SELF-INJECTIVE ALGEBRAS: COMPARISON WITH FRObenius ALGEBRAS

ROLF FARNSTEINER

Let \( \Lambda \) be a finite dimensional algebra, defined over a field \( k \). The category of finite dimensional left \( \Lambda \)-modules and the set of isoclasses of simple \( \Lambda \) modules will be denoted by \( \text{mod} \ \Lambda \) and \( \mathcal{S}(\Lambda) \), respectively. We will occasionally identify \( \mathcal{S}(\Lambda) \) with a complete set of its representatives. Given a simple \( \Lambda \)-module \( S \), we consider its projective cover \( P(S) \) and its injective envelope \( E(S) \). Recall that \( \text{Top}(P(S)) \cong S \) and \( \text{Soc}(E(S)) \cong S \).

In our lecture [2], we observed that \( \Lambda \) is self-injective if and only if \( S \mapsto \text{Soc}(P(S)) \) defines a permutation \( \nu : \mathcal{S}(\Lambda) \rightarrow \mathcal{S}(\Lambda) \), the so-called Nakayama permutation [2, Theorem]. In early articles, these algebras were referred to as quasi-Frobenius algebras (cf. [3]). The purpose of this lecture is to understand when a self-injective algebra is a Frobenius algebra. This class of algebras was introduced by Brauer and Nesbitt in [1], who provided the following characterization:

Lemma 1. Let \( \pi : \Lambda \rightarrow k \) be a linear map. Then the following statements are equivalent:

1. \( \ker \pi \) does not contain a non-zero left ideal.
2. The \( \Lambda \)-linear map \( \Phi_{\pi} : \Lambda \rightarrow \Lambda^* ; \ x \mapsto x.\pi \) is an isomorphism.
3. \( \ker \pi \) does not contain a non-zero right ideal.

Proof. (1) \( \Rightarrow \) (2). Consider the left ideal \( J := \ker \Phi_{\pi} \). Since
\[
0 = \Phi_{\pi}(x)(1) = \pi(x) \quad \forall \ x \in J,
\]
we obtain the injectivity and hence the bijectivity of \( \Phi_{\pi} \).

(2) \( \Rightarrow \) (3). Let \( J \subset \ker \pi \) be a right ideal. Given a linear form \( f \in \Lambda^* \), condition (2) provides an element \( x \in \Lambda \) such that \( f = x.\pi \). For \( y \in J \) we thus obtain
\[
f(y) = (x.\pi)(y) = \pi(yx) \in \pi(J) = (0).
\]
Consequently, \( J = (0) \).

(3) \( \Rightarrow \) (1). Since (1) \( \Rightarrow \) (3) holds for every algebra, application to the opposite algebra \( \Lambda^{\text{op}} \) yields the desired conclusion. \( \square \)

Definition. A \( k \)-algebra \( \Lambda \) is a Frobenius algebra if there exists a linear form \( \pi \in \Lambda^* \) such that \( \ker \pi \) contains no non-zero left ideals.

Given a linear form \( \pi : \Lambda \rightarrow k \), we consider the bilinear form
\[
( , )_{\pi} : \Lambda \times \Lambda \rightarrow k ; \ (a, b) := \pi(ab) \quad \forall \ a, b \in \Lambda.
\]
This form is associative, that is,
\[
(ax, b)_{\pi} = (a, xb)_{\pi} \quad \forall \ a, b, x \in \Lambda.
\]

Date: October 27, 2005.
Conversely, any associative form $( , ) : \Lambda \times \Lambda \rightarrow k$ is obtained in this fashion: $( , ) = ( , )_\pi$, where $\pi(a) = (a, 1)$.

**Corollary 2.** The algebra $\Lambda$ is a Frobenius algebra if and only if there exists a non-degenerate, associative form on $\Lambda$. □

If $( , ) : \Lambda \times \Lambda \rightarrow k$ is such a form, then there exists an automorphism $\mu : \Lambda \rightarrow \Lambda$ such that

$$(y, x) = (\mu(x), y) \quad \forall \, x, y \in \Lambda.$$ 

Another such automorphism $\mu'$, induced by a form $\{ , \}$, is related to $\mu$ via an invertible element $u \in \Lambda$, i.e.

$$\mu'(x) = u\mu(x)u^{-1} \quad \forall \, x \in \Lambda.$$ 

These automorphisms are referred to as Nakayama automorphisms of the Frobenius algebra $\Lambda$.

