Selected topics in representation theory 4
Koszul algebras and distributive lattices

WS 2005/06

1 The quadratic dual and lattices

We consider a graded algebra $A := TV/(I)$, where V is a finite dimensional k–vector space, TV is the tensor algebra of V over k and I is a subspace in $V \otimes V$. Since I generates an ideal in TV, we get

$$A_i = T^i V / \sum_r V \otimes ... \otimes V \otimes I \otimes V \otimes \ldots \otimes V,$$

and we define

$$W_i^{r+1} := V \otimes ... \otimes V \otimes I \otimes V \otimes \ldots \otimes V,$$

where we have r times the tensor product of V at the beginning. So we get subspaces $W_i^r \subseteq T^i V$ for $r = 1, \ldots, i-1$.
In a similar way we want to describe the graded dual B of $A^!$ defined by

$$B_i := (A_i^!)^*, \qquad B := \oplus_{i \geq 0} B_i.$$

Lemma. *For the graded pieces of B we obtain*

$$B_0 = k, \quad B_1 = V, \quad B_2 = I, \quad B_i = \bigcap_{r=1}^{i-1} W_i^r.$$

Proof. The assertion is obvious for B_0 and B_1. We consider B_2: Using the definition of the quadratic dual algebra we obtain an exact sequence

$$0 \longrightarrow I^\perp \longrightarrow V^* \otimes V^* \longrightarrow V^* \otimes V^*/I^\perp = A_2^! = B_2^* \longrightarrow 0.$$

Taking the dual space we obtain

$$0 \longrightarrow I = B_2 \longrightarrow V \otimes V \longrightarrow V \otimes V/I = A_2 \longrightarrow 0.$$

To prove the result for B_i, $i > 2$ we note that for a vector space U with two subspaces U_1 and U_2 we obtain

$$(U/(U_1 + U_2))^* = U_1^\perp \cap U_2^\perp,$$

where $U_i^\perp := \{\phi \in U^* \mid \phi(u) = 0 \quad \forall u \in U_i\}$. If we apply this formula to

$$B_i^* = T^i V^* / \sum V^* \otimes \ldots \otimes V^* \otimes I^\perp \otimes V^* \otimes \ldots \otimes V^* = T^i V^* / (\sum_{j=1}^{i-1} W_i^j)$$

we obtain the result. □

The subspaces W_i^r generate a lattice in $T^i V$ with respect to $+$ and $\cap$.

Definition. Let U be a vector space with a set of subspaces $\{U_i\}_{i\in I}$. The vector space U with subspaces $\{U_i\}_{i\in I}$ is called 3–*distributive* (or triple–distributive) if $(U_i + U_j) \cap U_l = U_i \cap U_l + U_j \cap U_l$ for all triples i, j, l in I. The lattice U is called *distributive* if for alle triples in the lattice (take three subspaces, each of them is obtained by a finite sequence of operations including $+$, $\cap$ and U_i) we have the previous triple–identity. Similarly, a set of subspaces $\{U_i\}_{i\in I}$ in U is called n–*distributive* (for $n \geq 3$) if each subset of n spaces generates a distributive lattice in U. A sequence of subspaces $\{U_i\}_{i=1}^{t}$ is called *linear–distributive* if the subspaces $U_1 \cap \ldots \cap U_{i-1}, U_i, U_{i+1} + \ldots + U_t$ form a distributive triple in U. Note that we need the total order on I to define it, however we can similarly define an analogeous notion for any poset I.

Example. Let $\dim U = 2$ and $\sharp I = 2$. The only non-trivial lattice consists of two one-dimensional subspaces U_1 and U_2. We can, after chosing an adapted basis, assume $U_1 = k(0,1)$ and $U_2 = k(1,0)$. Consequently the lattice is distributive.
Let $\dim U = 2$ and $\sharp I = 3$. We can again assume U_1 and U_2 are as above and $U_3 = k(1,1)$. This lattice is not distributive, since $U_1 + U_2 = k^2$ and $(U_1 + U_2) \cap U_3 = U_3$, whereas $U_1 \cap U_3 + U_2 \cap U_3 = \{0\}$.

