The ladder construction of Prüfer modules.

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Let R be any ring. We deal with (left) R-modules. Our aim is to consider pairs of maps $w, v\colon U \to V$ with w a proper monomorphism.

Let M be a module. If there exists an endomorphism ϕ of M which is surjective, locally nilpotent, and with non-zero kernel W of finite length, then M will be said to be a *Prüfer module* (with respect to ϕ, and with basis W).

1. The basic construction. A pair of exact sequences

$$0 \to U_0 \xrightarrow{w_0} U_1 \to W \to 0 \quad \text{and} \quad 0 \to K \to U_0 \xrightarrow{v_0} U_1 \to Q \to 0$$

yields a module U_2 and a pair of exact sequences

$$0 \to U_1 \xrightarrow{w_1} U_2 \to W \to 0 \quad \text{and} \quad 0 \to K \to U_1 \xrightarrow{v_1} U_2 \to Q \to 0$$

by forming the induced exact sequence of $0 \to U_0 \xrightarrow{w_0} U_1 \to W \to 0$ using the map v_0:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U_0 & \xrightarrow{w_0} & U_1 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \| & & \\
0 & \longrightarrow & U_1 & \xrightarrow{w_1} & U_2 & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q & = & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

2. The ladder. Using induction, we obtain in this way modules U_i and pairs of exact sequences

$$0 \to U_i \xrightarrow{w_i} U_{i+1} \to W \to 0 \quad \text{and} \quad 0 \to K \to U_i \xrightarrow{v_i} U_{i+1} \to Q \to 0$$

for all $i \geq 0$.

We may combine the pushout diagrams constructed inductively and obtain the following ladder of commutative squares:

$$\begin{array}{ccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
{\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots
\end{array}$$

We form the inductive limit $U_\infty = \bigcup_i U_i$ (along the maps w_i).

Since all the squares commute, the maps v_i induce a map $U_\infty \to U_\infty$ which we denote by v_∞:

$$\begin{array}{cccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & \bigcup_i U_i = U_\infty \\
{\scriptstyle v_0}\downarrow & & {\scriptstyle v_1}\downarrow & & {\scriptstyle v_2}\downarrow & & {\scriptstyle v_3}\downarrow & & & \downarrow{\scriptstyle v_\infty} \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots & \bigcup_i U_i = U_\infty
\end{array}$$

We also may consider the factor modules U_∞/U_0 and U_∞/U_1. The map $v_\infty\colon U_\infty \to U_\infty$ maps U_0 into U_1, thus it induces a map

$$\overline{v}\colon U_\infty/U_0 \longrightarrow U_\infty/U_1.$$

Claim. *The map $\overline{v}$ is an isomorphism.* Namely, the commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_{i-1}} & & \downarrow{\scriptstyle v_i} & & \| & & \\
0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

can be rewritten as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & U_i/U_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_{i-1}} & & \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle \overline{v}_i} & & \\
0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & U_{i+1}/U_i & \longrightarrow & 0
\end{array}$$

with an isomorphism $\overline{v}_i\colon U_i/U_{i-1} \to U_{i+1}/U_i$. The map $\overline{v}$ is a map from a filtered module with factors U_i/U_{i-1} (where $i \geq 1$) to a filtered module with factors U_{i+1}/U_i (again with $i \geq 1$), and the maps $\overline{v_i}$ are just those induced on the factors.

It follows: The composition of maps

$$U_\infty/U_0 \xrightarrow{p} U_\infty/U_1 \xrightarrow{\overline{v}^{-1}} U_\infty/U_0$$

with p the projection map is an epimorphism ϕ with kernel U_1/U_0. It is easy to see that ϕ is locally nilpotent.

Summery. *The maps v_i yield a map*

$$v_\infty \colon U_\infty \to U_\infty$$

with kernel K and cokernel Q. This map v_∞ induces an isomorphism $\overline{v}\colon U_\infty/U_0 \to U_\infty/U_1$. Composing the inverse of this isomorphism with the canonical projection p, we obtain an endomorphism ϕ

$$U_\infty/U_0 \xrightarrow{p} U_\infty/U_1 \xrightarrow{\overline{v}^{-1}} U_\infty/U_0$$

and U_∞/U_0 is a Prüfer module with respect to ϕ, with basis W.

