ALGEBRAIC TRANSFORMATION GROUPS
An Introduction

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(Preliminary version from February 20, 2005)

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Abstract. These are preliminary notes from a course "Introduction to algebraic geometry" which I taught in the Winter term 04/05. The plan is that they should finally become an appendix to an introductory book on Algebraic Transformation Groups. But it is unclear if this project will ever be completed!

In this appendix we concentrate on *affine* algebraic geometry which simplifies a lot the notational part and makes the subject much easier to access in a first attempt. One part, the relation between the Zariski topology and the $\mathbb{C}$-topology is still missing. With its help we are able to use certain compactness arguments replacing the corresponding results from projective geometry.

The appendix assumes a basic knowledge in commutative algebra. We give complete and quite elementary proofs for almost all statements. There is just one exception: I could not find an elementary proof that smooth points are normal. Also, the famous and very useful Serre Criterion for normality is stated without proof.

The notes are still in a very preliminary form. There are certainly a lot of misprints and some inaccuracies which have to be eliminated in the future versions. Of course, remarks and suggestions from all readers are very well-come.

Contents

APPENDIX A

BASICS FROM ALGEBRAIC GEOMETRY

1. AFFINE VARIETIES

Regular functions. Our base field is the field $\mathbb{C}$ of complex numbers. Every polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ can be regarded as a $\mathbb{C}$-valued function on $\mathbb{C}^n$ in the usual way:

$$a = (a_1, \dots, a_n) \mapsto p(a) = p(a_1, \dots, a_n).$$

These functions will be called *regular*. More generally, let V be a $\mathbb{C}$-vector space of dimension $\dim V = n < \infty$.

Definition 1.1. A $\mathbb{C}$-valued function $f\colon V \to \mathbb{C}$ is called *regular* if f is given by a polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ with respect to one and hence all bases of V. This means that for a given basis $v_1, \dots, v_n$ of V we have

$$f(a_1 v_1 + \cdots + a_n v_n) = p(a_1, \dots, a_n)$$

for a suitable polynomial p. The algebra of regular functions on V will be denoted by $\mathcal{O}(V)$.

By our definition, every choice of a basis $(v_1, v_2, \dots, v_n)$ of V defines an isomorphism $\mathbb{C}[x_1, \dots, x_n] \xrightarrow{\sim} \mathcal{O}(V)$ by identifying x_i with the ith coordinate function on V defined by the basis, i.e.,

$$x_i(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) := a_i.$$

Another way to express this is by remarking that the linear functions on V are regular and thus the dual space $V^* := \mathrm{Hom}(V, \mathbb{C})$ is a subspace of $\mathcal{O}(V)$. So if $(v_1, v_2, \dots, v_n)$ is a basis of V and $(x_1, x_2, \dots, x_n)$ the dual basis of V^* then $\mathcal{O}(V) = \mathbb{C}[x_1, x_2, \dots, x_n]$ and the linear functions x_i are algebraically independent.

Example 1.1. Denote by $M_n = M_n(\mathbb{C})$ the complex $n \times n$-matrices so that $\mathcal{O}(M_n) = \mathbb{C}[x_{ij} \mid 1 \le i, j \le n]$. Consider $\det(tE_n - X)$ as a polynomial in $\mathbb{C}[t, x_{ij}, i, j = 1, \dots, n]$ where $X := (x_{ij})$. Developing this as a polynomial in t we find

$$\det(tE_n - X) = t^n - s_1 t^{n-1} + s_2 t^{n-2} - \cdots + (-1)^n s_n$$

with polynomials s_k in the variables x_{ij}. This defines regular functions $s_k \in \mathcal{O}(M_n)$ which are homogeneous of degree k. Of course, we have $s_1(A) = \mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$ and $s_n(A) = \det(A)$ for any matrix $A \in M_n$.

The same construction applies to $\mathrm{End}(V)$ for any finite dimensional vector space V and defines regular function $s_k \in \mathcal{O}(\mathrm{End}(V))$.

Example 1.2. Consider the vector space P_n of unitary polynomials of degree n:

$$P_n := \{t^n - a_1 t^{n-1} + a_2 t^{n-2} - \cdots + (-1)^n a_n \mid a_1, \cdots, a_n \in \mathbb{C}\} \simeq \mathbb{C}^n.$$

There is a regular function $D \in \mathcal{O}(P_n)$, the *discriminant*, with the following property: *$D(p) = 0$ for a $p \in P_n$ if and only if p has a multiple root.*

PROOF. Expanding $\prod_{i=1}^n (t - y_i) = t^n - \sigma_1(y)t^{n-1} + \cdots + (-1)^n \sigma_n(y)$ we see that the polynomials $\sigma_j(y)$ are the *elementary symmetric polynomials* in n variables $y_1, \ldots, y_n$, i.e.

$$s_k := \sum_{i_1 < i_2 < \cdots < i_k} y_{i_1} y_{i_2} \cdots y_{i_k}.$$

Define $\tilde{D} := \prod_{i<j}(y_i - y_j)^2$. Since $\tilde{D}$ is symmetric it can be (uniquely) written as a polynomial in the elementary symmetric functions (see [**Ar91**, Chap. 14, Theorem 3.4]): $\tilde{D}(y_1, \ldots, y_n) = D(\sigma_1, \ldots, \sigma_n)$ with a suitable polynomial D. By construction, D has the required property. □

EXAMPLE 1.3. We denote by $\mathrm{Alt}_n \subseteq M_n$ the subspace of alternating matrices:

$$\mathrm{Alt}_n := \{A \in M_n \mid A^t = -A\}.$$

There is a regular function $\mathrm{Pf} \in \mathcal{O}(\mathrm{Alt}_{2m})$, the *Pfaffian*, with the following property: $\det(A) = \mathrm{Pf}(A)^2$ for all $A \in \mathrm{Alt}_{2m}$. Usually, the sign of the Pfaffian is determined by requiring that $\mathrm{Pf}(\begin{bmatrix} J & & \\ & \ddots & \\ & & J \end{bmatrix}) = 1$ where $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

PROOF. It is well-known that for any alternating matrix A with entries in an arbitrary field K there is a $g \in \mathrm{GL}_n(K)$ such that

$$gAg^t = \begin{bmatrix} J & & & & \\ & \ddots & & & \\ & & J & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}. \tag{1}$$

Now take $K = \mathbb{C}(x_{ij} \mid 1 \leq i < j \leq n = 2m)$ and put

$$A := \begin{bmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ -x_{12} & 0 & x_{23} & \cdots & x_{2n} \\ -x_{13} & -x_{23} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & & \ddots & \vdots \\ -x_{1n} & -x_{2n} & -x_{3n} & \cdots & 0 \end{bmatrix}.$$

Then there is a $g \in \mathrm{GL}_n(K)$ such that gAg^t has the form given in (1). It follows that the polynomial $\det(A) \in K[x_{ij} \mid 1 \leq i < j \leq n]$ equals $\det(g)^{-2}$, the square of a rational function, and the claim follows. □

EXERCISE 1.1. For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ denote by $\mathrm{ev}_a \colon \mathcal{O}(\mathbb{C}^n) \to \mathbb{C}$ the *evaluation map* $f \mapsto f(a)$. Then the kernel of ev_a is the maximal ideal

$$\mathfrak{m}_a := (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n).$$

EXERCISE 1.2. Let $W \subseteq \mathcal{O}(V)$ a finite dimensional subspace. Then the linear functions $\mathrm{ev}_v|_W$ for $v \in V$ span the dual space W^*.

Zero sets and Zariski topology. We now define the basic object of algebraic geometry, namely the zero set of regular functions. Let V be a finite dimensional vector space.

DEFINITION 1.2. If $f \in \mathcal{O}(V)$ then we define the *zero set* of f by

$$\mathcal{V}(f) := \{v \in V \mid f(v) = 0\} = f^{-1}(0).$$

More generally, the *zero set* of $f_1, f_2, \dots, f_s \in \mathcal{O}(V)$ or of a subset $S \subseteq \mathcal{O}(V)$ is defined by

$$\mathcal{V}(f_1, f_2, \dots, f_s) := \bigcap_{i=1}^{s} \mathcal{V}(f_i) = \{v \in V \mid f_1(v) = \dots = f_s(v) = 0\}$$

or

$$\mathcal{V}(S) := \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}.$$

REMARK 1.1. The following properties of zero sets follow immediately from the definition.

(1) Let $S \subseteq \mathcal{O}(V)$ and denote by $\mathfrak{a} := (S) \subseteq \mathcal{O}(V)$ the ideal generated by S. Then $\mathcal{V}(S) = \mathcal{V}(\mathfrak{a})$.
(2) If $S \subseteq T \subseteq \mathcal{O}(V)$ then $\mathcal{V}(S) \supseteq \mathcal{V}(T)$.
(3) For any family $(S_i)_{i \in I}$ of subset $S_i \subseteq \mathcal{O}(V)$ we have

$$\bigcap_{i \in I} \mathcal{V}(S_i) = \mathcal{V}(\bigcup_{i \in I} S_i).$$

EXAMPLE 1.4. (1) $\mathrm{SL}_n(\mathbb{C}) = \mathcal{V}(\det -1) \subseteq M_n(\mathbb{C})$.
(2) $\mathrm{O}_n(\mathbb{C}) = \mathcal{V}(\sum_{\nu=1}^{n} x_{i\nu} x_{j\nu} - \delta_{ij} \mid 1 \leq i \leq j \leq n)$.
(3) If $f = f(x, y) \in \mathbb{C}[x, y]$ is a non-constant polynomial in 2 variables, then $\mathcal{V}(f) \subseteq \mathbb{C}$ is called a *plane curve*. In order to visualize a plane curve, we usually draw a real picture $\subseteq \mathbb{R}^2$.