The following theorem is a standard result in lattice theory (cf. [3], 2.7 Theorem 19).

Theorem. *Let U, with subspaces U_i, a lattice as above. Then this lattice is distributive precisely when there exists a basis of U, so that each vector space U_i is generated by a part of this basis.*

Proof. Here we only show the easy conclusion, the other one is more technical.
Let $\{e_j\}$ be a basis of U, so that each U_i is generated by some elements e_j for some subset J $U^J := \langle e_j \mid j \in J\rangle$. Let I, J, K be three subsets, then

$$\begin{aligned}(U^I + U^J) \cap U^K &= U^{(I\cup J)\cap K} = U^{I\cap K \cup J\cap K} \\ &= U^I \cap U^K + U^J \cap U^K.\end{aligned}$$

□

2 Distributive triples and representations of $\mathbb{D}_4$

Let U be a vector space together with subspaces $U_1, \ldots, U_t$. These subspaces generate a lattice of subspaces in U. We are interested in distributive triples, n–distributive subspaces and linear–distributive subspaces. We can consider the subspaces of U in a natural way as representations of the subspace quiver. Then triples correspond to representations of $\mathbb{D}_4$, 4 subspaces correspond to representations of $\widetilde{\mathbb{D}}_4$ and t subspaces correspond to representations of the t–subspace quiver $Q(t)$.

Lemma. *The representation U associated to the t subspaces U_i decomposes into* $\dim U$ *indecomposable representations (these representations must be thin), precisely when the t subspaces U_i generate a distributive lattice in U.*

Proof. Assume the subspaces generate a distributive lattice, then there exists a basis of U compatible with all these subspaces, that is the intersection of this basis with each U_i is a basis of U_i. Consequently, U decomposes into $\dim U$ thin representations. Conversely, if there exists a non-thin direct summand, then the lattice is not distributive ny the above example. □

EXAMPLE. We show that there exist t–distributive sets of subspaces that are not $t+1$–distributive for each $t \geq 2$. Note that each set of subspaces is 1–distributive and 2–distributive (this is just representation theory of the quiver $\mathbb{A}_n$).
Consider a t–dimensional vector space U together with $t+1$ one–dimensional subspaces in general position. Then this set is t–distributive (the direct sum of any t subspaces is U) and not $t+1$–distributive: take for U_a the sum of $t-1$ subspaces, for U_b and U_c one of the remaining one–dimensional subspaces. Then $(U_a + U_b) \cap U_c \neq U_a \cap U_c + U_b \cap U_c$. Note that this set of subspaces is also not linear–distributive.

We show in section 4 that A is Koszul precisely when the subspaces W^i in T^dV form a linear–distributive set for all $d \geq 4$. Even stronger, one can show that A is Koszul precisely when the subspaces W^i in T^dV generate a distributive lattice (see [5]).

3 The Koszul complex in low degrees

In this section we consider the Koszul complex $B \otimes A, d$ in low degrees. With notation from above we have

$$A_i = T^iV / \sum_{r=1}^{i-1} W_i^r, \qquad B_i = \bigcap_{r=1}^{i-1} W_i^r.$$

DEGREE 1: The Koszul complex is exact:

$$\begin{array}{ccc} 0 \longrightarrow B_1 \simeq B_1 \otimes k & \longrightarrow & A_1 \simeq k \otimes A_1 \longrightarrow 0 \\ V & \simeq & V \end{array}$$

DEGREE 2: The Koszul complex is exact:

$$\begin{array}{ccccc} 0 \longrightarrow B_2 & \longrightarrow & B_1 \otimes A_1 & \longrightarrow & A_2 \longrightarrow 0 \\ I & \longrightarrow & V \otimes V & \longrightarrow & V \otimes V / I. \end{array}$$

