SELF-INJECTIVE ALGEBRAS: THE NAKAYAMA PERMUTATION

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Let $\Lambda$ be a finite dimensional algebra, defined over a field $k$. The category of finite dimensional left $\Lambda$-modules and the set of isoclasses of simple $\Lambda$ modules will be denoted by $\text{mod } \Lambda$ and $\mathcal{S}(\Lambda)$, respectively. We will occasionally identify $\mathcal{S}(\Lambda)$ with a complete set of its representatives. Given a simple $\Lambda$-module $S$, we consider its projective cover $P(S)$ and its injective envelope $E(S)$. Recall that $\text{Top}(P(S)) \cong S$ and $\text{Soc}(E(S)) \cong S$. Moreover, for every projective (injective) indecomposable $\Lambda$-module $Q$ there exists exactly one $S \in \mathcal{S}(\Lambda)$ with $Q \cong P(S)$ ($Q \cong E(S)$) (cf. [1, (I.4),(II.4)]).

What can be said about the structure of $\text{Soc}(P(S))$ (or $\text{Top}(E(S))$)? Let us consider simple examples.

**Examples.** (1) Let $\Lambda = k[1 \rightarrow 2]$. Setting $P_i := P(S_i)$, we obtain $\text{Soc}(P_1) = S_2 = \text{Soc}(P_2)$.
(2) If $\Lambda = k[1 \leftarrow 2 \rightarrow 3]$, then
$$\text{Soc}(P_1) = S_1 \ ; \ \text{Soc}(P_2) = S_1 \oplus S_3 \ ; \ \text{Soc}(P_3) = P_3.$$  

In general, one can thus not hope for a fixed pattern. We are going to introduce a class of algebras where such a pattern exists and show that this class is in fact determined by the presence of a correspondence between tops and socles of the principal indecomposable modules $P(S)$. These so-called *quasi-Frobenius algebras* were introduced and studied by Nakayama [3, 4], whose work was inspired by results of Brauer and Nesbitt [2]. Nowadays, the following notion is commonly used.

**Definition.** The algebra $\Lambda$ is *self-injective* if $\Lambda \in \text{mod } \Lambda$ is injective.

The principal result of this lecture was proved by Nakayama in the context of basic algebras.

**Theorem.** The following statements are equivalent:

1. The algebra $\Lambda$ is self-injective.
2. The rule $[S] \mapsto [\text{Soc}(P(S))]$ defines a permutation $\nu : \mathcal{S}(\Lambda) \to \mathcal{S}(\Lambda)$.

The permutation $\nu$ is referred to as the *Nakayama permutation* of the self-injective algebra $\Lambda$.

Given $M \in \text{mod } \Lambda$, its dual $M^* := \text{Hom}_k(M, k)$ has the structure of a right $\Lambda$-module. Thus, $M \mapsto M^*$ is a duality between the categories $\text{mod } \Lambda$ and $\text{mod } \Lambda^{\text{op}}$, where $\Lambda^{\text{op}}$ denotes the opposite algebra of $\Lambda$. In particular, $^*$ takes projectives to injectives and vice versa.

**Lemma 1.** Let $\Lambda$ be self-injective.

1. A $\Lambda$-module $M$ is projective if and only if it is injective.
2. $\text{Soc}(P(S))$ is simple for every $S \in \mathcal{S}(\Lambda)$.
3. $P(S) \cong E(\text{Soc}(P(S))$ for every $S \in \mathcal{S}(\Lambda)$.
4. Let $S, T$ be simple $\Lambda$-modules. If $\text{Soc}(P(S)) \cong \text{Soc}(P(T))$, then $S \cong T$.

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Proof. If $M$ is projective, then $M$ is a direct summand of a free module $\Lambda^n$. Thus, $M$ is, as a direct summand of an injective module, injective.

It follows that $\{[P(S)]; [S] \in \mathcal{S}(\Lambda)\}$ is a set of isoclasses of injective indecomposable $\Lambda$ modules of cardinality $|\mathcal{S}(\Lambda)|$. It thus coincides with the set of isoclasses of indecomposable injectives, and there exists a permutation $\nu: \mathcal{S}(\Lambda) \to \mathcal{S}(\Lambda)$ such that

$$P(S) \cong E(\nu(S)) \quad \forall \ [S] \in \mathcal{S}(\Lambda).$$

In particular, we have:

- Every indecomposable injective module is projective, so that (1) follows.
- $[\text{Soc}(P(S))] = [\text{Soc}(E(\nu(S)))] = [\nu(S)] \quad \forall \ [S] \in \mathcal{S}(\Lambda)$. Thus, we obtain (2)-(4).

\[\Box\]

Remarks. (a) Owing to (1), the class of self-injective algebras is stable under Morita equivalence.

(b) Since the right module $\Lambda \in \text{mod } \Lambda^{\text{op}}$ is projective, its dual is injective, hence projective, so that $\Lambda_\Lambda$ is injective. In other words, the algebra $\Lambda^{\text{op}}$ is self-injective.

By general theory, the principal indecomposable $\Lambda$ modules are of the form

$$P = \Lambda e,$$

where $e \in \Lambda$ is a primitive idempotent. If $M \in \text{mod } \Lambda$, then

$$\text{Hom}_\Lambda(P, M) \to eM ; \ f \mapsto f(e)$$

is an isomorphism of vector spaces, which is right $\Lambda$-linear in case $M$ is a $(\Lambda, \Lambda)$-bimodule.

