Gabriel-Roiter inclusions and irreducible monomorphisms.

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Let Λ be an artin algebra, and $\operatorname{mod}\Lambda$ the category of Λ-modules of finite length.

1. Proposition. *Assume $X \subset Y$ is a Gabriel-Roiter inclusion. Then there is an irreducible monomorphism $X \to M$ with M indecomposable and an epimorphism $M \to Y$ such that the composition $X \to M \to Y$ is injective* (and therefore also a Gabriel-Roiter inclusion.)

Proof: Let $u\colon X \to Y$ be the given Gabriel-Roiter inclusion. Let $f = (f_i)\colon X \to \bigoplus M_i$ be the source map, with all M_i indecomposable. We obtain maps $h_i\colon M_i \to Y$ such that $\sum h_i f_i = u$. The indices are $1 \le i \le t$. Assume $h_i f_i$ is not a monomorphism, for $i > s$, and a monomorphism for the remaining $1 \le i \le s$. Then also $v = \sum_{i>s} h_i f_i$ is in $\operatorname{Sing}(X,Y)$, whereas u is not, thus $u' = u - v = \sum_{1\le i\le s} h_i f_i$ is outside of $\operatorname{Sing}(X,Y)$, thus a monomorphism.

Let Y_i be the image of $h_i f_i$, for $1 \le i \le s$. All the following indices are $1 \le i \le s$. Assume that none of the maps h_i is surjective. Let Y_i be the image of h_i. Then $\mu(Y_i) < \mu(Y)$. Now under u' the module X embeds into $\bigoplus Y_i$, thus $\mu(X) \le \max \mu(Y_i) < \mu(Y)$. Since X is a Gabriel-Roiter submodule of Y, we must have $\mu(X) = \max \mu(Y_i)$, thus this embedding $X \to \bigoplus M_i \to \bigoplus Y_i$ is a split monomorphism, but then f itself is a split monomorphism — impossible. This shows that at least one of the h_i is surjective. But then f_i is a monomorphism (as an irreducible map, it is either mono or epi: if f_i would be epi, then $|X| > |M_i| \ge |Y|$, in contrast to the fact that X embeds into Y.)

2. Applications.

The proposition implies in particular the following: If X is a Gabriel-Roiter submodule of some module Y, then there has to exist an irreducible monomorphism $X \to M$ with M indecomposable. Consider for example the four-subspace quiver and let X be indecomposable preprojective of length 7: then all the irreducible maps starting in X are epimorphisms, thus X cannot be a Gabriel-Roiter submodule of any module.

Let p be the maximal length of an indecomposable projective module, let q be the maximal length of an indecomposable injective module, then we know that $|\tau^{-1}(X)| \le (pq-1)|X|$.

Corollary 1. *Let $X \to Y$ be a Gabriel-Roiter inclusion. Then $|Y| \le pq|X|$.*

Proof: The middle term X' of the Auslander-Reiten sequence starting in X has length at most $pq|X|$, and Y is a factor module of X'.

Another proof can be found in [R2], there it is shown that it implies the **Successor Lemma.**

Corollary 2. *Let M be an indecomposable module and $1 \le a < |M|$ a natural number. Then there exists an indecomposable submodule M' of M with length in the intervall $[a+1, pqa]$.*

Proof: Take a Gabriel-Roiter filtration $M_1 \subset \cdots \subset M_n = M$. Let i be maximal with $|M_i| \le a$. Then $1 \le i < n$, thus M_{i+1} exists and $a < |M_{i+1}| \le pq|M_i| \le pqa$.

Corollary 3. *Let M be an indecomposable module and assume all indecomposable proper submodules of M are of length at most b. Then $|M| \le pqb$.*

Proof: Let X be a Gabriel-Roiter submodule of M. By assumption, $|X| \le b$, thus $|M| \le pq|X| \le pqb$.

Reformulation: Let $\mathcal{N}$ be a class of indecomposable modules. Recall that a module M is said to be $\mathcal{N}$-critical provided it does not belong to $\operatorname{add}\mathcal{N}$, but any proper indecomposable submodule of M belongs to $\mathcal{N}$. Corollary 3 asserts the following: if all the modules in $\mathcal{N}$ are of length at most b, then any $\mathcal{N}$-critical module is of length at most pqb.

Observe that the last two corollaries do not refer at all to Gabriel-Roiter notions.

3. Examples: The take-off part of a generalised Kronecker algebra. We consider the finite dimensional hereditary algebras with s = 2, where s denotes the number of simple modules. We assume that Λ is representation-infinite. Let $P_1, P_2, \ldots$ be the sequence of preprojectives, with non-zero maps $P_i \to P_{i+1}$.