DEGREE 3: For simplicity we omitt the zeros and use both notations in this case. Moreover we also omit the subscript for W, since it is always 3. The complex is exact, it only needs a little argument:

$$\begin{array}{ccccccc} B_3 \longrightarrow & B_2 \otimes A_1 & \longrightarrow & B_1 \otimes A_2 & \longrightarrow A_3 \\ V \otimes V \otimes V \longrightarrow & I \otimes V & \longrightarrow & V \otimes V \otimes V / V \otimes I & \longrightarrow V \otimes V \otimes V / (V \otimes I + I \otimes V) \\ W^1 \cap W^2 \longrightarrow & W^1 & \longrightarrow & T^3V/W^2 & \longrightarrow T^3V/(W^1 + W^2). \end{array}$$

The complex is exact precisely when the cokernel of the first map coincides with the kernel of the last map. That is

$$W^1/(W^1 \cap W^2) \simeq (W^1 + W^2)/W^2,$$

which is obviously satisfied.

DEGREE 4: In degree 4 and higher we also introduce a short notation for the spaces $\bigcap W^i$ and the spaces $\sum W^i$ that appear in the Koszul complex. We define in degree d

$$X^i := W^1 \cap \ldots \cap W^i, \quad X^0 := X^{-1} := T^d V, \quad Y^i := W^i + \ldots + W^{d-1}, \quad Y^d := Y^{d+1} := \{0\}.$$

Then we get descending filtrations of $T^d V$ as follows

$$\{0\} \subseteq X^{d-1} \subseteq \ldots X^1 \subseteq X^0 = X^{-1} = T^d V \text{ and}$$

$$\{0\} = Y^{d+1} = Y^d \subseteq Y^{d-1} \subseteq \ldots \subseteq Y^2 \subseteq Y^1 \subseteq T^d V$$

Using this notation we obtain (in degree 4) a sequence

$$\begin{array}{ccccccc} B_4 & \longrightarrow & B_3 \otimes A_1 & \longrightarrow & B_2 \otimes A_2 & \longrightarrow & \\ W^1 \cap W^2 \cap W^3 & \longrightarrow & W^1 \cap W^2 & \longrightarrow & W^1/(W^1 \cap W^3) & \longrightarrow & \ldots \\ X^3/(X^3 \cap Y^5) & \longrightarrow & X^2/(X^2 \cap Y^4) & \longrightarrow & X^1/(X^1 \cap Y^3) & \longrightarrow & \end{array}$$

$$\begin{array}{ccccc} & \longrightarrow & B_1 \otimes A_3 & \longrightarrow & A_4 \\ \ldots & \longrightarrow & T^4V/(W^2 + W^3) & \longrightarrow & T^4V/(W^1 + W^2 + W^3) \\ & \longrightarrow & X^0/(X^0 \cap Y^2) & \longrightarrow & X^{-1}/(X^{-1} \cap Y^1) \end{array}$$

Since the Koszul complex is already exact in degree 3, it is everywhere exact, except, possibly, in position $B_2 \otimes A_2$. To show exactness in this place, we compute the cokernel and the kernel of the corresponding maps:

$$\begin{array}{rcl} \mathrm{Coker}\,(W^1 \cap W^2 \cap W^3 \longrightarrow W^1 \cap W^2) & = & (W^1 \cap W^2)/(W^1 \cap W^2 \cap W^3) \\ & \simeq & (W^1 \cap W^2 + W^1 \cap W^3)/(W^1 \cap W^3) \end{array}$$

$$\begin{array}{rcl} \mathrm{Ker}\,(W^1/(W^1 \cap W^3) \longrightarrow T^dV/(W^2 + W^3)) & = & \\ \mathrm{Ker}\,(W^1/(W^1 \cap W^3) \longrightarrow (W^1 + W^2 + W^3)/(W^2 + W^3)) & = & \\ \mathrm{Ker}\,(W^1/(W^1 \cap W^3) \longrightarrow W^1/(W^1 \cap (W^2 + W^3))) & = & (W^1 \cap (W^2 + W^3))/(W^1 \cap W^3). \end{array}$$

We finally see, that the Koszul complex is exact, precisely when $(W^1 \cap (W^2 + W^3))/(W^1 \cap W^3) = (W^1 \cap W^2 + W^1 \cap W^3)/(W^1 \cap W^3)$, that is the triple W^1, W^2, W^3 is distributive. So we have proven the following lemma.

