The ladder construction of Prüfer modules.

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Let $R$ be any ring. We deal with (left) $R$-modules. Our aim is to consider pairs of maps $w, v : U \rightarrow V$ with $w$ a proper monomorphism.

Let $M$ be a module. If there exists an endomorphism $\phi$ of $M$ which is surjective, locally nilpotent, and with non-zero kernel $W$ of finite length, then $M$ will be said to be a Prüfer module (with respect to $\phi$, and with basis $W$).

1. The basic construction. A pair of exact sequences

$$0 \rightarrow U_0 \xrightarrow{w_0} U_1 \rightarrow W \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K \rightarrow U_0 \xrightarrow{v_0} U_1 \rightarrow Q \rightarrow 0$$

yields a module $U_2$ and a pair of exact sequences

$$0 \rightarrow U_1 \xrightarrow{w_1} U_2 \rightarrow W \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K \rightarrow U_1 \xrightarrow{v_1} U_2 \rightarrow Q \rightarrow 0$$

by forming the induced exact sequence of $0 \rightarrow U_0 \xrightarrow{w_0} U_1 \rightarrow W \rightarrow 0$ using the map $v_0$:

$$\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
\downarrow & & & & & & & & \\
K & = & = & K & & & & & \\
\downarrow & & & & & & & & \\
0 & \rightarrow & U_0 & \xrightarrow{w_0} & U_1 & \rightarrow & W & \rightarrow & 0 \\
\downarrow \v_0 & & \downarrow \v_1 & & & & \| & & \\
0 & \rightarrow & U_1 & \xrightarrow{w_1} & U_2 & \rightarrow & W & \rightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
0 & = & = & Q & & & & & \\
\downarrow & & \downarrow & & & & & & \\
0 & 0 & & & & & & & 
\end{array}$$

2. The ladder. Using induction, we obtain in this way modules $U_i$ and pairs of exact sequences

$$0 \rightarrow U_i \xrightarrow{w_i} U_{i+1} \rightarrow W \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K \rightarrow U_i \xrightarrow{v_i} U_{i+1} \rightarrow Q \rightarrow 0$$
for all $i \geq 0$.

We may combine the pushout diagrams constructed inductively and obtain the following ladder of commutative squares:

$$
\begin{array}{cccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
\downarrow v_0 & & \downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots
\end{array}
$$

We form the inductive limit $U_\infty = \bigcup_i U_i$ (along the maps $w_i$).

Since all the squares commute, the maps $v_i$ induce a map $U_\infty \to U_\infty$ which we denote by $v_\infty$:

$$
\begin{array}{cccccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \cdots \\
\downarrow v_0 & & \downarrow v_1 & & \downarrow v_2 & & \downarrow v_3 & & \downarrow v_\infty \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & U_4 & \xrightarrow{w_4} & \cdots \\
\end{array}
$$

We also may consider the factor modules $U_\infty/U_0$ and $U_\infty/U_1$. The map $v_\infty : U_\infty \to U_\infty$ maps $U_0$ into $U_1$, thus it induces a map

$$
\overline{v} : U_\infty/U_0 \to U_\infty/U_1.
$$

Claim. The map $\overline{v}$ is an isomorphism. Namely, the commutative diagrams

$$
\begin{array}{ccccccc}
0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & W & \longrightarrow & 0 \\
& & \downarrow v_{i-1} & & \downarrow v_i & & \parallel & & \\
0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & W & \longrightarrow & 0
\end{array}
$$

can be rewritten as

$$
\begin{array}{ccccccc}
0 & \longrightarrow & U_{i-1} & \xrightarrow{w_{i-1}} & U_i & \longrightarrow & U_i/U_{i-1} & \longrightarrow & 0 \\
& & \downarrow v_{i-1} & & \downarrow v_i & & \downarrow \overline{v}_i & & \\
0 & \longrightarrow & U_i & \xrightarrow{w_i} & U_{i+1} & \longrightarrow & U_{i+1}/U_i & \longrightarrow & 0
\end{array}
$$

with an isomorphism $\overline{v}_i : U_i/U_{i-1} \to U_{i+1}/U_i$. The map $\overline{v}$ is a map from a filtered module with factors $U_i/U_{i-1}$ (where $i \geq 1$) to a filtered module with factors $U_{i+1}/U_i$ (again with $i \geq 1$), and the maps $\overline{v}_i$ are just those induced on the factors.

