{(a)}) \subset Y^{(a)}$ for all $a \in \mathbb{Q}$.

**Proof.** First, we will prove the following property by induction on $k$:

If $f(X_k) \subset Y_l \setminus Y_{l-1}$, then $\mu(Y_l/Y_{l-1}) \geq \mu(X_k/X_{k-1})$.

The claim in the lemma follows from this: given $a \in \mathbb{Q}$, we have $X^{(a)} = X_k$ for the index $k$ satisfying $\mu(X_k/X_{k-1}) \geq a > \mu(X_{k+1}/X_k)$ (by definition). Choosing
Thus, \( X \in \mathcal{F} \) belongs to \( \mathcal{F} \), and thus \( X \) belongs to \( \mathcal{F} \), as desired. Hence, the slope of \( Y \) is greater than the slope of \( X \), proving \( \text{Hom}(T, X) = 0 \) for some representation \( T \). Suppose \( X \) has a weight strictly less than \( a \), then certainly the slope of the (semistable) top factor \( X/X_{s-1} \) is strictly less than \( a \), too, thus it belongs to \( \mathcal{F} \). But the projection map \( X \to X/X_{s-1} \) is non-zero, a contradiction. Thus, \( X \) belongs to \( \mathcal{F} \).

Finally, assume \( \text{Hom}(\mathcal{F}_a, Y) = 0 \) for some representation \( Y \). If \( Y \) has a weight \( \geq a \), then certainly the slope of its (semistable) ss class subrepresentation \( Y_1 \) is \( \geq a \). Thus \( Y_1 \) belongs to \( \mathcal{F} \). But the inclusion \( Y_1 \to Y \) is non-zero, a contradiction. Thus, \( Y \) belongs to \( \mathcal{F} \).

The inclusion properties of the various torsion and free classes follows from the definitions.

\begin{align*}
\end{align*}

4 Torsion pairs from HN-filtration

Let us call the slopes \( \mu(X_1/X_0), \ldots, \mu(X_s/X_{s-1}) \) in the unique Harder-Narasimhan filtration of \( X \) the \textit{weights} of \( X \).

\begin{definition}
Given \( a \in \mathbb{Q} \), define \( \mathcal{T}_a \) as the class of all representations \( X \) all of whose weights are \( \geq a \), and define \( \mathcal{F}_a \) as the class of all representations \( X \) all of whose weights are \( < a \).
\end{definition}

\begin{lemma}
For each \( a \in \mathbb{Q} \), the pair \( (\mathcal{T}_a, \mathcal{F}_a) \) defines a torsion pair in \( \text{mod} \ k\mathbb{Q} \). For \( a < b \), we have \( \mathcal{T}_a \supseteq \mathcal{T}_b \) and \( \mathcal{F}_a \subseteq \mathcal{F}_b \).
\end{lemma}

\begin{proof}
Assume \( X \in \mathcal{T}_a \) and \( Y \in \mathcal{F}_a \). In the \( \mathbb{Q} \)-indexed Harder-Narasimhan filtration, we thus have \( X^{(b)} = X \) for all \( a \leq b \), and \( Y^{(b)} = 0 \) for all \( a < b \). But any morphism \( f : X \to Y \) is already zero, since the slope of \( X \) is greater than the slope of \( Y \), proving \( \text{Hom}(\mathcal{T}_a, \mathcal{F}_a) = 0 \).

Now assume \( \text{Hom}(X, \mathcal{F}_a) = 0 \) for some representation \( X \). Suppose \( X \) has a weight strictly less than \( a \), then certainly the slope of the (semistable) top factor in the Harder-Narasimhan filtration, \( X/X_{s-1} \) is strictly less than \( a \), too, thus it belongs to \( \mathcal{F}_a \). But the projection map \( X \to X/X_{s-1} \) is non-zero, a contradiction. Thus, \( X \) belongs to \( \mathcal{T}_a \).

Finally, assume \( \text{Hom}(\mathcal{T}_a, Y) = 0 \) for some representation \( Y \). If \( Y \) has a weight \( \geq a \), then certainly the slope of its (semistable) ss class subrepresentation \( Y_1 \) is \( \geq a \). Thus \( Y_1 \) belongs to \( \mathcal{T}_a \). But the inclusion \( Y_1 \to Y \) is non-zero, a contradiction. Thus, \( Y \) belongs to \( \mathcal{F}_a \).

The inclusion properties of the various torsion and free classes follows from the definitions.

\end{proof}
References