LEMMA 1.1. *Let V be a finite dimensional vector space and let $\mathfrak{a}, \mathfrak{b}$ be ideals in $\mathcal{O}(V)$ and $(\mathfrak{a}_i \mid i \in I)$ a family of ideals of $\mathcal{O}(V)$.*

(1) *If $\mathfrak{a} \subseteq \mathfrak{b}$ then $\mathcal{V}(\mathfrak{a}) \supseteq \mathcal{V}(\mathfrak{b})$.*
(2) $\bigcap_{i \in I} \mathcal{V}(\mathfrak{a}_i) = \mathcal{V}(\sum_{i \in I} \mathfrak{a}_i)$.
(3) $\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b}) = \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{V}(\mathfrak{a} \cdot \mathfrak{b})$.
(4) $\mathcal{V}(0) = V$ *and* $\mathcal{V}(1) = \emptyset$.

PROOF. Properties (1) and (2) follow from Remark 1, and property (4) is easy. So we are left with property (3). Since $\mathfrak{a} \supseteq \mathfrak{a} \cap \mathfrak{b} \supseteq \mathfrak{a} \cdot \mathfrak{b}$, it follows from (1) that $\mathcal{V}(\mathfrak{a}) \subseteq \mathcal{V}(\mathfrak{a} \cap \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a} \cdot \mathfrak{b})$. So it remains to show that $\mathcal{V}(\mathfrak{a} \cdot \mathfrak{b}) \subseteq \mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b})$. If $v \in V$ does not belong to $\mathcal{V}(\mathfrak{a}) \cup \mathcal{V}(\mathfrak{b})$ then there are functions $f \in \mathfrak{a}$ and $h \in \mathfrak{b}$ such that $f(v) \neq 0 \neq h(v)$. Since $f \cdot h \in \mathfrak{a} \cdot \mathfrak{b}$ and $(f \cdot h)(v) \neq 0$ we see that $v \notin \mathcal{V}(\mathfrak{a} \cdot \mathfrak{b})$, and the claim follows. □

DEFINITION 1.3. The lemma shows that the subsets $\mathcal{V}(\mathfrak{a})$ where $\mathfrak{a}$ runs through the ideals of $\mathcal{O}(V)$ form the closed sets of topology on V which is called *Zariski topology*. From now on all topological terms like “open”, “closed”, “neighborhood”, “continuous”, etc. will refer to the Zariski topology.

EXAMPLE 1.5. (1) The *nilpotent cone* $N \subseteq M_n$ consisting of all nilpotent matrices is closed and is a *cone*, i.e. stable under multiplication with scalars. E.g. for $n = 2$ we have

$$N = \mathcal{V}(x_{11} + x_{22}, x_{11}x_{22} - x_{12}x_{21}) \subseteq M_2.$$

(2) The subset $M_n^{(r)} \subseteq M_n$ of matrices of rank $\leq r$ are closed cones.
(3) The set of polynomials $f \in P_n$ with a multiple root is closed (see Example 1.2).
(4) The closed subsets of $\mathbb{C}$ are the finite sets together with $\mathbb{C}$. So the non-empty open sets of $\mathbb{C}$ are the cofinite sets.

EXERCISE 1.3. A regular function $f \in \mathcal{O}(V)$ is called *homogeneous of degree d* if $f(tv) = t^d f(v)$ for all $t \in \mathbb{C}$ and all $v \in V$.

(1) A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous of degree d as a regular function on $\mathbb{C}^n$ if and only if all monomials occurring in f have degree d.
(2) Assume that the ideal $\mathfrak{a} \subseteq \mathcal{O}(V)$ is generated by homogeneous functions. Then the zeros set $\mathcal{V}(\mathfrak{a}) \subseteq V$ is a cone.
(3) Conversely, if $X \subseteq V$ is a cone, then the ideal $I(X)$ can be generated by homogeneous functions.

EXERCISE 1.4. Show that the subset $A := \{(n, m) \in \mathbb{C}^2 \mid n, m \in \mathbb{Z} \text{ and } m \geq n \geq 0\}$ is Zariski-dense in $\mathbb{C}^2$.

DEFINITION 1.4. Let $X \subseteq V$ be a closed subset. A *regular function on X* is defined to be the restriction of a regular function on V:

$$\mathcal{O}(X) := \{f|_X \mid f \in \mathcal{O}(V)\}.$$

The kernel of the (surjective) restriction map $\text{res}\colon \mathcal{O}(V) \to \mathcal{O}(X)$ is called the *ideal* of X:

$$I(X) := \{f \in \mathcal{O}(V) \mid f(x) = 0 \text{ for all } x \in X\}.$$

Thus we have a canonical isomorphism $\mathcal{O}(V)/I(X) \xrightarrow{\sim} \mathcal{O}(X)$.

REMARK 1.2. Every finite dimensional $\mathbb{C}$-vector space V carries a natural topology given by the choice of a norm or a hermitian scalar product. We will call it the *$\mathbb{C}$-topology*. Since all polynomials are continuous with respect to the $\mathbb{C}$-topology we see that the $\mathbb{C}$-topology is finer than the Zariski topology.

EXERCISE 1.5. Show that every non-empty open set in $\mathbb{C}^n$ is dense in the $\mathbb{C}$-topology. (Hint: Reduce to the case $n = 1$ where the claim follows from Example 1.5(4).)

REMARK 1.3. In the Zariski topology the finite sets are closed. This follows from the fact that for any two different points $v, w \in V$ one can find a regular function $f \in \mathcal{O}(V)$ such that $f(v) = 0$ and $f(w) \neq 0$. (One says that the regular functions *separate the points*.) But the Zariski topology is not Hausdorff (see the following exercise).

EXERCISE 1.6. Let $U, U' \subseteq \mathbb{C}^n$ be two non-empty open sets. Then $U \cap U'$ is non-empty, too. In particular, the Zariski topology is not Hausdorff.

EXERCISE 1.7. Consider a polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ of the form $f = x_0 - p(x_1, \ldots, x_n)$, and let $X = \mathcal{V}(f)$ be its zero set. Then $I(X) = (f)$ and $\mathcal{O}(X) \simeq \mathbb{C}[x_1, \ldots, x_n]$.

Hilbert's Nullstellensatz. The famous Nullstellensatz of Hilbert appears in many different forms which are all more or less equivalent. We will give some of them which are suitable for our purposes.

DEFINITION 1.5. If $\mathfrak{a}$ is an ideal of an arbitrary ring R, its *radical* $\sqrt{\mathfrak{a}}$ is defined by

$$\sqrt{\mathfrak{a}} := \{r \in R \mid r^m \in \mathfrak{a} \text{ for some } m > 0\}.$$

The ideal $\mathfrak{a}$ is *perfect* if $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

It is easy to see that $\sqrt{\mathfrak{a}}$ is an ideal and that $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$. Moreover, $\sqrt{\mathfrak{a}} = R$ implies that $\mathfrak{a} = R$.

THEOREM 1.1 (Hilbert's Nullstellensatz). *Let $\mathfrak{a} \subseteq \mathcal{O}(V)$ be an ideal and $X := \mathcal{V}(\mathfrak{a}) \subseteq V$ its zero set. Then*

$$I(X) = I(\mathcal{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

A first consequence is that every strict ideal has a non-empty zero set, because $X = \mathcal{V}(\mathfrak{a}) = \emptyset$ implies that $\sqrt{\mathfrak{a}} = I(X) = \mathcal{O}(V)$ and so $\mathfrak{a} = \mathcal{O}(V)$.

COROLLARY 1.1. *For every ideal $\mathfrak{a} \neq \mathcal{O}(V)$ we have $\mathcal{V}(\mathfrak{a}) \neq \emptyset$.*

If $\mathfrak{m} \subseteq \mathbb{C}[x_1, \dots, x_n]$ is a maximal ideal and $a = (a_1, \dots, a_n) \in \mathcal{V}(\mathfrak{m})$ then $\mathfrak{m} \supseteq (x_1 - a_1, \dots, x_n - a_n)$ and so these two are equal.

COROLLARY 1.2. *Every maximal ideal $\mathfrak{m}$ of $\mathbb{C}[x_1, \dots, x_n]$ is of the form*

$$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n).$$

Another way to express this is by using the evaluation map ev_v (see Exercise 1.1).

COROLLARY 1.3. *Every maximal ideal of $\mathcal{O}(V)$ equals the kernel of the evaluation map $\mathrm{ev}_v \colon \mathcal{O}(V) \to \mathbb{C}$ at a suitable $v \in V$.*

EXERCISE 1.8. If $X \subseteq V$ is a closed subset and $\mathfrak{m} \subseteq \mathcal{O}(X)$ a maximal ideal then $\mathcal{O}(X)/\mathfrak{m} = \mathbb{C}$. Moreover, $\mathfrak{m} = \ker(\mathrm{ev}_x \colon f \mapsto f(x))$ for a suitable $x \in X$.