It follows: The composition of maps

$$
\begin{array}{cccc}
U_\infty/U_0 & \xrightarrow{p} & U_\infty/U_1 & \xrightarrow{\overline{v}_1} & U_\infty/U_0
\end{array}
$$

with $p$ the projection map is an epimorphism $\phi$ with kernel $U_1/U_0$. It is easy to see that $\phi$ is locally nilpotent.
Summery. The maps \( v_i \) yield a map

\[ v_\infty : U_\infty \to U_\infty \]

with kernel \( K \) and cokernel \( Q \). This map \( v_\infty \) induces an isomorphism \( \overline{\varphi} : U_\infty/U_0 \to U_\infty/U_1 \). Composing the inverse of this isomorphism with the canonical projection \( p \), we obtain an endomorphism \( \phi \)

\[ U_\infty/U_0 \overset{p}{\to} U_\infty/U_1 \overset{\overline{\varphi}^{-1}}{\to} U_\infty/U_0 \]

and \( U_\infty/U_0 \) is a Prüfer module with respect to \( \phi \), with basis \( W \).

(Using a terminology introduced for string algebras, we also can say: \( U_\infty \) is expanding, \( U_\infty/U_0 \) is contracting.)

If necessary, we will use the following notation: \( U_i(w; v) = U_i \), for all \( i \in \mathbb{N} \) and also for \( i = \infty \), and \( P(w; v) = U_\infty/U_0 \) for the Prüfer module (here, \( w = w_0 \), \( v = v_0 \)). Since \( P(w; v) \) is a Prüfer module with basis the cokernel \( W \) of \( w \), we will sometimes write \( W[n] = U_n/U_0 \).

Examples.

(1) The classical example: Let \( R = \mathbb{Z} \), and also \( U_0 = U_1 = \mathbb{Z} \). Maps \( \mathbb{Z} \to \mathbb{Z} \) are given by the multiplication with some integer \( n \), thus we denote it just by \( n \). Let \( w_0 = 2 \) and \( v_0 = n \). If \( n \) is odd, then \( P(2; n) \) is the ordinary Prüfer group for the prime \( 2 \), and \( U_\infty(2; n) = \mathbb{Z}[\frac{1}{2}] \) (the subring of \( \mathbb{Q} \) generated by \( \frac{1}{2} \)). If \( n \) is even, then \( P(2; n) \) is an elementary abelian 2-group.

(2) Let \( R = K(2) \) be the Kronecker algebra over some field \( k \). Let \( U_0 \) be simple projective, \( U_1 \) indecomposable projective of length 3 and \( w_0 : U_0 \to U_1 \) a non-zero map (one of the indecomposable modules of length 2). The module \( P(w_0; v_0) \) is the Prüfer module for \( W \) if and only if \( v_0 \notin kw_0 \), otherwise it is a direct sum of copies of \( W \).

(3) Trivial cases: First, let \( w \) be a split monomorphism. Then the Prüfer module with respect to any map \( \alpha : U_0 \to U_1 \) is just the countable sum of copies of \( W \). Second, let \( w : U_0 \to U_1 \) be an arbitrary monomorphism, let \( \beta : U_1 \to U_1 \) be an endomorphism. Then \( P(w; \beta w) \) is the countable sum of copies of \( W \).

(4) Assume there exists a split monomorphism \( \alpha : U_0 \to U_1 \), say \( U_1 = U_0 \oplus X \) and \( \alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix} : U_0 \to U_1 \). Then

\[ 0 \to U_0 \overset{w}{\to} U_0 \oplus X \to W \to 0 \]

is a Riedtmann-Zwara sequence, thus \( W \) is a degeneration of \( X \). According to Zwara, there is \( n_0 \) such that \( W[n+1] \simeq W[n] \oplus X \) for all \( n \geq n_0 \).
The chessboard. Assume now that both maps \( w_0, v_0 : U_0 \to U_1 \) are monomorphisms. Then we get the following arrangement of commutative squares:

\[
\begin{array}{cccccc}
U_0 & \xrightarrow{w_0} & U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \ldots \\
v_0 & \downarrow & v_1 & \downarrow & v_2 & \downarrow & v_3 & \downarrow & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & U_3 & \xrightarrow{w_3} & \ldots \\
v_0 & \downarrow & v_1 & \downarrow & v_2 & \downarrow & \\
U_1 & \xrightarrow{w_1} & U_2 & \xrightarrow{w_2} & \ldots \\
v_0 & \downarrow & v_1 & \downarrow & \\
U_1 & \xrightarrow{w_1} & \ldots \\
v_0 & \downarrow & \\
\ldots & \\
\end{array}
\]

We see both horizontally as well as vertically ladders: the horizontal ladders yield \( U_\infty(w_0; v_0) \) (and its endomorphism \( v_\infty \)); the vertical ladders yield \( U_\infty(v_0; w_0) \) (and its endomorphism \( w_\infty \)).