(Using a terminology introduced for string algebras, we also can say: U_∞ is *expanding,* U_∞/U_0 is *contracting.*)

If necessary, we will use the following notation: $U_i(w;v) = U_i$, for all $i \in \mathbb{N}$ and also for $i = \infty$, and $P(w;v) = U_\infty/U_0$ for the Prüfer module (here, $w = w_0,\ v = v_0$). Since $P(w;v)$ is a Prüfer module with basis the cokernel W of w, we will sometimes write $W[n] = U_n/U_0$.

Examples.

(1) The classical example: Let $R = \mathbb{Z}$, and also $U_0 = U_1 = \mathbb{Z}$. Maps $\mathbb{Z} \to \mathbb{Z}$ are given by the multiplication with some integer n, thus we denote it just by n. Let $w_0 = 2$ and $v_0 = n$. *If n is odd, then $P(2;n)$ is the ordinary Prüfer group for the prime* 2, and $U_\infty(2;n) = \mathbb{Z}[\frac{1}{2}]$ (the subring of $\mathbb{Q}$ generated by $\frac{1}{2}$). If n is even, then $P(2;n)$ is an elementary abelian 2-group.

(2) Let $R = K(2)$ be the Kronecker algebra over some field k. Let U_0 be simple projective, U_1 indecomposable projective of length 3 and $w_0 \colon U_0 \to U_1$ a non-zero map with cokernel W (one of the indecomposable modules of length 2). The module $P(w_0;v_0)$ is the Prüfer module for W if and only if $v_0 \notin kw_0$, otherwise it is a direct sum of copies of W.

(3) Trivial cases: First, let w be a split monomorphism. Then the Prüfer module with respect to any map $\alpha\colon U_0 \to U_1$ is just the countable sum of copies of W. Second, let $w\colon U_0 \to U_1$ be an arbitrary monomorphism, let $\beta\colon U_1 \to U_1$ be an endomorphism. Then $P(w;\beta w)$ is the countable sum of copies of W.

(4) Assume there exists a split monomorphism $\alpha\colon U_0 \to U_1$, say $U_1 = U_0 \oplus X$ and $\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : U_0 \to U_1$. Then

$$0 \to U_0 \xrightarrow{w} U_0 \oplus X \to W \to 0$$

is a Riedtmann-Zwara sequence, thus W is a degeneration of X. According to Zwara, there is n_0 such that $W[n+1] \simeq W[n] \oplus X$ for all $n \geq n_0$.

The chessboard. Assume now that both maps $w_0, v_0 \colon U_0 \to U_1$ are monomorphisms. Then we get the following arrangement of commutative squares:

$$
\begin{array}{ccccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
{\scriptstyle v_0}\big\downarrow & & {\scriptstyle v_1}\big\downarrow & & {\scriptstyle v_2}\big\downarrow & & {\scriptstyle v_3}\big\downarrow & & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots & & \\
{\scriptstyle v_0}\big\downarrow & & {\scriptstyle v_1}\big\downarrow & & {\scriptstyle v_2}\big\downarrow & & & & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & \cdots & & & & \\
{\scriptstyle v_0}\big\downarrow & & {\scriptstyle v_1}\big\downarrow & & & & & & \\
U_1 & \xrightarrow{w_1} & \cdots & & & & & & \\
{\scriptstyle v_0}\big\downarrow & & & & & & & & \\
\cdots & & & & & & & &
\end{array}
$$

We see both horizontally as well as vertically ladders: the horizontal ladders yield $U_\infty(w_0; v_0)$ (and its endomorphism v_∞); the vertical ladders yield $U_\infty(v_0; w_0)$ (and its endomorphism w_∞).

Let Λ be an artin algebra.

3. First application: Degenerations.

Proposition 1. *Let U, V be modules, and let W and W' be cokernels of monomorphisms $U \to V$. If $\operatorname{Ext}^1(W, W) = 0$, then there exists a module X and an exact sequence*

$$0 \to X \to X \oplus W \to W' \to 0.$$

Note that the existence of an exact sequence of the form $0 \to X \to X \oplus W \to W' \to 0$ may be interpreted as asserting that W' is a *degeneration* of W, according to Riedtmann and Zwara [Z].

Corollary. *Let U, V be modules, and let W and W' be cokernels of monomorphisms $U \to V$. Assume that both $\operatorname{Ext}^1(W, W) = 0$ and $\operatorname{Ext}^1(W', W') = 0$. Then the modules W and W' are isomorphic.*

Both assertions are well-known in case k is an algebraically closed field: in this case, the conclusion of proposition 1 just asserts that W' is a degeneration of W in the sense of algebraic geometry. The main point here is to deal with the general case when Λ is an arbitrary artin algebra. Our interest in this question was raised by a series of lectures by Sverre Smalø at the Mar del Plata conference, March 2006. The corollary stated above (under the additional assumptions that V is projective and that $w(U), w'(U)$ are contained in the radical of V) is due to Bautista and Perrez [BP] and this result was presented by Smalø with a new proof [S] at Mar del Plata.