PROOF OF THEOREM 1.1. Let $\mathfrak{m} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a maximal ideal and denote by $K := \mathbb{C}[x_1, \dots, x_n]/\mathfrak{m}$ its residue class field. Then K contains $\mathbb{C}$ and has a countable $\mathbb{C}$-basis, because $\mathbb{C}[x_1, \dots, x_n]$ does. If $K \neq \mathbb{C}$ and $p \in K \setminus \mathbb{C}$ then p is transcendental over $\mathbb{C}$. It follows that the elements $(\frac{1}{p-a} \mid a \in \mathbb{C})$ from K form a non-countable set of linearly independent elements over $\mathbb{C}$. This contradiction shows that $K = \mathbb{C}$. Thus $x_i + \mathfrak{m} = a_i + \mathfrak{m}$ for a suitable $a_i \in \mathbb{C}$ (for $i = 1, \dots, n$), and so $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$. This proves Corollary 1.2 (and Corollary 1.3).

To get the theorem, we use the so-called trick of Rabinowich. Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal and assume that the polynomial f vanishes on $\mathcal{V}(\mathfrak{a})$. Now consider the polynomial ring $R := \mathbb{C}[x_0, x_1, \dots, x_n]$ in $n+1$ variables and the ideal $\mathfrak{b} := (\mathfrak{a}, 1 - x_0 f)$ generated by $1 - x_0 f$ and the elements of $\mathfrak{a}$. Clearly, $\mathcal{V}(\mathfrak{b}) = \emptyset$ and so $1 \in \mathfrak{b}$. This means that we can find an equation of the form

$$\sum_i h_i f_i + h(1 - x_0 f) = 1$$

where $f_i \in \mathfrak{a}$ and $h_i, h \in R$. Now we substitute $\frac{1}{f}$ for x_0 and find

$$\sum_i h_i(\frac{1}{f}, x_1, \ldots, x_n) f_i = 1.$$

Clearing denominators finally gives $\sum_i \tilde{h}_i f_i = f^m$, i.e., $f^m \in \mathfrak{a}$, and the claim follows. $\square$

Corollary 1.4. *For any ideal* $\mathfrak{a} \subseteq \mathcal{O}(V)$ *and its zero set* $X := \mathcal{V}(\mathfrak{a})$ *we have* $\mathcal{O}(X) = \mathcal{O}(V)/\sqrt{\mathfrak{a}}$.

Exercise 1.9. Let $\mathfrak{a} \subseteq R$ be an ideal of a (commutative) ring R. Then $\mathfrak{a}$ is perfect if and only if the residue class ring $R/\mathfrak{a}$ has no nilpotent elements different from 0.

Example 1.6. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an arbitrary polynomial and consider its decomposition into irreducible factors: $f = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$. Then $\sqrt{(f)} = (p_1 p_2 \cdots p_s)$ and so the ideal (f) is perfect if and only if the polynomial f it is square-free. In particular, if $f \in \mathbb{C}[x_1, \ldots, x_n]$ is irreducible, then $\mathcal{O}(\mathcal{V}(f)) \simeq \mathbb{C}[x_1, \ldots, x_n]/(f)$. A closed subset of the form $\mathcal{V}(f)$ is called a *hypersurface*.

Example 1.7. We have $\mathcal{O}(\mathrm{SL}_n(\mathbb{C})) \simeq \mathcal{O}(M_n)/(\det -1)$ because the polynomial $\det -1$ is irreducible.
In fact, if $\det -1 = f_1 \cdot f_2$ then each factor f_i is linear in the variables which occur. But if $x_{i_0 j_0}$ occurs in f_1 then all the variables $x_{i j_0}$ and $x_{i_0 j}$ have to occur in f_1, too, since $\det -1$ does not contain products of the form $x_{i j_0} x_{i' j_0}$ and $x_{i_0 j} x_{i_0 j'}$. This implies that all variables occur in f_1, hence f_2 is a constant.

Example 1.8. Consider the plane curve $C := \mathcal{V}(y^2 - x^3)$ which is called *Neil's parabola*. Then $\mathcal{O}(C) \simeq \mathbb{C}[x,y]/(y^2 - x^3) \xrightarrow{\sim} \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$ where the second isomorphism is given by $\rho\colon x \mapsto t^3, y \mapsto t^2$.

Proof. Clearly, $y^2 - x^3 \in \ker \rho$. For any $f \in \mathbb{C}[x,y]$ we can write $f = f_0(x) + f_1(x)y + h(x,y)(y^2 - x^3)$. If $f \in \ker \rho$ then $0 = \rho(f) = f_0(t^2) + f_1(t^2)t^3$ and so $f_0 = f_1 = 0$. This shows that $\ker \rho = (y^2 - x^3)$, and the claim follows. $\square$

Exercise 1.10. Let $C \subseteq \mathbb{C}^2$ be the plane curve defined by $y - x^2 = 0$. Then $I(C) = (y - x^2)$ and $\mathcal{O}(C)$ is a polynomial ring in one variable.

Exercise 1.11. Let $D \subseteq \mathbb{C}^2$ be the zero set of $xy - 1$. Then $\mathcal{O}(D)$ is not isomorphic to a polynomial ring, but there is an isomorphism $\mathcal{O}(D) \xrightarrow{\sim} \mathbb{C}[t, t^{-1}]$.

Exercise 1.12. Consider the "parametric curve"

$$Y := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}.$$

Then Y is closed in $\mathbb{C}^3$. Find generators for the ideal $I(Y)$ and show that $\mathcal{O}(Y)$ is isomorphic to the polynomial ring in one variable.

Another important consequence of the "Nullstellensatz" is a one-to-one correspondence between closed subsets of $\mathbb{C}^n$ and perfect ideals of the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$.

Corollary 1.5. *The map* $X \mapsto I(X)$ *defines a inclusion-reversing bijection*

$$\{X \subseteq V \text{ closed}\} \xrightarrow{\sim} \{\mathfrak{a} \subseteq \mathcal{O}(V) \text{ perfect ideal}\}$$

whose inverse map is given by $\mathfrak{a} \mapsto \mathcal{V}(\mathfrak{a})$. *Moreover, for any finitely generated reduced* $\mathbb{C}$*-algebra* R *there is a closed subset* $X \subseteq \mathbb{C}^n$ *for some* n *such that* $\mathcal{O}(X)$ *is isomorphic to* R

Proof. The first part is clear since $\mathcal{V}(I(X)) = X$ and $I(\mathcal{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for any closed subset $X \subseteq V$ and any ideal $\mathfrak{a} \subseteq \mathcal{O}(V)$.

If R is a reduced and finitely generated $\mathbb{C}$-Algebra, $R = \mathbb{C}[f_1, \ldots, f_n]$, then $R \simeq \mathbb{C}[x_1, x_2, \ldots, x_n]/\mathfrak{a}$ where $\mathfrak{a}$ is the kernel of the homomorphism defined by $x_i \mapsto f_i$. Since R is reduced we have $\sqrt{\mathfrak{a}} = \mathfrak{a}$ and so $\mathcal{O}(\mathcal{V}(\mathfrak{a})) \simeq \mathbb{C}[x_1, \ldots, x_n]/\mathfrak{a} \simeq R$. □

Exercise 1.13. Let $X \subseteq V$ be a closed subset and $f \in \mathcal{O}(X)$ a regular function such that $f(x) \neq 0$ for all $x \in X$. Then f is invertible in $\mathcal{O}(X)$, i.e. the $\mathbb{C}$-valued function $x \mapsto f(x)^{-1}$ is regular on X.

Exercise 1.14. Every closed subset $X \subseteq \mathbb{C}^n$ is quasi-compact, i.e., every covering of X by open sets contains a finite covering.

Exercise 1.15. Let $X \subseteq \mathbb{C}^n$ be a closed subset. Assume that there are no non-constant invertible regular function on X. Then every non-constant $f \in \mathcal{O}(X)$ attains all values in $\mathbb{C}$, i.e. $f\colon X \to \mathbb{C}$ is surjective.

Affine varieties. We have seen in the previous section that every closed subset $X \subseteq V$ (or $X \subseteq \mathbb{C}^n$) is equipped with an algebra of $\mathbb{C}$-valued functions, namely the coordinate ring $\mathcal{O}(X)$. We first remark that $\mathcal{O}(X)$ determines the topology of X. In fact, define for every ideal $\mathfrak{a} \subseteq \mathcal{O}(X)$ *the zero set in* X by

$$\mathcal{V}_X(\mathfrak{a}) := \{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

Clearly, we have $\mathcal{V}_X(\mathfrak{a}) = \mathcal{V}(\tilde{\mathfrak{a}}) \cap X$ where $\tilde{\mathfrak{a}} \subseteq \mathcal{O}(V)$ is an ideal which maps surjectively onto $\mathfrak{a}$ under the restriction map. This shows that the sets $\mathcal{V}_X(\mathfrak{a})$ are the closed sets of the topology on X induced by the Zariski-topology of V. Moreover, the coordinate ring $\mathcal{O}(X)$ also determines the points of X since they are in one-to-one correspondence with the maximal ideals of $\mathcal{O}(X)$:

$$x \in X \mapsto \mathfrak{m}_x := \ker \operatorname{ev}_x \subseteq \mathcal{O}(X)$$

where $\operatorname{ev}_x\colon \mathcal{O}(X) \to \mathbb{C}$ is the evaluation map $f \mapsto f(x)$. This leads to the following definition of an "abstract" zero set, not embedded in a vector space.

Definition 1.6. A set Z together with a $\mathbb{C}$-algebra $\mathcal{O}(Z)$ of $\mathbb{C}$-valued functions on Z is called an *affine variety* if there is a closed subset $X \subseteq \mathbb{C}^n$ for some n and a bijection $\varphi\colon Z \xrightarrow{\sim} X$ which identifies $\mathcal{O}(X)$ with $\mathcal{O}(Z)$, i.e., $\varphi^*\colon \mathcal{O}(X) \to \mathcal{O}(Z)$ given by $f \mapsto f \circ \varphi$, is an isomorphism.