Let \( \Lambda \) be an artin algebra.

3. First application: Degenerations.

Proposition 1. Let \( U, V \) be modules, and let \( W \) and \( W' \) be cokernels of monomorphisms \( U \to V \). If \( \text{Ext}^1(W, W) = 0 \), then there exists a module \( X \) and an exact sequence

\[ 0 \to X \to X \oplus W \to W' \to 0. \]

Note that the existence of an exact sequence of the form \( 0 \to X \to X \oplus W \to W' \to 0 \) may be interpreted as asserting that \( W' \) is a degeneration of \( W \), according to Riedtmann and Zwara [Z].

Corollary. Let \( U, V \) be modules, and let \( W \) and \( W' \) be cokernels of monomorphisms \( U \to V \). Assume that both \( \text{Ext}^1(W, W) = 0 \) and \( \text{Ext}^1(W', W') = 0 \). Then the modules \( W \) and \( W' \) are isomorphic.

Both assertions are well-known in case \( k \) is an algebraically closed field: in this case, the conclusion of proposition 1 just asserts that \( W' \) is a degeneration of \( W \) in the sense of algebraic geometry. The main point here is to deal with the general case when \( \Lambda \) is an arbitrary artin algebra. Our interest in this question was raised by a series of lectures by Sverre Smalø at the Mar del Plata conference, March 2006. The corollary stated above (under the additional assumptions that \( V \) is projective and that \( w(U), w'(U) \) are contained...
in the radical of $V$) is due to Bautista and Perrez [BP] and this result was presented by Smalø with a new proof [S] at Mar del Plata.

**Lemma.** Let $W$ be a module with $\operatorname{Ext}^1(W,W) = 0$. Let $U_0 \subset U_1 \subset U_2 \subset \cdots$ be a sequence of inclusions of modules with $U_i/U_{i-1} = W$ for all $i \geq 1$. Then there is a natural number $n_0$ such that $U_n \subset U_{n+1}$ is a split monomorphism for all $n \geq n_0$.

Lemma is well-known, it is based on the fact that $\operatorname{Ext}^1(W,U_0)$ when considered as a $k$-module is of finite length. A proof will be given below. Let us use it in order to finish the proof of proposition 1.

We apply Lemma to the chain of inclusions

$$U_0 \xrightarrow{w_0} U_1 \xrightarrow{w_1} U_2 \xrightarrow{w_2} \cdots$$

and see that there is $n$ such that $w_n: U_n \to U_{n+1}$ splits. This shows that $U_{n+1}$ is isomorphic to $U_n \oplus W$. But we also have the exact sequence

$$0 \to U_n \xrightarrow{v_n} U_{n+1} \to W' \to 0.$$

Replacing $U_{n+1}$ by $U_n \oplus W$, we see that we get an exact sequence of the form

$$0 \to U_n \xrightarrow{u_n} U_n \oplus W \to W' \to 0$$

(a Riedtmann-Zwara sequence), as asserted.

**Proof of Corollary.** It is well-known that the existence of exact sequences

$$0 \to X \to X \oplus W \to W' \to 0 \quad \text{and} \quad 0 \to Y \to Y \oplus W' \to W \to 0$$

implies that the modules $W$- and $W'$- are isomorphic. But in our case we just have to change one line in the proof of proposition 1 in order to get the required isomorphism. Thus, assume that both $\operatorname{Ext}^1(W,W) = 0$ and $\operatorname{Ext}^1(W',W') = 0$. Choose $n$ such that both the inclusion maps

$$w_n: U_n \to U_{n+1} \quad \text{and} \quad v_n: U_n \to U_{n+1}$$

split. Then $U_{n+1}$ is isomorphic both to $U_n \oplus W$ and to $U_n \oplus W'$, thus it follows from the Krull-Remak-Schmidt theorem that $W$ and $W'$ are isomorphic.

**Remark.** Assume that $w, w': U, V$ are monomorphisms with cokernels $W$ and $W'$, respectively, and that $\operatorname{Ext}^1(W,W) = 0$ and $\operatorname{Ext}^1(W',W') = 0$. Then $w$ splits if and only if $w'$ splits.

Proof: According to the corollary, we can assume $W = W'$. Assume that $w$ splits, thus $V$ is isomorphic to $U \oplus W$. Look at the exact sequence $0 \to U \xrightarrow{w'} V \to W \to 0$. If it does not split, then $\dim \operatorname{End}(V) < \dim \operatorname{End}(U \oplus W)$, but $V$ is isomorphic to $U \oplus W$.