The Nakayama permutation is a combinatorial tool that does not provide any information concerning the endomorphism rings of $S$ and $\nu(S)$ (these are actually isomorphic). For algebras over algebraically closed fields, this is of course not a problem. However, a better understanding of $\nu$ necessitates a module theoretic (functorial) description.

Definition. Let $\Lambda$ be a $k$-algebra. The functor

$$\mathcal{N}: \{\text{mod } \Lambda \to \text{mod } \Lambda; \ M \mapsto \text{Hom}_\Lambda(M, \Lambda)^*\}$$

is called the Nakayama functor of $\Lambda$.

The above observations yield

$$\mathcal{N}(\Lambda e) \cong (e\Lambda)^*,$$

so that $\mathcal{N}$ sends indecomposable projectives to indecomposable injectives. (However, it does in general not send indecomposables to indecomposables.)

Lemma 2. Let $\Lambda$ be a $k$-algebra that affords a Nakayama permutation $\nu: \mathcal{S}(\Lambda) \to \mathcal{S}(\Lambda)$. Then the following statements hold:

(1) $\mathcal{N}(P(S)) \cong E(S) \quad \forall \ S \in \mathcal{S}(\Lambda)$.
(2) $\mathcal{N}(S) \cong \nu^{-1}(S) \quad \forall \ S \in \mathcal{S}(\Lambda)$. 

Proof. (1) Let \( S \) be a simple \( \Lambda \)-module. Pick a primitive idempotent \( e_S \in \Lambda \) with \( P(S) = \Lambda e_S \). Since
\[
\text{Hom}_\Lambda(e_S \Lambda, S^*) \cong (S^* e_S) \cong (e_S S)^* \cong \text{Hom}_\Lambda(P(S), S)^* \neq (0),
\]
it follows that the principal indecomposable right \( \Lambda \)-module \( e_S \Lambda \) has top \( S^* \). As the duality \(^*\) maps tops to socles, we obtain
\[
S \cong \text{Top}(e_S \Lambda)^* \cong \text{Soc}((e_S \Lambda)^*) \cong \text{Soc}(N(\Lambda e_S)) \cong \text{Soc}(N(P(S))),
\]
so that the indecomposable injective \( \Lambda \)-module \( N(P(S)) \) is isomorphic to \( E(S) \).

(2) Let \( S, T \) be simple \( \Lambda \)-modules, \( e_S, e_T \) the corresponding primitive idempotents. Given any \( \Lambda \)-module \( M \), we have
\[
\dim_k \text{Hom}_\Lambda(P(T), M) = [M : T] \dim_k \text{End}_\Lambda(T),
\]
where \([M : T]\) denotes the multiplicity of \( T \) in \( M \). From the \( k \)-vector space isomorphisms
\[
\text{Hom}_\Lambda(P(T), N(S)) \cong e_T N(S) \cong e_T \text{Hom}_\Lambda(S, \Lambda)^* \cong (\text{Hom}_\Lambda(S, \Lambda)e_T)^*
\cong \text{Hom}_\Lambda(S, \Lambda e_T)^* \cong \text{Hom}_\Lambda(S, \nu(T))^*
\cong \begin{cases} 
\text{End}_\Lambda(S) & \text{if } T \cong \nu^{-1}(S) \\
(0) & \text{otherwise}
\end{cases},
\]
we see that \( \nu^{-1}(S) \) is the only composition factor of \( N(S) \). By the same token, we have
\[
[M(S) : \nu^{-1}(S)] \dim_k \text{End}_\Lambda(\nu^{-1}(S)) = \dim_k \text{Hom}_\Lambda(P(\nu^{-1}(S)), N(S)) = \dim_k \text{End}_\Lambda(S).
\]
By applying this formula successively to the modules \( \nu^{-i}(S) \), we obtain, observing \( \nu^{-n}(S) \cong S \) for some \( n \in \mathbb{N} \), a natural number \( m \in \mathbb{N} \) such that
\[
\dim_k \text{End}_\Lambda(S) = m[M(S) : \nu^{-1}(S)] \dim_k \text{End}_\Lambda(S).
\]
Thus, \([N(S) : \nu^{-1}(S)] = 1 \), and \( N(S) \cong \nu^{-1}(S) \).

Proof of the Theorem. In view of Lemma 1, it suffices to verify \((2) \Rightarrow (1)\). Let \( \ell(M) \) denote the length of the \( \Lambda \)-module \( M \). The Nakayama functor is right exact and Lemma 2 ensures that it takes simples to simples. Induction on \( \ell(M) \) then implies
\[
\ell(N(M)) \leq \ell(M) \quad \forall M \in \text{mod } \Lambda.
\]
Lemma 2 now yields
\[
\ell(E(S)) \leq \ell(P(S)) \quad \forall S \in \text{S}(\Lambda).
\]
Since \( \text{Soc}(P(S)) = \nu(S) \), we have an embedding \( \iota_S : P(S) \hookrightarrow E(\nu(S)) \). Iteration gives rise to a chain
\[
\ell(E(S)) \leq \ell(P(S)) \leq \ell(E(\nu(S))) \leq \ell(P(\nu(S)) \leq \ell(E(\nu^2(S)) \leq \cdots \leq \ell(E(S)),
\]
so that \( \ell(P(S)) = \ell(E(\nu(S))) \). As a result, \( \iota_S \) is bijective, showing that \( P(S) \) is injective. This implies the self-injectivity of \( \Lambda \). \( \square \)

References