Given an automorphism $\alpha : \Lambda \rightarrow \Lambda$ and a $\Lambda$-module $M$, we denote by $M^{(\alpha)}$ the $\Lambda$-module with underlying $k$-space $M$ and action

$$a.m := \alpha^{-1}(a)m \quad \forall \, a \in \Lambda, m \in M.$$ 

Since twisting $M$ by an inner automorphism reproduces $M$, the Nakayama automorphisms induce naturally equivalent automorphisms on mod $\Lambda$.

Let $( , ) : \Lambda \times \Lambda \rightarrow k$ be a non-degenerate associative form with Nakayama automorphism $\mu$; then $\gamma := \mu \otimes \text{id}_\Lambda$ is an automorphism of the enveloping algebra $\Lambda^e := \Lambda \otimes_k \Lambda^{\text{op}}$. The map

$$\Psi : \Lambda^\gamma^{-1} \rightarrow \Lambda^* ; \quad \Psi(x)(y) = (x, y)$$

is an isomorphism of $\Lambda^e$-modules. Let us look at left linearity. For $r, x, y \in \Lambda$ we have

$$\Psi(r.x)(y) = \Psi(\mu(r)x)(y) = (\mu(r)x, y) = (\mu(r), xy) = (xy, r) = (x, yr) = \Psi(x)(yr) = (r, \Psi(x))(y).$$

As a result, we have

$$\Lambda^* \otimes_\Lambda M \cong M^{(\mu^{-1})} \quad \forall \, M \in \text{mod } \Lambda.$$ 

Watt’s theorem tells us that the functor $M \mapsto \Lambda^* \otimes_\Lambda M$ is naturally isomorphic to the Nakayama functor (see [2] for the definition). Here is a low brow argument involving adjoint isomorphisms: We have the following isomorphisms of $\Lambda$-modules:

$$\Lambda^* \otimes_\Lambda M \cong (\Lambda^* \otimes_\Lambda M)^{**} \cong \text{Hom}_k(\Lambda^* \otimes_\Lambda M, k)^* \cong \text{Hom}_\Lambda(M, \text{Hom}_k(\Lambda^*, k))^* \cong \text{Hom}_\Lambda(M, \Lambda)^* = \mathcal{N}(M).$$

As an upshot, we obtain natural isomorphisms

$$\mathcal{N}(M) \cong M^{(\mu^{-1})}.$$ 

By combining this with [2, Lemma 2] we conclude that the Nakayama permutation is given by

$$\nu(S) \cong S^{(\mu)} \quad \forall \, S \in \mathcal{S}(\Lambda).$$

We have thus verified one implication of our main result:

**Theorem 3.** Let $\Lambda$ be a self-injective algebra. Then the following statements are equivalent:

1. $\Lambda$ is a Frobenius algebra.
2. $\dim_k \text{Soc}(P(S)) = \dim_k S \quad \forall \, S \in \mathcal{S}(\Lambda).$
Proof. \((2) \Rightarrow (1)\). Since \(\Lambda\) is self-injective, we have a Nakayama permutation \(\nu : S(\Lambda) \rightarrow S(\Lambda)\).

Since \(N(S) \cong \nu^{-1}(S)\), \(N\) induces an injection \(\text{End}_\Lambda(S) \hookrightarrow \text{End}_\Lambda(\nu^{-1}(S))\), so that iteration implies
\[
\text{End}_\Lambda(S) \cong \text{End}_\Lambda(\nu(S)) \quad \forall \ S \in S(\Lambda).
\]

Writing \(\Lambda = \bigoplus_{S \in S(\Lambda)} n_S P(S)\), application of \(\text{Hom}_\Lambda(-, S)\) yields
\[
(*) \quad n_S = \frac{\dim_k S}{\dim_k \text{End}_\Lambda(S)} = \frac{\dim_k \nu(S)}{\dim_k \text{End}_\Lambda(\nu(S))} = n_\nu(S).
\]

In view of \(P(S) \cong E(\nu(S))\) and [2, Lemma 2] we thus have
\[
(\Lambda_\Lambda)^* \cong N(\Lambda) \cong \bigoplus_{S \in S(\Lambda)} n_S E(S) \cong \bigoplus_{S \in S(\Lambda)} n_S P(S) \cong \Lambda.
\]

If \(\Phi : \Lambda \rightarrow \Lambda^*\) is the corresponding isomorphism of \(\Lambda\)-modules, then \(\pi := \Phi(1)\) renders \(\Lambda\) a Frobenius algebra.

Corollary 4. Every self-injective, basic algebra \(\Lambda\) is a Frobenius algebra.

Proof. Returning to the proof of Theorem 3 we recall that our present assumption means \(n_S = 1\) for every \(S \in S(\Lambda)\). Equation \((*)\) then implies \(\dim_k S = \dim_k \nu(S)\), so that \(\Lambda\) is a Frobenius algebra.

References