Proof of Lemma. An assertion equivalent to Lemma was used for example by Roiter in his proof of the first Brauer-Thrall conjecture, a corresponding proof can be found in [R]. We include here a slightly different proof.

5
Applying the functor $\text{Hom}(W, -)$ to the short exact sequence $0 \rightarrow U_{i-1} \overset{w_{i-1}}{\rightarrow} U_i \rightarrow W \rightarrow 0$, we obtain the exact sequence

$$\text{Ext}^1(W, U_{i-1}) \rightarrow \text{Ext}^1(W, U_i) \rightarrow \text{Ext}^1(W, W).$$

Since the latter term is zero, we see that we have a sequence of surjective maps

$$\text{Ext}^1(W, U_0) \rightarrow \text{Ext}^1(W, U_1) \rightarrow \cdots \rightarrow \text{Ext}^1(W, U_i) \rightarrow \cdots,$$

being induced by the inclusion maps $U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i \rightarrow \cdots$. The maps between the Ext-groups are $k$-linear. Since $\text{Ext}^1(W, U_0)$ is a $k$-module of finite length, the sequence of surjective maps must stabilize: there is some $n_0$ such that the inclusion $U_n \rightarrow U_{n+1}$ induces an isomorphism

$$\text{Ext}^1(W, U_n) \rightarrow \text{Ext}^1(W, U_{n+1})$$

for all $n \geq n_0$. Now we consider also some Hom-terms: the exactness of

$$\text{Hom}(W, U_{n+1}) \rightarrow \text{Hom}(W, W) \rightarrow \text{Ext}^1(W, U_n) \rightarrow \text{Ext}^1(W, U_{n+1})$$

shows that the connecting homomorphism is zero, and thus that the map $\text{Hom}(W, U_{n+1}) \rightarrow \text{Hom}(W, W)$ (induced by the projection map $p: U_{n+1} \rightarrow W$) is surjective. But this means that there is a map $h \in \text{Hom}(W, U_{n+1})$ with $ph = 1_W$, thus $p: U_{n+1} \rightarrow W$ is a split epimorphism and therefore the inclusion map $U_n \rightarrow U_{n+1}$ is a split monomorphism.

**Remark.** In general, there is no actual bound on the number $n_0$. However, in case of dealing with the chain of inclusions

$$U_0 \overset{w_0}{\rightarrow} U_1 \overset{w_1}{\rightarrow} U_2 \overset{w_n}{\rightarrow} \cdots$$

such a bound exists, namely the length of $\text{Ext}^1(W, U_0)$ as a $k$-module, or, even better, the length of $\text{Ext}^1(W, U_0)$ as an $E$-module, where $E = \text{End}(W)$.

**Proof:** Look at the surjective maps

$$\text{Ext}^1(W, U_0) \rightarrow \text{Ext}^1(W, U_1) \rightarrow \cdots \rightarrow \text{Ext}^1(W, U_i) \rightarrow \cdots,$$

being induced by the maps $U_n \overset{w_n}{\rightarrow} U_{n+1}$ (and these maps are not only $k$-linear, but even $E$-linear). Assume that $\text{Ext}^1(W, U_n) \rightarrow \text{Ext}^1(W, U_{n+1})$ is bijective, for some $n$. As we have seen above, this implies that the sequence

$$(*) \quad 0 \rightarrow U_n \overset{w_n}{\rightarrow} U_{n+1} \rightarrow W \rightarrow 0$$

splits. Now the map $w_{n+1}$ is obtained from $(*)$ as the induced exact sequence using the map $w'_n$. With $(*)$ also any induced exact sequence will split. Thus $w_{n+1}$ is a split monomorphism (and $\text{Ext}^1(W, U_{n+1}) \rightarrow \text{Ext}^1(W, U_{n+2})$ will be bijective, again). Thus, as soon as we get a bijection $\text{Ext}^1(W, U_n) \rightarrow \text{Ext}^1(W, U_{n+1})$ for some $n$, then also all the following maps $\text{Ext}^1(W, U_m) \rightarrow \text{Ext}^1(W, U_{m+1})$ with $m > n$ are bijective.
**Example.** Consider the $D_4$-quiver with subspace orientation:

```
 a <---- b
    |
 c <---- d
```

and let $\Lambda$ be its path algebra over some field $k$. We denote the indecomposable $\Lambda$-modules by the corresponding dimension vectors. Let

$$U_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad W' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Note that a map $w_0: U_0 \to U_1$ with cokernel $W$ exists only in case the base-field $k$ has at least 3 elements; of course, there is always a map $w'_0: U_0 \to U_1$ with cokernel $W'$.

