ℓ-adic cohomology

Étale cohomology was invented by Grothendieck and his collaborators in the 1960s. As far as I know, the aim was to prove the Weil conjectures, which Deligne did in 1974. In this note I try to give a very brief introduction to étale cohomology, especially to ℓ-adic cohomology. This talk lacks proofs, and completeness is by no means intended (one would need way more than 45 minutes). The proper setup for this would be schemes, but for sake of simplicity I will only consider varieties.

Sheaf cohomology

Let X be a topological space. Let X_{open} be the category whose objects are the open subsets of X, and the morphisms are the inclusions. A presheaf on X is a contravariant functor $\mathcal{P}$ from X_{open} to the category of Abelian groups. One usually writes ϱ_V^U for the restriction homomorphism $\mathcal{P}(V \subseteq U) : \mathcal{P}(U) \to \mathcal{P}(V)$. A presheaf $\mathcal{F}$ is called a sheaf if for any open covering $U = \bigcup_{i \in I} U_i$ the following two properties hold:

(i) If $f, f' \in \mathcal{F}(U)$ satisfy $\varrho_{U_i}^U(f) = \varrho_{U_i}^U(f')$ for all $i \in I$, then $f = f'$.

(ii) For $f_i \in \mathcal{F}(U_i)$ with $\varrho_{U_i \cap U_j}^{U_i}(f_i) = \varrho_{U_i \cap U_j}^{U_j}(f_j)$ for all $i, j \in I$ there exists $f \in \mathcal{F}(U)$ such that $\varrho_{U_i}^U(f) = f_i$ for all $i \in I$.

The basic example is that of a constant sheaf: for an Abelian group A, we define $\mathcal{F}_A(U) := A^{\pi_0(U)}$, where $\pi_0(U)$ denotes the set of connected components of U. The restrictions are diagonal embeddings: for U connected and $V \subseteq U$, where $V = \coprod_{j \in J} V_j$ is a decomposition into connected components, $\varrho_V^U : A \to A^J, a \mapsto (a, \ldots, a)$. The constant sheaf is often denoted by A rather than $\mathcal{F}_A$. If X is a variety over an algebraically closed field $\mathbb{F}$, there is the structure sheaf $\mathcal{O}_X$. Here, $\mathcal{O}_X(U) := \mathrm{Hom}(U, \mathbb{A}_1)$ is the ring of polynomial functions from U to the one dimensional affine space over $\mathbb{F}$. Of course, the structure sheaf is a sheaf of rings.

Let $Sh(X)$ denote the category of sheaves on X. Grothendieck has shown that this is an Abelian category with enough injectives. The global section functor $\Gamma : Sh(X) \to Ab, \mathcal{F} \mapsto \mathcal{F}(X)$ is left exact. Its i^{th} right derived functor is denoted by $H^i(X, \cdot)$. If X is paracompact (i.e. every open covering admits a locally finite refinement), then $H^i(X, \mathbb{Z})$ coincides with singular cohomology, as defined in classical algebraic topology.[1] However, sheaf cohomology is useless for varieties, since there are too few open subsets:

Theorem 1 (Grothendieck)
If X is irreducible (i.e. any two non-empty open subsets intersect non-trivially), then $H^i(X, A) = 0$ for all $i > 0$ and any constant sheaf A. □

Étale cohomology

Let X and Y be non-singular varieties over an algebraically closed field $\mathbb{F}$. A morphism $f : X \to Y$ is étale, if the map $\partial f_x : T_x X \to T_{f(x)} Y$ of tangent spaces is an isomorphism for all $x \in X$.[2] Now

[1] This should hold in many more cases I am not aware of.
[2] This can easily be generalized to singular varieties. Just use the tangent cone instead of the tangent space.

let X_{et} be the category whose objects are étale morphisms $U \to X$, and whose morphisms are commutative triangles:

$$\begin{array}{ccc} U & \longrightarrow & U' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Obviously, embeddings of open subspaces are étale, so X_{open} is a full subcategory of X_{et}. As before, a presheaf on X is a contravariant functor $\mathscr{P} : X_{et} \to Ab$. To define what a sheaf is, we need a generalization of the intersection of open subsets. The obivous choice is the pull-back or fibered product:

$$\begin{array}{ccc} U \times_X U' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