The functions from $\mathcal{O}(Z)$ are called *regular*, and the algebra $\mathcal{O}(Z)$ is called *coordinate ring of* Z or *algebra of regular functions on* Z.

The affine variety Z has a natural topology, the *Zariski-topology*, the closed sets being the *zero sets*

$$\mathcal{V}_Z(\mathfrak{a}) := \{z \in Z \mid f(z) = 0 \text{ for all } f \in \mathfrak{a}\}$$

where $\mathfrak{a}$ runs through the ideals of $\mathcal{O}(Z)$. If follows from what we said above that the bijection $\rho\colon Z \xrightarrow{\sim} X$ is a homeomorphism with respect to the Zariski-topology.

Clearly, every closed subset $X \subseteq V$ or $X \subseteq \mathbb{C}^n$ together with its coordinate ring $\mathcal{O}(X)$ is an affine variety. More generally, if X is an affine variety and $Y \subseteq X$ a closed subset, then Y together with the restrictions $\mathcal{O}(Y) := \{f|_Y \mid f \in \mathcal{O}(X)\}$ is an affine variety, called a *closed subvariety.*

Less trivial examples are the following.

EXAMPLE 1.9. Let M be a finite set and define $\mathcal{O}(M) := \operatorname{Maps}(M, \mathbb{C})$ to be the set of all $\mathbb{C}$-valued functions on M. Then $(M, \mathcal{O}(M))$ is an affine variety and $\mathcal{O}(M)$ is isomorphic to a product of copies of $\mathbb{C}$. This follows immediately from the fact that any finite subset $N \subseteq \mathbb{C}^n$ is closed and that $\mathcal{O}(N) = \operatorname{Maps}(N, \mathbb{C})$.

EXAMPLE 1.10. Let X be a set and define the *symmetric product* $\operatorname{Sym}_n(X)$ to be the set of unordered n-tuples of elements from X, i.e.,

$$\operatorname{Sym}_n(X) = X \times X \times \cdots \times X/\sim$$

where $(a_1, a_2, \ldots, a_n) \sim (b_1, b_2, \ldots, b_n)$ if and only if one is a permutation of the other.

In case $X = \mathbb{C}$ we define $\mathcal{O}(\operatorname{Sym}_n(\mathbb{C}))$ to be the symmetric polynomials in n variables and claim that $\operatorname{Sym}_n(\mathbb{C})$ is an affine variety.

To see this consider the map

$$\Phi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad a = (a_1, \ldots, a_n) \mapsto (\sigma_1(a), \sigma_2(a), \ldots, \sigma_n(a))$$

where $\sigma_1, \ldots, \sigma_n$ are the elementary symmetric polynomials (see Example 1.2). It is easy to see that Φ is surjective and that $\Phi(a) = \Phi(b)$ if and only if $a \sim b$. Thus, Φ defines a bijection $\varphi\colon \operatorname{Sym}_n(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^n$, and the pull-back of the regular functions on $\mathbb{C}^n$ are the symmetric polynomials: $\varphi^*\colon \mathbb{C}[x_1, \ldots, x_n] \xrightarrow{\sim} \mathcal{O}(\operatorname{Sym}_n(\mathbb{C}))$.

EXERCISE 1.16. Let Z be an affine variety with coordinate ring $\mathcal{O}(Z)$. Then every polynomial $f \in \mathcal{O}(Z)[t]$ with coefficients in $\mathcal{O}(Z)$ defines a function on the product $Z \times \mathbb{C}$ in the usual way:

$$f = \sum_{k=0}^{m} f_k t^k\colon (z, a) \mapsto \sum_{k=0}^{m} f_k(z) a^k \in \mathbb{C}$$

Show that $Z \times \mathbb{C}$ together with $\mathcal{O}(Z)[t]$ is an affine variety.
(Hint: For any ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ there is a canonical isomorphism $\mathbb{C}[x_1, \ldots, x_n, t]/(\mathfrak{a}) \xrightarrow{\sim} (\mathbb{C}[x_1, \ldots, x_n]/\mathfrak{a})[t]$.)

EXERCISE 1.17. For any affine variety Z there is a inclusion-reversing bijection

$$\{A \subseteq Z \text{ closed}\} \xrightarrow{\sim} \{\mathfrak{a} \subseteq \mathcal{O}(Z) \text{ perfect ideal}\}$$

given by $A \mapsto I(A) := \{f \in \mathcal{O}(Z) \mid f|_A = 0\}$ (cf. Corollary 1.5).

For the last example we start with a reduced and finitely generated $\mathbb{C}$-algebra R. Denote by $\operatorname{spec} R$ the set of maximal ideas of R:

$$\operatorname{spec} R := \{\mathfrak{m} \mid \mathfrak{m} \subseteq R \text{ a maximal ideal}\}.$$

We know from the "Nullstellensatz" (see Exercise 1.8) that $R/\mathfrak{m} = \mathbb{C}$ for all maximal ideals $\mathfrak{m} \in \operatorname{spec} R$. This allows to identify the elements from R with $\mathbb{C}$-valued functions on $\operatorname{spec} R$: For $f \in R$ and $\mathfrak{m} \in \operatorname{spec} R$ we define

$$f(\mathfrak{m}) := f + \mathfrak{m} \in R/\mathfrak{m} = \mathbb{C}.$$

Proposition 1.1. *Let R be a reduced and finitely generated $\mathbb{C}$-algebra. Then the set of maximal ideals* $\operatorname{spec} R$ *together with the algebra R considered as functions on* $\operatorname{spec} R$ *is an affine variety.*

Proof. We have already seen earlier that every such algebra R is isomorphic to the coordinate ring of a closed subset $X \subseteq \mathbb{C}^n$. The claim then follows by using the bijection $X \xrightarrow{\sim} \operatorname{spec} \mathcal{O}(X)$, $x \mapsto \mathfrak{m}_x = \ker \mathrm{ev}_x$, and remarking that for $f \in \mathcal{O}(X)$ and $x \in X$ we have $f(x) = \mathrm{ev}_x(f) = f + \mathfrak{m}_x$, by definition. □

Exercise 1.18. Denote by C_n the n-tuples of complex numbers up to sign, i.e., $C_n := \mathbb{C}^n/\sim$ where $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if $a_i = \pm b_i$ for all i. Then every polynomial in $\mathbb{C}[x_1^2, x_2^2, \ldots, x_n^2]$ is a well-defined function on C_n. Show that C_n together with these functions is an affine variety.
(Hint: Consider the map $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$, $(a_1, \ldots, a_n) \mapsto (a_1^2, \ldots, a_n^2)$ and proceed like in Example 1.10.)

Although every affine variety is "isomorphic" to a closed subset of $\mathbb{C}^n$ for a suitable n, there are many advantages to look at these objects and not only at closed subsets. In fact, an affine variety can be identified with many different closed subsets of this form (see the following Exercise 1.19), and depending on the questions we are asking one of them might be more useful than another. We will even see in the following section that certain open subsets are affine varieties in a natural way.

On the other hand, whenever we want to prove some statements for an affine variety X we can always assume that $X = \mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$ so that the regular functions on X appear as restrictions of polynomial functions. This will be helpful in the future and quite often simplify the arguments.

Exercise 1.19. Let X be an affine variety. Show that every choice of a generating system $f_1, f_2, \ldots, f_n \in \mathcal{O}(X)$ of the algebra $\mathcal{O}(X)$ consisting of n elements defines an identification of X with a closed subset $\mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$.
(Hint: Consider the map $X \to \mathbb{C}^n$ given by $x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$.)

Special open sets. Let X be an affine variety and $f \in \mathcal{O}(X)$. Define the open set $X_f \subseteq X$ by
$$X_f := X \setminus \mathcal{V}_X(f) = \{x \in X \mid f(x) \neq 0\}.$$
An open set of this form is called a *special open set.*

Lemma 1.2. *The special open sets of an affine variety X form a basis of the topology.*

Proof. If $U \subseteq X$ is open and $x \in U$, then $X \setminus U$ is closed and does not contain x. Thus, there is a regular function $f \in \mathcal{O}(X)$ vanishing on $X \setminus U$ such that $f(x) \neq 0$. This implies $x \in X_f \subseteq U$. □

Given a special open set $X_f \subseteq X$ we see that $f(x) \neq 0$ for all $x \in X_f$ and so the function $\frac{1}{f}$ is well-defined on X_f.

Proposition 1.2. *Denote by $\mathcal{O}(X_f)$ the algebra of functions on X_f generated by $\frac{1}{f}$ and the restrictions $h|_{X_f}$ of regular functions h on X:*
$$\mathcal{O}(X_f) := \mathbb{C}[\frac{1}{f}, \{h|_{X_f} \mid h \in \mathcal{O}(X)\}] = \mathcal{O}(X)|_{X_f}[\frac{1}{f}].$$

Then $(X_f, \mathcal{O}(X_f))$ *is an affine variety and* $\mathcal{O}(X_f) \xrightarrow{\sim} \mathcal{O}(X)[t]/(f \cdot t - 1)$.