We have $\dim \text{Ext}^1(W, U_0) = 2$, and it turns out that the module $U_2$ is the following:

$$U_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$ 

The pushout diagram involving the modules $U_0$, $U_1$ (twice) and $U_2$ is constructed as follows: denote by $\mu_a, \mu_b, \mu_c$ monomorphisms $U_0 \to U_1$ which factor through the indecomposable projective modules $P(a), P(b), P(c)$, respectively. We can assume that $\mu_c = -\mu_a - \mu_b$, so that a mesh relation is satisfied. Denote the 3 summands of $U_2$ by $M_a, M_b, M_c$, with non-zero maps $\nu_a: U_1 \to M_a$, $\nu_b: U_1 \to M_b$, $\nu_c: U_1 \to M_c$, such that $\nu_a \mu_a = 0$, $\nu_b \mu_b = 0$, $\nu_c \mu_c = 0$. There is the following commutative square, for any $q \in k$, we are interested when $q \notin \{0, 1\}$:

$$
\begin{array}{ccc}
U_0 & \xrightarrow{w_0 = \mu_a + q \mu_b} & U_1 \\
\text{v}_0 = \mu_a & \downarrow & \text{v}_1 = \begin{pmatrix} 0 \\ \nu_b \\ \nu_c \end{pmatrix} \\
U_1 & \xrightarrow{w_1 = \begin{pmatrix} \nu_a \\ \nu_b \\ (1-q) \nu_c \end{pmatrix}} & U_2
\end{array}
$$

(the only calculation which has to be done concerns the third entries: $\nu_c(\mu_a + q \mu_b) = (1-q)\nu_c \mu_a$). Note that $w_1$ (as well as $w'_1$) does not split.

But now we deal with a module $U_2$ such that $\text{Ext}^1(W, U_2) = 0$. This implies that $U_3$ is isomorphic to $U_2 \oplus W$. Thus the next pushout construction yields an exact sequence of the form

$$0 \to U_2 \to U_2 \oplus W \to W' \to 0.$$

**Proposition 2.** Let $w, w' : U \to V$ be monomorphisms with cokernel $W, W'$, respectively. Assume $\text{End}(W)$ is a brick, $W, W'$ are non-isomorphic, and $\dim \text{End}(W) = \dim \text{End}(W')$. Then $\Lambda$ is not of finite representation type.

Proof: Let $\mathcal{F} = \mathcal{F}(W)$ be the full category of modules with a filtration with factors isomorphic to $W$. This is an abelian category with a unique simple object. It is sufficient to show that $\mathcal{F}$ has infinitely many isomorphism classes of indecomposable objects. If not, then $\mathcal{F}$ is a serial category, say with $l$ indecomposable objects. It follows that the $\mathcal{F}$-length of any object in $\mathcal{F}$ is bounded by $l$ times its socle length.

We consider the chain of inclusions $U_0 \subset U_1 \subset U_2 \subset \cdots$ corresponding to $w$ (thus, with all factors isomorphic to $W$). Claim: one of the inclusions has to split! Note that $U_i/U_0$ is an object of $\mathcal{F}$-length $i$. Denote by $s(i)$ the $\mathcal{F}$-socle length of $U_i/U_0$. We see

$$1 = s(1) \leq s(2) \leq \cdots$$

with $i \leq l \cdot s(i)$, thus $s(i) \geq i/l$. In particular, this is an unbounded sequence. Let $U'_i$ be the submodule of $U_i$ containing $U_0$ such that $U'_i/U_0$ is the $\mathcal{F}$-socle of $U_i/U_0$. The chain $U_0 \subset U'_1 \subset U'_2 \subset U'_3 \subset \cdots$ is a sequence of extensions of $U_0$ by direct sums of copies of $W$, thus after a while all the inclusions split. Let $n$ be an index such that $U'_n \subset U'_{n+1}$ is a proper inclusion which splits. Then $U_n + U'_{n+1} = U_{n+1}$ and the splitting of the inclusion $U'_n \subset U'_{n+1}$ implies the splitting of $U_n \subset U_{n+1}$ as we wanted to show.

But the splitting of $w_n$ implies that $W'$ is a degeneration of $W$. Since $\dim \text{End}(W) = \dim \text{End}(W')$, it follows that $W$ and $W'$ are isomorphic, a contradiction.

References.