Proposition. For $n \ge 2$, $\mathcal{A}(I_n) = \{P_n\}$.

For $n = 2$, the assertion is true according to the general description of I_2. For $n > 2$, we use induction. We have to consider three cases:

Case 1. Consider first a bimodule ${}_FM_G$ with dimensions (a, b) where $a, b \ge 2$. Then all the non-zero maps $P_n \to P_{n+1}$ are monomorphisms. Also, since all the irreducible maps ending in P_n are monomorphisms, *the monomorphisms $P_{n-1} \to P_n$ are Gabriel-Roiter inclusions.*

Consider some $n > 2$ and assume the assertion is true for $n-1$. Since there is a Gabriel-Roiter inclusion $P_{n-1} \to P_n$, it follows that $I_n = I_{n-1} \cup \{t\}$ with $t \ge |P_n|$. Thus let Y be indecomposable with $\mu(Y) = I_n$, let X be a Gabriel-Roiter submodule of Y. Then $\mu(Y) = I_{n-1}$, thus by induction $X = P_{n-1}$. But now we can apply proposition 1 above which shows that Y is a factor module of P_n. Since $|Y| = t \ge |P_n|$, we see that $Y = P_n$.

Case 2. $G \subset F$, and $M = {}_GF_F$. Let $a = [F : G]$. Then we deal with the preprojectives

$$P_1 = (1, 0) \to P_2 = (a, 1) \to P_3 = (a-1) \to \cdots$$

with $\operatorname{End}(P_{2i-1}) = G$, and $\operatorname{End}(P_{2n}) = F$.

- *The non-zero maps $P_{2n-1} \to P_{2n}$ are injective and are Gabriel-Roiter inclusions.*
- The non-zero maps $P_{2n} \to P_{2n+1}$ are surjective.
- *The non-zero maps $P_{2n-1} \to P_{2n+1}$ are injective and are Gabriel-Roiter inclusions.*

Consider some $2n$ and assume the assertion is true for $2n-1$. The argument is the same as in Case 1, using Proposition 1.

Also, consider some $2n+1$ and assume the assertion is true for $2n-1$ and $2n$. Since the irreducible maps starting in P_{2n} are epi, we see that I_{2n+1} cannot start with I_{2n}. Since there are Gabriel-Roiter inclusions $P_{2n-1} \to P_{2n+1}$, we see that $I_{2n+1} = I_{2n-1} \cup \{t\}$ with $|P_{2n}| > t \geq |P2n+1|$.

Thus let Y be indecomposable with $\mu(Y) = I_{2n+1}$, let X be a Gabriel-Roiter submodule of Y. Then $\mu(Y) = I_{2n-1}$, thus by induction $X = P_{2n-1}$. But now we can apply proposition 2 above which shows that Y is a factor module of P_{2n1}. Since $|Y| = t \geq |P_{2n+1}|$, we see that $Y = P_{2n+1}$.

Case 3. $G \subset F$, and $M = {}_F F_G$. Let $a = [F : G]$. Then we deal with the preprojectives

$$P_1 = (1,0) \to P_2 = (1,1) \to P_3 = (a-1,a) \to P_4 = (a-2,a-1) \cdots$$

with $\operatorname{End}(P_{2i-1}) = F$, and $\operatorname{End}(P_{2n} = G$.

The non-zero maps $P_{2n-1} \to P_{2n}$ are surjective, for $n \geq 2$, whereas $P_1 \to P_2$ is injective (and this is a Gabriel-Roiter inclusion).

- *The non-zero maps $P_{2n} \to P_{2n+1}$ are injective and are Gabriel-Roiter inclusions.*
- *The non-zero maps $P_{2n} \to P_{2n+2}$ are injective and are Gabriel-Roiter inclusions.*

Proof: As in case 2, but taking into account the additional Gabriel-Roiter inclusion $P_1 \to P_2$.

Consequence. Consider for example the 3-Kronecker quiver. Let $\mathcal{N}$ be the class of projective modules. A mdoule M is said to be $\mathcal{N}$-critical provided it does not belong to $\mathcal{N}$, but any proper submodule of M belongs to $\mathcal{N}$. Claim: any $\mathcal{N}$-critical module is a factor module of P_3, thus of length at most 11 and its top is of length at most 3. (Note that the proper factor modules of P_2 are also factor modules of P_3.)