PROOF. Let $X = \mathcal{V}(\mathfrak{a}) \subseteq \mathbb{C}^n$ and define

$$\tilde{X} := \mathcal{V}(\mathfrak{a}, f \cdot x_{n+1} - 1) \subseteq \mathbb{C}^{n+1}.$$

It is easy to see that the projection $\mathrm{pr}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ onto the first n coordinates induces a bijective map $\tilde{X} \xrightarrow{\sim} X_f$ whose inverse $\varphi\colon X_f \xrightarrow{\sim} \tilde{X}$ is given by

$$\varphi(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n)^{-1}).$$

The following commutative diagram now shows that $\varphi^*(\mathcal{O}(\tilde{X}))$ is generated by $\varphi^*(x_{n+1}) = \frac{1}{f}$ and the restrictions $h|_{X_f}$ $(h \in \mathcal{O}(X))$.

$$\begin{tikzcd}
\tilde{X} \arrow[r, equal] & \tilde{X} \arrow[r, "\subseteq", "\text{closed}"'] \arrow[d] & \mathbb{C}^{n+1} \arrow[d, "\mathrm{pr}"] \\
X_f \arrow[u, "\varphi", "\simeq"'] \arrow[r, "\subseteq", "\text{open}"'] & X \arrow[r, "\subseteq", "\text{closed}"'] & \mathbb{C}^n
\end{tikzcd}$$

This proves the first claim. For the second, we have to show that the canonical homomorphism $\mathcal{O}(X)[t]/(f \cdot t - 1) \to \mathcal{O}(\tilde{X})$ is an isomorphism. In other words, every function $h = \sum_{i=0}^m h_i t^i \in \mathcal{O}(X)[t]$ which vanishes on $\tilde{X}$ is divisible by $f \cdot t - 1$. Since $f|_{\tilde{X}}$ is invertible we first obtain $\sum_i h_i f^{m-i} = 0$ which implies

$$h = h - t^m \sum_{i=0}^{m} h_i f^{m-i} = \sum_{i=0}^{m-1} h_i t^i (1 - f^{m-i} t^{m-i}),$$

and the claim follows. $\square$

EXAMPLE 1.11. The group $\mathrm{GL}_n(\mathbb{C})$ is a special open set of $M_n(\mathbb{C})$, hence $\mathrm{GL}_n(\mathbb{C})$ is an affine variety with coordinate ring $\mathcal{O}(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}[\{x_{ij} \mid 1 \le i, j \le n\}, \frac{1}{det}]$. In particular, $\mathbb{C}^* := \mathrm{GL}_1 = \mathbb{C} \setminus \{0\}$ is an affine variety with coordinate ring $\mathbb{C}[x, x^{-1}]$.

EXERCISE 1.20. Let R be an arbitrary $\mathbb{C}$-algebra. For any element $s \in R$ define $R_s := R[x]/(s \cdot x - 1)$.

(1) Describe the kernel of the canonical homomorphism $\iota\colon R \to R_s$.
(2) Prove the universal property: *For any homomorphism* $\rho\colon R \to A$ *such that* $\rho(s)$ *is invertible in* A *there is a unique homomorphism* $\bar{\rho}\colon R_s \to A$ *such that* $\bar{\rho} \circ \iota = \rho$.
(3) What happens if s is a zero divisor and what if s is invertible?

Decomposition into irreducible components. We start with a purely topological notion.

DEFINITION 1.7. A topological space T is called *irreducible* if it cannot be decomposed in the form $T = A \cup B$ where $A, B \subsetneqq T$ are strict closed subsets. Equivalently, every non-empty open subset is dense.

LEMMA 1.3. *Let* $X \subseteq \mathbb{C}^n$ *be a closed subset. Then the following are equivalent:*

(i) X *is irreducible.*
(ii) $I(X)$ *is a prime ideal.*
(iii) $\mathcal{O}(X)$ *is a domain, i.e., has no zero-divisor.*

PROOF. (i)⇒(ii): If $I(X)$ is not prime we can find two polynomials $f, h \in \mathbb{C}[x_1, \dots, x_n] \setminus I(X)$ such that $f \cdot h \in I(X)$. This implies that $X \subseteq \mathcal{V}(f \cdot h) = \mathcal{V}(f) \cup \mathcal{V}(h)$, but X is neither contained in $\mathcal{V}(f)$ nor in $\mathcal{V}(h)$. Thus $X = (\mathcal{V}(g) \cap X) \cup (\mathcal{V}(h) \cap X)$ is a decomposition into strict closed subsets, contradicting the assumption.

(ii)⇒(iii): This is clear since $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]/I(X)$.

(iii)⇒(i): If $X = A \cup B$ is a decomposition into strict closed subsets there are non-zero functions $f, h \in \mathcal{O}(X)$ such that $f|_A = 0$ and $h|_B = 0$. But then $f \cdot h$ vanishes on all of X and so $f \cdot h = 0$. This contradicts the assumption that $\mathcal{O}(X)$ does not contain zero-divisor. $\square$

EXAMPLE 1.12. Let $f \in \mathbb{C}[x_1, \dots, x_n]$. Then the hypersurface $\mathcal{V}(f)$ is irreducible if and only if f is a power of an irreducible polynomial. This follows from Example 1.6 and the fact that the ideal (f) is prime if and only if f is irreducible. More generally, if $f = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ is the primary decomposition, then

$$\mathcal{V}(f) = \mathcal{V}(p_1) \cup \mathcal{V}(p_2) \cup \cdots \cup \mathcal{V}(p_n)$$

where each $\mathcal{V}(p_i)$ is irreducible, and this decomposition is *irredundant*, i.e., no term can be dropped.

THEOREM 1.2. *Every affine variety X is a finite union of irreducible closed subsets X_i:*

$$X = X_1 \cup X_2 \cup \cdots \cup X_s. \tag{2}$$

If this decomposition is irredundant, then the X_i's are the maximal irreducible subsets of X and are therefore uniquely determined.

The unique irredundant decomposition of an affine variety X in the form (2) is called *irreducible decomposition* and the X_i's are called the *irreducible components.*

For the proof of the theorem above we first recall that a $\mathbb{C}$-algebra R is called *Noetherian* if the following equivalent conditions hold:

(i) *Every ideal of R is finitely generated.*
(ii) *Every strictly ascending chain of ideals of R terminates.*
(iii) *Every non-empty set of ideals of R contains maximal elements.*

(The easy proofs are left to the reader; for the equivalence of (ii) and (iii) one has to use Zorn's Lemma.)

The famous "Basissatz" of Hilbert implies that every finitely generated $\mathbb{C}$-algebra is Noetherian (see [**Ar91**, Chap. 12, Theorem 5.18]). In particular, this holds for the coordinate ring $\mathcal{O}(X)$ of any affine variety X. Using the inclusion reversing bijection between closed subsets of X and perfect ideals of $\mathcal{O}(X)$ (see Corollary 1.5 and Exercise 1.17) we get the following result.

PROPOSITION 1.3. *Let X be an affine variety. Then*

(1) *Every closed subset $A \subseteq X$ is of the form $\mathcal{V}_X(f_1, f_2, \dots, f_r)$.*
(2) *Every strictly descending chain of closed subsets of X terminates.*
(3) *Every non-empty set of closed subsets of X contains minimal elements.*

REMARK 1.4. It is easy to see that for an arbitrary topological space T the properties (2) and (3) from the previous proposition are equivalent. If they hold then T is called *Noetherian.*

Proof of Theorem 1.2. We first show that such a decomposition exists. Consider the following set

$$\mathcal{M} := \{A \subseteq X \mid A \text{ closed and not a finite union of irreducible closed subsets}\}.$$

If $\mathcal{M} \neq \emptyset$ then it contains a minimal element A_0. Since A_0 is not irreducible, we can find strict closed subset $B, B' \subsetneq A_0$ such that $A_0 = B \cup B'$. But then $B, B' \notin \mathcal{M}$ and so both are finite unions of irreducible closed subsets. Hence A_0 is a finite union of irreducible closed subsets, too, contradicting the assumption.

To show the uniqueness let $X = X_1 \cup X_2 \cup \cdots \cup X_s$ where all X_i are irreducible closed subsets and assume that the decomposition is irredundant. Then, clearly, every X_i is maximal. Let $Y \subseteq X$ be a maximal irreducible closed subset. Then $Y = (Y \cap X_1) \cup (Y \cap X_2) \cup \cdots \cup (Y \cap X_s)$ and so $Y = Y \cap X_j$ for some j. It follows that $Y \subseteq X_j$ and so $Y = X_j$ because of maximality. □

Remark 1.5. The algebraic counterpart to the decomposition into irreducible components is the following statement about radical ideals in finitely generated algebras R: *Every radical ideal $\mathfrak{a} \subseteq R$ is a finite intersection of prime ideals:*

$$\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_s.$$

If this intersection is irredundant then the $\mathfrak{p}_i$'s are the minimal prime ideals containing $\mathfrak{a}$. (The easy proof is left to the reader.)

Example 1.13. Consider the two hypersurfaces $H_1 := \mathcal{V}(xy - z)$, $H_2 := \mathcal{V}(xz - y^2)$ in $\mathbb{C}^3$ and their intersection $X := H_1 \cap H_2$. Then

$$X = \mathcal{V}(y, z) \cup C \quad \text{where } C := \{(t, t^2, t^3) \mid t \in \mathbb{C}\},$$

and this is the irreducible decomposition.

In fact, it is obvious that the x-axis $\mathcal{V}(y, z)$ is closed and irreducible and belongs to X, and the same holds for the curve C (see Exercise 1.12). If $(a, b, c) \in X \setminus \mathcal{V}(y, z)$ then either b or c is $\neq 0$. But then $b \neq 0$ because $ab = c$. Hence $a = cb^{-1}$ and so $b^2 = ac = c^2 b^{-1}$ which implies that $c^2 = b^3$. Thus $b = (cb^{-1})^2$ and $c = (cb^{-1})^3$, i.e. $(a, b, c) \in C$.