Lemma. *Let W be a module with $\operatorname{Ext}^1(W,W) = 0$. Let $U_0 \subset U_1 \subset U_2 \subset \cdots$ be a sequence of inclusions of modules with $U_i/U_{i-1} = W$ for all $i \geq 1$. Then there is a natural number n_0 such that $U_n \subset U_{n+1}$ is a split monomorphism for all $n \geq n_0$.*

Lemma is well-known, it is based on the fact that $\operatorname{Ext}^1(W, U_0)$ when considered as a k-module is of finite length. A proof will be given below. Let us use it in order to finish the proof of proposition 1.

We apply Lemma to the chain of inclusions

$$U_0 \xrightarrow{w_0} U_1 \xrightarrow{w_1} U_2 \xrightarrow{w_2} \cdots$$

and see that there is n such that $w_n \colon U_n \to U_{n+1}$ splits. This shows that U_{n+1} is isomorphic to $U_n \oplus W$. But we also have the exact sequence

$$0 \to U_n \xrightarrow{v_n} U_{n+1} \to W' \to 0.$$

Replacing U_{n+1} by $U_n \oplus W$, we see that we get an exact sequence of the form

$$0 \to U_n \xrightarrow{v_n} U_n \oplus W \to W' \to 0$$

(a Riedtmann-Zwara sequence), as asserted.

Proof of Corollary. It is well-known that the existence of exact sequences

$$0 \to X \to X \oplus W \to W' \to 0 \quad \text{and} \quad 0 \to Y \to Y \oplus W' \to W \to 0$$

implies that the modules W- and W' are isomorphic. But in our case we just have to change one line in the proof of proposition 1 in order to get the required isomorphism. Thus, assume that both $\operatorname{Ext}^1(W,W) = 0$ and $\operatorname{Ext}^1(W',W') = 0$. Choose n such that **both** the inclusion maps

$$w_n \colon U_n \to U_{n+1} \quad \text{and} \quad v_n \colon U_n \to U_{n+1}$$

split. Then U_{n+1} is isomorphic both to $U_n \oplus W$ and to $U_n \oplus W'$, thus it follows from the Krull-Remak-Schmidt theorem that W and W' are isomorphic.

Remark. *Assume that $w, w' \colon U, V$ are monomorphisms with cokernels W and W', respectively, and that $\operatorname{Ext}^1(W,W) = 0$ and $\operatorname{Ext}^1(W',W') = 0$. Then w splits if and only if w' splits.*

Proof: According to the corollary, we can assume $W = W'$. Assume that w splits, thus V is isomorphic to $U \oplus W$. Look at the exact sequence $0 \to U \xrightarrow{w'} V \to W \to 0$. If it does not split, then $\dim \operatorname{End}(V) < \dim \operatorname{End}(U \oplus W)$, but V is isomorphic to $U \oplus W$.

Proof of Lemma. An assertion equivalent to Lemma was used for example by Roiter in his proof of the first Brauer-Thrall conjecture, a corresponding proof can be found in [R]. We include here a slightly different proof.

Applying the functor $\mathrm{Hom}(W,-)$ to the short exact sequence $0 \to U_{i-1} \xrightarrow{w_{i-1}} U_i \to W \to 0$, we obtain the exact sequence

$$\mathrm{Ext}^1(W,U_{i-1}) \to \mathrm{Ext}^1(W,U_i) \to \mathrm{Ext}^1(W,W).$$

Since the latter term is zero, we see that we have a sequence of surjective maps

$$\mathrm{Ext}^1(W,U_0) \to \mathrm{Ext}^1(W,U_1) \to \cdots \to \mathrm{Ext}^1(W,U_i) \to \cdots,$$

being induced by the inclusion maps $U_0 \to U_1 \to \cdots \to U_i \to \cdots$. The maps between the Ext-groups are k-linear. Since $\mathrm{Ext}^1(W,U_0)$ is a k-module of finite length, the sequence of surjective maps must stabilize: there is some n_0 such that the inclusion $U_n \to U_{n+1}$ induces an isomorphism