Lemma. *The Koszul comples in degree at most 4 is exact precisely when W^1, W^2, and W^3 is a distributive triple of subspaces in T^4V.*

4 The Koszul complex and the lattice

Now we consider the Koszul complex in arbitrary degree. We first compute the terms of the Koszul complex using the subsapces X^i and Y^i defined above. It turns out, that the differential is the unique natural map comming from the two filtrations of T^dV:

Theorem. *1) The Koszul complex is the complex*

$$\cdots \longrightarrow X^i/(X^i \cap Y^{i+2}) \xrightarrow{d_i} X^{i-1}/(X^{i-1} \cap Y^{i+1}) \xrightarrow{d_{i-1}} X^{i-2}/(X^{i-2} \cap Y^i) \longrightarrow \cdots,$$

with the natural maps.
2.) The kernel of d_i is $(X^i \cap Y^{i+1})/(X^i \cap Y^{i+2})$ and the cokernel of d_{i+1} is $X^i/(X^i \cap Y^{i+1})$.
3.) The Koszul complex splits of into short exact sequences

$$0 \longrightarrow (X^i \cap Y^{i+1})/(X^i \cap Y^{i+2}) \longrightarrow X^i/(X^i \cap Y^{i+2}) \longrightarrow X^i/(X^i \cap Y^{i+1}) \longrightarrow 0$$

and the Koszul complex is exact precisely when $(X^i \cap Y^{i+1})/(X^i \cap Y^{i+2}) = X^{i+1}/(X^{i+1} \cap Y^{i+2})$.
4) The Koszul complex is exact in degree d precisely when the subspaces $W^i_d, i = 1, \dots, d-1$ form a linear-distributive set of subspaces in T^dV. This condition is equivalent to

$$X^i \cap Y^{i+1} = X^{i+1} + X^i \cap Y^{i+2}.$$

Proof. First we note that (we denote natural isomorphisms by "=")

$$\begin{aligned} X^{i+1}/(X^{i+1} \cap Y^{i+2}) &= X^{i+1}/(X^{i+1} \cap X^i \cap Y^{i+2}) \\ &= (X^{i+1} + X^i \cap Y^{i+2})/(X^i \cap Y^{i+2}) \\ &= (X^i \cap W^{i+1} + X^i \cap Y^{i+2})/(X^i \cap Y^{i+2}), \end{aligned}$$

$$(X^i \cap Y^{i+1})/(X^i \cap Y^{i+2}) = (X^i \cap (W^{i+1} + Y^{i+2}))/(X^i \cap Y^{i+2}),$$

and both sides are equal precisely when the triples X^i, W^{i+1}, Y^{i+2} are distributive (as subspaces in T^dV). In any case we have a natural injective map inducing the differential in the Koszul complex

$$X^{i+1}/(X^{i+1} \cap Y^{i+2}) \longrightarrow (X^i \cap Y^{i+1})/(X^i \cap Y^{i+2}).$$

If we replace the terms in the Koszul complex by the terms X^i and Y^j and combining the formulas above we proved the theorem. □

The following stronger result is proven in [5].

Theorem. *The quadratic algebra $A = TV/(I)$ is Koszul precisely when the subspaces W^i generate a distributive lattice in T^dV for all $d \geq 4$.*

References

[1] A. A. Beilinson, V. A. Ginsburg, V. V. Schechtman, *Koszul duality.* J. Geom. Phys. **5** (1988), no. 3, 317–350.

[2] A. A. Beilinson, V. A. Ginsburg, W. Soergel, *Koszul duality patterns in representation theory.* J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[3] G. Grätzer, *Lattice theory.* W. H. Freeman and Company, San Francisco 1971.

[4] Yu. I. Manin, *Some remarks on Koszul algebras and quantum groups.* Ann. Inst. Fourier (Grenoble) **37** (1987), no. 4, 191–205.

[5] A. Polishchuk, L. Positselski, *Quadratic algebras.* University Lecture Series, **37**. American Mathematical Society, Providence, RI, 2005.