Another way to see this is by looking at the coordinate ring:

$$\mathbb{C}[x, y, z]/(xy - z, xz - y^2) \xrightarrow{\sim} \mathbb{C}[x, y]/(x^2 y - y^2).$$

Now $(x^2 y - y^2) = y(x - y^2) = (y) \cap (x - y^2)$ and the ideals (y) and $(x - y^2)$ are obviously prime with residue class ring isomorphic to a polynomial ring in one variable. This shows again that X has two irreducible components, both with coordinate ring isomorphic to $\mathbb{C}[t]$.

Exercise 1.21. The closed subvariety $X := \mathcal{V}(x^2 - yz, xz - x) \subseteq \mathbb{C}^3$ has three irreducible components. Describe the corresponding prime ideals in $\mathbb{C}[x, y, z]$.

Example 1.14. The group $\mathrm{O}_2 := \{A \in M_2 \mid AA^t = E\}$ has two irreducible components, namely $\mathrm{SO}_2 := \mathrm{O}_2 \cap \mathrm{SL}_2$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathrm{SO}_2$, and the two components are disjoint.

In fact, the condition $AA^t = E$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ implies that $\begin{bmatrix} a \\ b \end{bmatrix} = \pm \begin{bmatrix} d \\ -c \end{bmatrix}$. Since $\det \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a^2 + b^2$ we see that $\mathrm{SO}_2 = \{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a^2 + b^2 = 1\}$ is irreducible as well as $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathrm{SO}_2 = \{\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \mid a^2 + b^2 = 1\}$ and the claim follows immediately.

Exercise 1.22. Let $G \subseteq \mathrm{GL}_n$ be a closed subgroup.

(1) The irreducible components of G are disjoint.
(2) The irreducible component of G containing E is a normal subgroup G^0.
(3) The irreducible components are the cosets of G^0.

Exercise 1.23. Let $X = \bigcup X_i$ be the decomposition into irreducible components and let $U_i \subseteq X_i$ be open subsets. Then $U := \bigcup U_i$ is open in X. It is dense in X if and only if all U_i are non-empty.

Rational functions and local rings. If X is an irreducible affine variety then $\mathcal{O}(X)$ is a domain by Lemma 1.3. Therefore, we can form the field of fractions of $\mathcal{O}(X)$ which is called *the field of rational functions* on X and will be denoted by $\mathbb{C}(X)$. Clearly, if $X = \mathbb{C}^n$ then $\mathbb{C}(X) = \mathbb{C}(x_1, x_2, \ldots, x_n)$, the rational function field. An irreducible affine variety X is called *rational* if its field of rational functions $\mathbb{C}(X)$ is isomorphic to a rational function field.

A *rational function* $f \in \mathbb{C}(X)$ can be regarded as a function "defined almost everywhere" on X. In fact, we say that f *is defined in* $x \in X$ if there are $p, q \in \mathcal{O}(X)$ such that $f = \frac{p}{q}$ and $q(x) \neq 0$.

Exercise 1.24. If $f \in \mathbb{C}(\mathbb{C}^2) = \mathbb{C}(x, y)$ is defined in $\mathbb{C}^2 \setminus \{(0,0)\}$ then f is regular.

Exercise 1.25. Let f be a rational function on the irreducible affine variety X and denote by $\mathrm{Def}(f) \subseteq X$ the set of points where f is defined.

(1) The set $\mathrm{Def}(f)$ is open in X.
(2) If $\mathrm{Def}(f) = X$ then f is regular on X. (Hint: Look at the "ideal of denominators" $\mathfrak{a} := \{p \in \mathcal{O}(X) \mid p \cdot f \in \mathcal{O}(X)\}$.)

Example 1.15. Consider the irreducible plane curve $C := \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$ and put $\bar{x} := x|_C$ and $\bar{y} := y|_C$. Then the rational function $f := \frac{\bar{y}}{\bar{x}} \in \mathbb{C}(C)$ is not defined in $(0,0)$. The interesting point here is that f has a continuous extension to all of C with value 0 at $(0,0)$, even in the $\mathbb{C}$-topology.

Proof. There is an isomorphism $\mathcal{O}(C) \xrightarrow{\sim} \mathbb{C}[t^2, t^3]$ (see Example 1.8) which maps $\bar{x}$ to t^2 and $\bar{y}$ to t^3, and so $f = \frac{\bar{y}}{\bar{x}}$ is mapped to t. Since $t \notin \mathbb{C}[t^2, t^3]$ the first claim follows from Exercise 1.25(2) above. The second part is easy because the map $\mathbb{C} \to C \colon t \mapsto (t^2, t^3)$ is a homeomorphism even in the $\mathbb{C}$-topology. □

Assume that X is irreducible and let $x \in X$. Define

$$\mathcal{O}_{X,x} := \{r \in \mathbb{C}(X) \mid f \text{ is defined in } x\}.$$

It is easy to see that $\mathcal{O}_{X,x}$ is the *localization* of $\mathcal{O}(X)$ at the maximal ideal $\mathfrak{m}_x$. (For the definition of the localization of a ring at a prime ideal and, more generally, for the construction of rings of fractions we refer to [**Eis95**, I.2.1].) This example motivates the following definition.

Definition 1.8. Let X be an affine variety and $x \in X$ an arbitrary point. Then the localization $\mathcal{O}(X)_{\mathfrak{m}_x}$ of the coordinate ring $\mathcal{O}(X)$ at the maximal ideal in x is called the *local ring of* X *at* x. It will be denoted by $\mathcal{O}_{X,x}$, its unique maximal ideal by $\mathfrak{m}_{X,x}$.

We will see later that the local ring of X at x completely determines X in a neighborhood of x.

Exercise 1.26. If X is irreducible, then $\mathcal{O}(X) = \bigcap_{x \in X} \mathcal{O}_{X,x}$.

Exercise 1.27. Let X be an affine variety, $x \in X$ a point and $X' \subseteq X$ the union of irreducible components of X passing through x. Then the restriction map induces a natural isomorphism $\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{X',x}$.

Exercise 1.28. Let R be an algebra and $\mu \colon R \to R_S$ the canonical map $r \mapsto \frac{r}{1}$ where R_S is the localization at a multiplicatively closed subset $1 \in S \subseteq R$ $(0 \notin S)$.

(1) If $\mathfrak{a} \subseteq R$ is an ideal and $\mathfrak{a}_S := R_S\,\mu(\mathfrak{a}) \subseteq R_S$ then

$$\mu^{-1}(\mu(\mathfrak{a})) = \mu^{-1}(\mathfrak{a}_S) = \{b \in R \mid sb \in \mathfrak{a} \text{ for some } s \in S\}.$$

Moreover, $(R/\mathfrak{a})_{\bar{S}} \xrightarrow{\sim} R_S/\mathfrak{a}_S$ where $\bar{S}$ is the image of S in $R/\mathfrak{a}$.
(Hint: For the second assertion use the universal property of the localization.)

(2) If $\mathfrak{m} \subseteq R$ is a maximal ideal and $S := R \setminus \mathfrak{m}$, then $\mathfrak{m}_S$ is the maximal ideal of R_S and the natural maps $R/\mathfrak{m}^k \xrightarrow{\sim} R_S/\mathfrak{m}_S^k$ are isomorphisms for all $k \geq 1$.
(Hint: The image $\bar{S}$ in $R/\mathfrak{m}^k$ consists of invertible elements.)

Exercise 1.29. Let $p < q$ be positive integers with no common divisor and define $C_{p,q} := \{(t^p, t^q) \mid t \in \mathbb{C}\} \subseteq \mathbb{C}^2$. Then $C_{p,q}$ is a closed irreducible plane curve which is rational, i.e. $\mathbb{C}(C_{p,q}) \simeq \mathbb{C}(t)$. Moreover, $\mathcal{O}(C_{p,q})$ is a polynomial ring if and only if $p = 1$.

Exercise 1.30. Consider the *elliptic curve* $E := \mathcal{V}(y^2 - x(x^2-1)) \subseteq \mathbb{C}^2$. Show that E is not rational, i.e. that $\mathbb{C}(E)$ is not isomorphic to $\mathbb{C}(t)$. (Hint: If $\mathbb{C}(E) = \mathbb{C}(t)$ then there are rational functions $f(t), h(t)$ which satisfy the equation $f(t)^2 = h(t)(h(t)^2 - 1)$.)

2. MORPHISMS

Morphisms and comorphisms. In the previous sections we have defined and discussed the main objects of algebraic geometry, the affine varieties. Now we have to introduce the "regular maps" between affine varieties which should be compatible with the concept of regular functions.

DEFINITION 2.1. Let X, Y be affine varieties. A map $\varphi\colon X \to Y$ is called *regular* or *a morphism* if the pull-back of a regular function on Y is regular on X:

$$f \circ \varphi \in \mathcal{O}(X) \text{ for all } f \in \mathcal{O}(Y).$$

Thus we obtain a homomorphism $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ of $\mathbb{C}$-algebras given by $\varphi^*(f) := f \circ \varphi$, which is called *comorphism* of φ.

A morphism φ is called an *isomorphism* if φ is bijective and the inverse map φ^{-1} is also a morphism. If, in addition, $Y = X$ then φ us called an *automorphism*.

EXERCISE 2.1. Let $g \in \mathrm{GL}_n$ be an invertible matrix. Then left multiplication $A \mapsto gA$, right multiplication $A \mapsto Ag$ and conjugation $A \mapsto gAg^{-1}$ are automorphisms of M_n.