$$\mathrm{Ext}^1(W,U_n) \to \mathrm{Ext}^1(W,U_{n+1})$$

for all $n \geq n_0$. Now we consider also some Hom-terms: the exactness of

$$\mathrm{Hom}(W,U_{n+1}) \to \mathrm{Hom}(W,W) \to \mathrm{Ext}^1(W,U_n) \to \mathrm{Ext}^1(W,U_{n+1})$$

shows that the connecting homomorphism is zero, and thus that the map $\mathrm{Hom}(W,U_{n+1}) \to \mathrm{Hom}(W,W)$ (induced by the projection map $p\colon U_{n+1} \to W$) is surjective. But this means that there is a map $h \in \mathrm{Hom}(W,U_{n+1})$ with $ph = 1_W$, thus $p\colon U_{n+1} \to W$ is a split epimorphism and therefore the inclusion map $U_n \to U_{n+1}$ is a split monomorphism.

Remark. In general, there is no actual bound on the number n_0. However, in case of dealing with the chain of inclusions

$$U_0 \xrightarrow{w_0} U_1 \xrightarrow{w_1} U_2 \xrightarrow{w_n} \cdots$$

such a bound exists, namely the length of $\mathrm{Ext}^1(W,U_0)$ as a k-module, or, even better, the length of $\mathrm{Ext}^1(W,U_0)$ as an E-module, where $E = \mathrm{End}(W)$.

Proof: Look at the surjective maps

$$\mathrm{Ext}^1(W,U_0) \to \mathrm{Ext}^1(W,U_1) \to \cdots \to \mathrm{Ext}^1(W,U_i) \to \cdots,$$

being induced by the maps $U_n \xrightarrow{w_n} U_{n+1}$ (and these maps are not only k-linear, but even E-linear). Assume that $\mathrm{Ext}^1(W,U_n) \to \mathrm{Ext}^1(W,U_{n+1})$ is bijective, for some n. As we have seen above, this implies that the sequence

$$(*) \qquad 0 \to U_n \xrightarrow{w_n} U_{n+1} \to W \to 0$$

splits. Now the map w_{n+1} is obtained from $(*)$ as the induced exact sequence using the map w_n'. With $(*)$ also any induced exact sequence will split. Thus w_{n+1} is a split monomorphism (and $\mathrm{Ext}^1(W,U_{n+1}) \to \mathrm{Ext}^1(W,U_{n+2})$ will be bijective, again). Thus, as soon as we get a bijection $\mathrm{Ext}^1(W,U_n) \to \mathrm{Ext}^1(W,U_{n+1})$ for some n, then also all the following maps $\mathrm{Ext}^1(W,U_m) \to \mathrm{Ext}^1(W,U_{m+1})$ with $m > n$ are bijective.

Example. Consider the D_4-quiver with subspace orientation:

$$\begin{array}{ccc} & & b \\ & \swarrow & \\ a & \leftarrow & c \\ & \nwarrow & \\ & & d \end{array}$$

and let Λ be its path algebra over some field k. We denote the indecomposable Λ-modules by the corresponding dimension vectors. Let

$$U_0 = {\scriptstyle\begin{array}{cc} & 0 \\ 1 & 0 \\ & 0 \end{array}}, \quad U_1 = {\scriptstyle\begin{array}{cc} & 1 \\ 2 & 1 \\ & 1 \end{array}}, \quad W = {\scriptstyle\begin{array}{cc} & 1 \\ 1 & 1 \\ & 1 \end{array}}, \quad W' = {\scriptstyle\begin{array}{cc} & 0 \\ 1 & 1 \\ & 1 \end{array}} \oplus {\scriptstyle\begin{array}{cc} & 1 \\ 0 & 0 \\ & 0 \end{array}}.$$

Note that a map $w_0 \colon U_0 \to U_1$ with cokernel W exists only in case the base-field k has at least 3 elements; of course, there is always a map $w_0' \colon U_0 \to U_1$ with cokernel W'.