It is easy to see that the corresponding map $U \times_X U' \to X$ is étale. Now define a presheaf $\mathcal{F}$ to be a sheaf if the two conditions from classical sheaf theory are satisfied, where "open covering" is replaced by "étale covering", and the intersection is replaced by the fibered product. Here, an étale covering of $U \to X$ is a set of étale maps $f_i : U_i \to U$ such that $U = \bigcup_{i \in I} f_i(U_i)$. The basic example is again the constant sheaf $\mathcal{F}_A$ for an Abelian group A. It is defined – again – to be $\mathcal{F}_A(U \to X) := A^{\pi_0(U)}$. The classical structure sheaf can be extended to the étale structure sheaf $\mathcal{O}_{X_{et}}$, defined by $\mathcal{O}_{X_{et}}(U \to X) := \mathcal{O}_U(U)$. Let $Sh_{et}(X)$ be the category of étale sheaves on X. As above, this is an Abelian category with enough injectives, and the global section functor

$$\Gamma : Sh_{et}(X) \to Ab, \mathcal{F} \mapsto \mathcal{F}(X \to X)$$

is left exact. The i^{th} right derived functor $H^i(X, \mathcal{F})$ of Γ is called i^{th} étale cohomology group of X with coefficients in $\mathcal{F}$.

Let $\mathscr{P}$ be an étale presheaf on X. Then there exist $a\mathscr{P} \in Sh_{et}(X)$ and a morphism $a : \mathscr{P} \to a\mathscr{P}$ such that every morphism $\mathscr{P} \to \mathcal{F}$ for a sheaf $\mathcal{F}$ factors uniquely through a. The sheaf $a\mathscr{P}$ is called the sheafification of $\mathscr{P}$. A sheaf is called a torsion sheaf if it is the sheafification of a presheaf $\mathscr{P}$, where each $\mathscr{P}(U \to X)$ is a torsion group.

Theorem 2

Let X be a variety over an algebraically closed field. Then $H^i(X, \mathcal{F}) = 0$ for all torsion sheafs $\mathcal{F} \in Sh_{et}(X)$ and all $i > 2\dim(X)$. □

A variety X is called complete, if for each variety Y, the projection $X \times Y \to Y$ is a closed map. Étale cohomology behaves nicer if only complete varieties are considered. For example, there is the following theorem:

Theorem 3

Let X be complete variety, and let A be a finite Abelian group. Then $H^i(X, A)$ is finite.[3] □

[3] This holds for all constructible sheaves, not just for constant ones. However, I did not want to introduce yet another kind of sheaves.

Cohomology with compact support

Nagata has shown that for any variety X, there exists a complete variety $\overline{X}$ and an open immersion $j : X \to \overline{X}$ such that $j(X)$ is dense in $\overline{X}$. By abuse of notation, $\overline{X}$ is called a compactification of X. Now let $\mathcal{F} \in Sh_{et}(X)$. Let $j_!\mathcal{F} \in Sh_{et}(\overline{X})$ be the sheafification of

$$j_!\mathcal{F}(\alpha : U \to \overline{X}) := \begin{cases} 0 & \alpha(U) \nsubseteq j(X) \\ \mathcal{F}(U) & \alpha(U) \subseteq j(X) \end{cases}$$

This procedure is called "extension by zero". If $\mathcal{F}$ is a torsion sheaf, $H^i_c(X, \mathcal{F}) := H^i_c(\overline{X}, j_!\mathcal{F})$ is called the i^{th} cohomology group of X with compact support and coefficients in $\mathcal{F}$. Although compactifications are far from being unique, this can be shown not to depend on the chosen compactification. The disadvantage of cohomology with compact support is that it is not a derived functor. However, it has a number of nice properties, e.g. it satisfies an open-closed exact sequence and a Künneth formula. Also, the cohomology groups $H^i_c(X, A)$ are finite for a finite Abelian group A. However, to prove the Weil conjectures (and for applications in representation theory), it is necessary to have cohomology with coefficients in characteristic zero.