EXAMPLE 2.1. A map $\varphi = (f_1, f_2, \ldots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ is regular if and only if the components f_i are polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. This is clear, since $\varphi^*(y_j) = f_j$ where $y_1, y_2, \ldots, y_m$ are the coordinate functions on $\mathbb{C}^m$.

More generally, let X be an affine variety and a $\varphi = (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$ a map. *Then φ is a morphism if and only if the components f_j are regular functions on X.* (This is clear since $f_j = \varphi^*(y_j)$.)

If a morphism $\varphi = (f_1, f_2, \ldots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ maps a closed subset $X \subseteq \mathbb{C}^n$ into a closed subset $Y \subseteq \mathbb{C}^m$ then the induced map $\bar{\varphi}\colon X \to Y$ is clearly a morphism, just by definition. This holds in a slightly more general setting, as claimed in the next exercise.

EXERCISE 2.2. Let $\varphi\colon X \to Y$ be a morphism. If $X' \subseteq X$ and $Y' \subseteq Y$ are closed subvarieties such that $\varphi(X') \subseteq Y'$ then the induced map $\varphi'\colon X' \to Y'$, $x \mapsto \varphi(x)$, is again a morphism. The same holds if X' and Y' are special open sets.

These examples have the following converse which will be useful in many applications.

LEMMA 2.1. *Let $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ be closed subvarieties and let $\varphi\colon X \to Y$ be a morphism. Then there are polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ such that the following diagram commutes:*

$$\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{\Phi:=(f_1,\ldots,f_m)} & \mathbb{C}^m \\
\uparrow{\scriptstyle\subseteq} & & \uparrow{\scriptstyle\subseteq} \\
X & \xrightarrow{\varphi} & Y
\end{array}$$

PROOF. Let $y_1, \ldots, y_m$ denote the coordinate functions on $\mathbb{C}^m$. Put $\bar{y}_j := y_j|_Y$ and define $\bar{f}_j := \varphi^*(\bar{y}_j) \in \mathcal{O}(X)$, $j = 1, \ldots, m$. Since $\mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n]/I(X)$ we can find representatives $f_j \in \mathbb{C}[x_1, \ldots, x_n]$, i.e. $\bar{f}_j = f_j + I(X)$. We claim that the morphism $\Phi := (f_1, \ldots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ satisfies the requirements of the lemma. In fact, let $a \in X \subseteq \mathbb{C}^n$ and set $\varphi(a) =: b = (b_1, \ldots, b_m)$. Then

$$b_j = y_j(b) = \bar{y}_j(b) = \bar{y}_j(\varphi(a)) = \varphi^*(\bar{y}_j)(a) = \bar{f}_j(a) = f_j(a),$$

and so $\varphi(a) = \Phi(a)$. $\square$

Example 2.2. The morphism $t \mapsto (t^2, t^3)$ from $\mathbb{C} \to \mathbb{C}^2$ induces a bijective morphism $\mathbb{C} \to C := \mathcal{V}(y^2 - x^3)$ which is not an isomorphism (see Example 1.8).

Similarly there is a morphism $\psi\colon \mathbb{C} \to D := \mathcal{V}(y^2 - x^2 - x^3)$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. This time ψ is surjective, but not injective since $\psi(1) = \psi(-1) = (0, 0)$.

Exercise 2.3. (1) Every morphism $\mathbb{C} \to \mathbb{C}^*$ is constant.
(2) Describe all morphisms $\mathbb{C}^* \to \mathbb{C}^*$.
(3) Every non-constant morphism $\mathbb{C} \to \mathbb{C}$ is surjective.
(4) An injective morphism $\mathbb{C} \to \mathbb{C}$ is an isomorphism, and the same holds for morphisms $\mathbb{C}^* \to \mathbb{C}^*$.

Exercise 2.4. **The graph of a morphism.** Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ be a morphism and define

$$\Gamma\varphi := \{(a, \varphi(a)) \in \mathbb{C}^{n+m}\}.$$

which is called the *graph* of the morphism φ. Show that $\Gamma\varphi$ is closed in $\mathbb{C}^{n+m}$, that the projection $\mathrm{pr}_{\mathbb{C}^n}\colon \mathbb{C}^{n+m} \to \mathbb{C}^n$ induces an isomorphism $p\colon \Gamma\varphi \xrightarrow{\sim} \mathbb{C}^n$ and that $\varphi = \mathrm{pr}_{\mathbb{C}^m} \circ p^{-1}$.

Proposition 2.1. *Let X, Y be affine varieties. The map $\varphi \mapsto \varphi^*$ induces a bijection*

$$\mathrm{Mor}(X, Y) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{C}}(\mathcal{O}(Y), \mathcal{O}(X)).$$

between the morphisms from X to Y and the algebra homomorphism from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.

Remark 2.1. The mathematical term used in the situation above is that of a *contravariant functor* from the category of affine varieties and morphisms to the category of finitely generated reduced $\mathbb{C}$-algebras and homomorphism, given by $X \mapsto \mathcal{O}(X)$ and $\varphi \mapsto \varphi^*$. In particular, we have $\varphi^*(\mathrm{Id}_X) = \mathrm{Id}_{\mathcal{O}(X)}$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ whenever the expressions make sense. The proposition above then says that this functor is an *equivalence*, the inverse functor being $R \mapsto \operatorname{spec} R$ defined in Proposition 1.1. It will be helpful to keep this "functorial point of view" in mind although it will not play an important role in the following.

Proof. (a) If $\varphi_1^* = \varphi_2^*$ then, for all $f \in \mathcal{O}(Y)$ and all $x \in X$, we get

$$f(\varphi_1(x)) = \varphi_1^*(f)(x) = \varphi_2^*(f)(x) = f(\varphi_2(x)).$$

Hence, $\varphi_1(x) = \varphi_2(x)$ since the regular functions separate the points (Remark 1.3).

(b) Let $\rho\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ be an algebra homomorphism. We want to construct a morphism $\varphi\colon X \to Y$ such that $\varphi^* = \rho$. For this we can assume that $Y \subseteq \mathbb{C}^m$ is a closed subvariety. Let $\bar{y}_j := y_j|_Y$ be the restrictions of the coordinate functions on $\mathbb{C}^m$ and define $f_j := \rho(\bar{y}_j) \in \mathcal{O}(X)$. Then we get a morphism $\Phi := (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$ such that $\Phi^*(y_j) = f_j$ (see Example 2.1). If $h = h(y_1, \ldots, y_m) \in I(Y)$ then

$$h(f_1, \ldots, f_m) = h(\rho(\bar{y}_1), \ldots, \rho(\bar{y}_m)) = \rho(h(\bar{y}_1, \ldots, \bar{y}_m)) = 0$$

because $h(\bar{y}_1, \ldots, \bar{y}_m) = h|_Y = 0$ by assumption. Therefore $h(\Phi(a)) = 0$ for all $a \in X$ and all $h \in I(Y)$ and so $\Phi(X) \subseteq Y$. This shows that Φ induces a morphism $\varphi\colon X \to Y$ such that $\varphi^*(\bar{y}_j) = \Phi^*(y_j) = f_j = \rho(\bar{y}_j)$, and so $\varphi^* = \rho$. $\square$

EXAMPLE 2.3. Let X be an affine variety, V a finite dimensional vector space and $\varphi\colon X \to V$ a morphism. The linear functions on V form a subspace $V^* \subseteq \mathcal{O}(V)$ which generates $\mathcal{O}(X)$. Therefore, the induced linear map $\varphi^*|_{V^*}\colon V^* \to \mathcal{O}(X)$ completely determines φ^*, and we get a bijection

$$\operatorname{Mor}(X, V) \xrightarrow{\sim} \operatorname{Hom}(V^*, \mathcal{O}(X)) \quad \varphi \mapsto \varphi^*|_{V^*}.$$

The second assertion follows from Proposition 2.1 and the well-known "Substitution Principle" for polynomials rings (see [**Ar91**, Chap. 10, Proposition 3.4]).

EXERCISE 2.5. Show that for an affine variety X the morphisms $X \to \mathbb{C}^*$ correspond bijectively to the invertible functions on X.

EXERCISE 2.6. Let X, Y be affine varieties and $\varphi\colon X \to Y$, $\psi\colon Y \to X$ morphisms such that $\psi \circ \varphi = \mathrm{Id}_X$. Then $\varphi(X) \subseteq Y$ is closed and $\varphi\colon X \xrightarrow{\sim} \varphi(X)$ is an isomorphism.

Images, pre-images and fibers. It is easy to see that morphisms are continuous. In fact, the Zariski topology is the finest topology such that regular functions are continuous, and since morphisms are defined by the condition that the pull-back of a regular function is again regular, it immediately follows that morphisms are continuous. We will get this result again from the next proposition where we describe images and preimages of closed subsets under morphisms.