We have $\dim \operatorname{Ext}^1(W, U_0) = 2$, and it turns out that the module U_2 is the following:

$$U_2 = {\scriptstyle\begin{array}{cc} & 0 \\ 1 & 1 \\ & 1 \end{array}} \oplus {\scriptstyle\begin{array}{cc} & 1 \\ 1 & 0 \\ & 1 \end{array}} \oplus {\scriptstyle\begin{array}{cc} & 1 \\ 1 & 1 \\ & 0 \end{array}}.$$

The pushout diagram involving the modules U_0, U_1 (twice) and U_2 is constructed as follows: denote by μ_a, μ_b, μ_c monomorphisms $U_0 \to U_1$ which factor through the indecomposable projective modules $P(a), P(b), P(c)$, respectively. We can assume that $\mu_c = -\mu_a - \mu_b$, so that a mesh relation is satisfied. Denote the 3 summands of U_2 by M_a, M_b, M_c, with non-zero maps $\nu_a \colon U_1 \to M_a$, $\nu_b \colon U_1 \to M_b$, $\nu_c \colon U_1 \to M_c$, such that $\nu_a\mu_a = 0$, $\nu_b\mu_b = 0$, $\nu_c\mu_c = 0$. There is the following commutative square, for any $q \in k$, we are interested when $q \notin \{0, 1\}$:

$$\begin{array}{ccc} U_0 & \xrightarrow{w_0 = \mu_a + q\mu_b} & U_1 \\ {\scriptstyle v_0 = \mu_a}\big\downarrow & & \big\downarrow{\scriptstyle v_1 = \begin{bmatrix} 0 \\ \nu_b \\ \nu_c \end{bmatrix}} \\ U_1 & \xrightarrow[w_1 = \begin{bmatrix} \nu_a \\ \nu_b \\ (1-q)\nu_c \end{bmatrix}]{} & U_2 \end{array}$$

(the only calculation which has to be done concerns the third entries: $\nu_c(\mu_a + q\mu_b) = (1-q)\nu_c\mu_a$). Note that w_1 (as well as w_1') does not split.

But now we deal with a module U_2 such that $\operatorname{Ext}^1(W, U_2) = 0$. This implies that U_3 is isomorphic to $U_2 \oplus W$. Thus the next pushout construction yields an exact sequence of the form

$$0 \to U_2 \to U_2 \oplus W \to W' \to 0.$$

4. Second application: Non-degeneration.

Proposition 2. *Let* $w, w' : U \to V$ *be monomorphisms with cokernel* W, W'*, respectively. Assume* $\operatorname{End}(W)$ *is a brick,* W*,* W' *are non-isomorphic, and* $\dim \operatorname{End}(W) = \dim \operatorname{End}(W')$*. Then* Λ *is not of finite representation type.*

Proof: Let $\mathcal{F} = \mathcal{F}(W)$ be the full category of modules with a filtration with factors isomorphic to W. This is an abelian category nwith a unique simple object. It is sufficient to show that $\mathcal{F}$ has infinitely many isomorphism clases of indecomposable objects. If not, then $\mathcal{F}$ is a serial category, say with l indecomposable objects. It follows that the $\mathcal{F}$-length of any object in $\mathcal{F}$ is bounded by l times its socle length.

We consider the chain of inclusions $U_0 \subset U_1 \subset U_2 \subset \cdots$ corresponding to w (thus, with all factors isomorphic to W). Claim: one of the inclusions has to split! Note that U_i/U_0 is an object of $\mathcal{F}$-length i. Denote by $s(i)$ the $\mathcal{F}$-socle length of U_i/U_0. We see

$$1 = s(1) \le s(2) \le \cdots$$

with $i \le l \cdot s(i)$, thus $s(i) \ge i/l$. In particular, this is an unbounded sequence. Let U_i' be the submodule of U_i containing U_0 such that U_i'/U_0 is the $\mathcal{F}$-socle of U_i/U_0. The chain $U_0 \subset U_1' \subseteq U_2' \subseteq U_3' \subseteq \cdots$ is a sequence of extensions of U_0 by direct sums of copies of W, thus after a while all the inclusions split. Let n be an index such that $U_n' \subset U_{n+1}'$ is a proper inclusion which splits. Then $U_n + U_{n+1}' = U_{n+1}$ and the splitting of the inclusion $U_n' \subset U_{n+1}'$ implies the splitting of $U_n \subset U_{n+1}$ as we wanted to show.

But the splitting of w_n implies that W' is a degeneration of W. Since $\dim \operatorname{End}(W) = \dim \operatorname{End}(W')$, it follows that W and W' are isomorphic, a contradiction.

References.

[BP] Raymundo Bautista and Efren Perez: On modules and complexes without self-extensions. Communications in Algebra (to appear)

[R] Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005). 726-748.

[S] Smalø: Homological conjectures and degenerations of modules. Lectures. Mar del Plata 2006.

[Z] Zwara. G. Zwara: A degeneration-like order for modules. Arch. Math. 71 (1998), 437-444