4. A further example: *Calculation of I_4 for the algebra with vertices a, b, c, two arrows $a \leftarrow b$, and two arrows $b \leftarrow c$.*

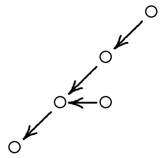
Let Y be indecomposable with $\mu(Y) = I_4$. Then Y is a factor module of the preprojective module $M = (12,7,2)$. This modules has a submodule U which is the direct sum of two copies of $(3,2,0)$. Under the epimorphism $M \to Y$, the socle of U has to go to zero, since the socle of U is contained in the kernel of any map $U \to Y$ (since Y is $\mathcal{N}$-critical, with $\mathcal{N}$ the class of projective modules). Now $Y/(\operatorname{soc} U)$ is the direct sum of two copies of $P(c)$ and $S(b)$, thus M is a factor module of $P(c) \oplus P(c)$, say $P(c) \oplus P(c)/V$. If V is of length 1, then Y would contain $P(b)^4/V$ and this contains a submodule $(3,2,0)$, impossible. Thus V is of length at least 2. However, there is such an indecomposable module which is $\mathcal{N}$-critical, namely the module $(6,4,2)$ with 7-dimensional socle - it lies in the component which contains $(0,1,0)$ - actually, there is an Auslander-Reiten sequence $(0,1,0) \to (6,4,2) \to (6,3,2)$.

5. Remarks. The proposition asserts that for any Gabriel-Roiter inclusion $X \to Y$, there exists an irreducible monomorphism $X \to M$ with M indecomposable and an epimorphism $M \to Y$ such that the composition $X \to M \to Y$ is a Gabriel-Roiter inclusion.

Assume conversely that an irreducible monomorphism $X \to M$ is given.

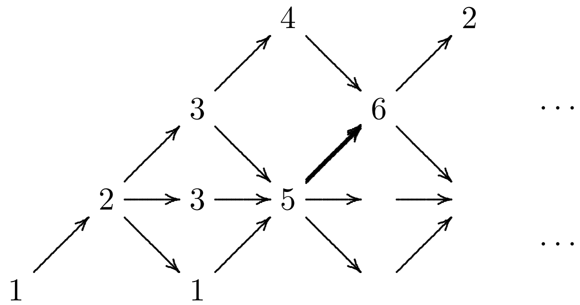
(a) Then there does not have to exist an epimorphism $M \to Y$ such that the composition $X \to M \to Y$ is a Gabriel-Roiter inclusion.

Example 1: Here is an example of an irreduble monomorphism $X \to M$, with M/X simple such that $\mu(X) = \{1,2,3,5\}, \mu(M) = \{1,2,3,4,6\}$. The quiver and the dimension vectors:



$$X = \begin{smallmatrix} & & & 0 \\ & & 1 & \\ & 2 & 1 & \\ 1 & & & \end{smallmatrix} \qquad M = \begin{smallmatrix} & & & 1 \\ & & 1 & \\ & 2 & 1 & \\ 1 & & & \end{smallmatrix}$$

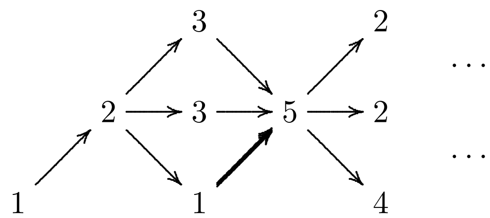
Here is the start of the Auslander-Reiten quiver:



The fat arrow indicates an irreducible monomorphism $X \to M$, and this is not a Gabriel-Roiter inclusion, since $\mu(M) = \{1,2,3,4,6\}$. Since Y/X is simple, there cannot exist any proper epimorphism $M \to Y$ such that that the composition $X \to M \to Y$ is still a monomorphism.

Example 2. Consider the D_4-quiver with one sink, the sink being a leaf. Let X be the simple module for the branching point, Y the largest indecomposable. Then $X \subset Y$ is irreducible, but not a Gabriel-Roiter inclusion (with respect to the ordinary weights), the factor modules Y/U do not yield Gabriel-Roiter inclusions $X \to Y \to Y/U$.

(b) In case there is an irreducible monomorphism $X \to M$, there may be several epimorphisms $M \to Y_i$ such that the compositions $X \to M \to Y_i$ are Gabriel-Roiter inclusions, as follows:



The fat arrow indicates an irreducible monomorphism $X \to M$ which is not a Gabriel-Roiter inclusion, since $\mu(M) = \{1,2,3,5\}$. There are two proper epimorphism $M \to Y_i$ such that that the composition $X \to M \to Y_i$ is still a monomorphism, with $|Y_i| = 2$. Both these monomorphisms $X \to Y_i$ are Gabriel-Roiter inclusions.

References.

[R1] Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[R2] Ringel: Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure. To appear in the Proceedings of the Queretaro Workshop 2004. Contemporary Mathematics. Amer.Math.Soc.