ℓ-adic cohomology

Let ℓ be a prime, $\mathbb{Z}_\ell$ be the ℓ-adic integers, and let $\mathbb{Q}_\ell$ be the field of ℓ-adic numbers. Also, let $\mathbb{F}$ be an algebraically closed field of characteristic different from ℓ, and let X be a variety over $\mathbb{F}$. Since $\mathbb{Z}_\ell$ is equal to the inverse limit $\varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$, one defines $H^i(X, \mathbb{Z}_\ell) := \varprojlim H^i(X, \mathbb{Z}/\ell^n\mathbb{Z})$ and $H^i(X, \mathbb{Q}_\ell) := H^i(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and similarly for cohomology with compact support. Then $H^i_c(X, \mathbb{Z}_\ell)$ is a finitely generated $\mathbb{Z}_\ell$-module, hence $H^i_c(X, \mathbb{Q}_\ell)$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space. It is often easier to work with cohomology with coefficients in $\mathbb{Q}_\ell$, since the torsion part of $H^i_c(X, \mathbb{Z}_\ell)$ is difficult to handle.[4]

Let $g \in \mathrm{Aut}(X)$ be an automorphism of finite order. This induces an automorphism of $H^i_c(X, \mathbb{Q}_\ell)$, which will also be denoted by g. The sum $\mathcal{L}(g, X) := \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(g, H^i_c(X, \mathbb{Q}_\ell))$ is called the Lefschetz number of g on X. Here, $d := \dim(X)$, and $\mathrm{Tr}(g, V)$ denotes the trace of the endomorphism g of V. The Lefschetz numbers were used by Deligne and Lusztig to construct virtual characters of finite groups of Lie type. They can be defined without ℓ-adic cohomology, as will be shown now.

Theorem 4 (Grothendieck's trace formula)
Suppose that $\mathbb{F}$ is the algebraic closure of the finite field $\mathbb{F}_q$, and that X is defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F.[5] *Then*

$$|X^F| = \sum_{i=0}^{2\dim(X)} \mathrm{Tr}(F, H^i_c(X, \mathbb{Q}_\ell)),$$

where $X^F := \{x \in X \mid F(x) = x\}$. □

[4] See the paper "Catégories dérivées et variétés de Deligne-Lusztig" from Bonnafé and Rouquier, where they prove that Jordan decomposition is a Morita equivalence. The main difficulty is to show that the relevant cohomology groups are torsion-free.

[5] The easiest example is the affine space, where F takes every component to the q^{th} power.

Corollary 5
Let the assumptions be as above, and let $g \in \mathrm{Aut}(X)$ be a rational (i.e. g commutes with F) automorphism of finite order. Also, let $R(t) := -\sum_{n=1}^{\infty} |X^{gF^n}| t^n$. Then $\mathcal{L}(g, X) = R(t)|_{t=\infty}$.

Proof: For $0 \leq i \leq 2d$, where $d := \dim(X)$, let α_{ij} and β_{ij} be the eigenvalues of g and F, respectively, on $H_c^i(X, \mathbb{Q}_\ell)$ for $1 \leq j \leq \dim H_c^i(X, \mathbb{Q}_\ell)$. Since F and g commute, they can be transformed into triangular form simultaneously. Also, gF^n is a Frobenius endomorphism of X, hence Grothendieck's trace formula yields

$$R(t) = -\sum_{n=1}^{\infty}\sum_{i=0}^{2d}\sum_{j}\alpha_{ij}\beta_{ij}^n t^n = -\sum_{i,j}\alpha_{ij}\sum_{n=1}^{\infty}(\beta_{ij}t)^n = -\sum_{i,j}\alpha_{ij}\frac{\beta_{ij}t}{1-\beta_{ij}t}.$$

Thus $R(t)$ is a rational function, where numerator and denominator have the same degree. Hence one can define $R(t)|_{t=\infty} := R(\frac{1}{t})|_{t=0} = \sum_{i,j}\alpha_{ij} = \mathcal{L}(g, X)$. □

Corollary 6
The Lefschetz number $\mathcal{L}(g, X)$ is an integer independent of ℓ.

Proof: Lefschetz numbers are algebraic integers by definition. They are independent of ℓ and values of rational functions with coefficients in $\mathbb{Q}$ by the above corollary. □