PROPOSITION 2.2. *Let $\varphi\colon X \to Y$ be a morphism of affine varieties.*

(1) *If $B := \mathcal{V}_Y(S) \subseteq Y$ is the closed subset defined by $S \subseteq \mathcal{O}(Y)$ then $\varphi^{-1}(B) = \mathcal{V}_X(\varphi^*(S))$. In particular, φ is continuous.*
(2) *Let $A := \mathcal{V}(\mathfrak{a}) \subseteq X$ be the closed subset defined by the ideal $\mathfrak{a} \subseteq \mathcal{O}(X)$. Then the closure of the image $\varphi(A)$ is defined by ${\varphi^*}^{-1}(\mathfrak{a}) \subseteq \mathcal{O}(Y)$:*

$$\overline{\varphi(A)} = \mathcal{V}_Y({\varphi^*}^{-1}(\mathfrak{a})).$$

PROOF. For $x \in X$ we have

$$x \in \varphi^{-1}(B) \iff \varphi(x) \in B \iff f(\varphi(x)) = 0 \text{ for all } f \in S,$$

and this is equivalent to $\varphi^*(f)(x) = 0$ for all $f \in S$, hence to $x \in \mathcal{V}_X(\varphi^*(S))$, proving the first claim.

For the second claim, let $f \in \mathcal{O}(Y)$. Then

$$f|_{\overline{\varphi(A)}} = 0 \iff f|_{\varphi(A)} = 0 \Leftrightarrow \varphi^*(f)|_A = 0 \iff \varphi^*(f) \in I(A) = \sqrt{\mathfrak{a}}$$

The latter is equivalent to the condition that a power of f belongs to ${\varphi^*}^{-1}(\mathfrak{a})$. Thus the zero set of ${\varphi^*}^{-1}(\mathfrak{a})$ equals the closed set $\overline{\varphi(A)}$. □

EXERCISE 2.7. If $\varphi_1, \varphi_2\colon X \to Y$ are two morphisms, then the "kernel of coincidence"

$$\ker(\varphi_1, \varphi_2) := \{x \in X \mid \varphi_1(x) = \varphi_2(x)\} \subseteq X$$

is closed in X

EXERCISE 2.8. Let $\varphi\colon X \to Y$ be a morphism of affine varieties.

(1) If X is irreducible, then $\overline{\varphi(X)}$ is irreducible.
(2) Every irreducible component of X is mapped into an irreducible component of Y.

EXERCISE 2.9. Let $\varphi\colon X \xrightarrow{\sim} X$ be an automorphism and $Y \subseteq X$ a closed subset such that $\varphi(Y) \subseteq Y$. Then $\varphi(Y) = Y$ and $\varphi|_Y\colon Y \to Y$ is an automorphism, too.

A special case of pre-images are the fibers of a morphism $\varphi\colon X \to Y$. Let $y \in Y$. Then

$$\varphi^{-1}(y) := \{x \in X \mid \varphi(x) = y\}$$

is called the *fiber of* $y \in Y$. By the proposition above, the fiber of y is a closed subvariety of X defined by $\varphi^*(\mathfrak{m}_y)$:

$$\varphi^{-1}(y) = \mathcal{V}_X(\varphi^*(\mathfrak{m}_y)).$$

Of course, the fiber of a point $y \in Y$ can be empty. In algebraic terms this means that $\varphi^*(\mathfrak{m}_y)$ generates the unit ideal $(1) = \mathcal{O}(X)$.

EXERCISE 2.10. Let $\varphi\colon X \to Y$ be a morphism of affine varieties. If $U \subseteq Y$ is a special open set then so is $\varphi^{-1}(U)$.

EXERCISE 2.11. Describe the fibers of the morphism $\varphi\colon M_2 \to M_2$, $A \mapsto A^2$. (Hint: Use the fact that $\varphi(gAg^{-1}) = g\varphi(A)g^{-1}$ for $g \in \mathrm{GL}_2$.)

DEFINITION 2.2. Let $\varphi\colon X \to Y$ be a morphism of affine varieties and consider the fiber $F := \varphi^{-1}(y)$ of a point $y \in \varphi(X) \subseteq Y$. Then the fiber F is called *reduced* if $\varphi^*(\mathfrak{m}_y)$ generates a perfect ideal in $\mathcal{O}(X)$, i.e. if

$$\sqrt{\mathcal{O}(X)\cdot\varphi^*(\mathfrak{m}_y)} = \mathcal{O}(X)\cdot\varphi^*(\mathfrak{m}_y).$$

The fiber F is called *reduced in the point* $x \in F$ if this holds in the local ring $\mathcal{O}_{X,x}$, i.e.

$$\sqrt{\mathcal{O}_{X,x}\cdot\varphi^*(\mathfrak{m}_y)} = \mathcal{O}_{X,x}\cdot\varphi^*(\mathfrak{m}_y).$$

EXAMPLE 2.4. Look again at the morphism $\varphi\colon \mathbb{C} \to C := \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$, $t \mapsto (t^2, t^3)$. Then φ^* is the injection $\mathcal{O}(C) \xrightarrow{\sim} \mathbb{C}[t^2, t^3] \hookrightarrow \mathbb{C}[t]$ and so

$$\mathbb{C}[t]\cdot\varphi^*(\mathfrak{m}_{(0,0)}) = (t^2, t^3) \subsetneq \sqrt{(t^2, t^3)} = (t).$$

Thus the zero fiber $\varphi^{-1}(0)$ is not reduced. On the other hand, all other fibers are reduced since φ induces an isomorphism of $\mathbb{C}^*$ with the special open set $C \setminus \{(0,0)\}(= C_{\bar{x}} = C_{\bar{y}})$, where the inverse map is given by $(a, b) \mapsto \frac{b}{a}$.

EXERCISE 2.12. Show that all fibers of the morphism $\psi\colon \mathbb{C} \to D := \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$, $t \mapsto (t^2 - 1, t(t^2 - 1))$, are reduced and that ψ induces an isomorphism $\mathbb{C} \setminus \{1, -1\} \xrightarrow{\sim} D \setminus \{(0,0)\}$.

EXERCISE 2.13. Consider the following morphism $\varphi\colon \mathrm{SL}_2 \to \mathbb{C}^3$, $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := (ab, ad, cd)$.

(1) The image of φ is a closed hypersurface $H \subseteq \mathbb{C}^3$.
(2) The fibers of φ are the left cosets of the subgroup $T := \{\begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \mid t \in \mathbb{C}^*\}$.
(3) All fibers are reduced.

(Hint: Show that the left multiplication with some $g \in \mathrm{SL}_2$ induces an automorphism λ_g of H and isomorphisms $\varphi^{-1}(y) \xrightarrow{\sim} \varphi^{-1}(\lambda_g(y))$ for all $y \in H$. This implies that it suffices to study just one fiber, e.g. $\varphi^{-1}(\varphi(E))$.)

EXERCISE 2.14. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $\varphi(x, y) := (x, xy)$.
(1) $\varphi(\mathbb{C}^2) = \mathbb{C}^2 \setminus \{(0, y) \mid y \neq 0\}$ which is not locally closed.
(2) What happens with the lines parallel to the x-axis or parallel to the y-axis?

(3) $\varphi^{-1}(0) = y$-axis. Is this fiber reduced?
(4) φ induces an isomorphism $\mathbb{C}^2 \setminus y\text{-axis} \xrightarrow{\sim} \mathbb{C}^2 \setminus y\text{-axis}$

Dominant morphisms. Let $\varphi\colon X \to Y$ be a morphism of affine varieties, x a point of X and $y := \varphi(x)$ its image in Y. Then $\varphi^*(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$, and so φ^* induces a local homomorphism

$$\varphi_x^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$

(A homomorphism between local rings is called *local* if it maps the maximal ideal into the maximal ideal.)

The next proposition tells us that, in a neighborhood of a point $x \in X$, a morphism φ is uniquely determined by the local homomorphism φ_x^*.

PROPOSITION 2.3. (1) *If $\varphi, \psi\colon X \to Y$ are two morphisms such that $\varphi(x) = \psi(x)$ and $\varphi_x^* = \psi_x^*$ for some $x \in X$, then φ and ψ coincide on every irreducible component of X passing through x.*
(2) *If $x \in X$, $y \in Y$ and if $\rho\colon \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}$ is an isomorphism, then there exist special open sets $X' \subseteq X$ and $Y' \subseteq Y$ containing x and y, respectively, and an isomorphism $\psi\colon X' \xrightarrow{\sim} Y'$ such that $\psi_x^* = \rho$.*

PROOF. (1) Let R be a finitely generated reduced $\mathbb{C}$-algebra and $\mathfrak{m} \subseteq R$ a maximal ideal. The canonical map $\mu\colon R \to R_{\mathfrak{m}}$ is injective if and only if $\mathfrak{m}$ contains all minimal prime ideals of R. (In fact, $\ker \mu = \{r \in R \mid sr = 0 \text{ for some } s \in R \setminus \mathfrak{m}\}$.)

Denote by $\bar{X} \subseteq X$ the union of irreducible components passing through x and by $\bar{Y} \subseteq Y$ the union of irreducible components passing through $\varphi(x)$. Then $\varphi(\bar{X}) \subseteq \bar{Y}$, because the image of an irreducible component of X is contained in an irreducible component of Y (see Exercise 2.8). Thus we obtain a morphism $\bar{\varphi}\colon \bar{X} \to \bar{Y}$ with the following commutative diagram of $\mathbb{C}$-algebras and homomorphisms which shows that $\bar{\varphi}$ is completely determined by φ_x^*:

$$\begin{array}{ccccc}
\mathcal{O}(Y) & \longrightarrow & \mathcal{O}(\bar{Y}) & \xrightarrow{\subseteq} & \mathcal{O}_{\bar{Y},\varphi(x)} = \mathcal{O}_{Y,\varphi(x)} \\
\Big\downarrow \varphi^* & & \Big\downarrow \bar{\varphi}^* & & \Big\downarrow \varphi_x^* \\
\mathcal{O}(X) & \longrightarrow & \mathcal{O}(\bar{X}) & \xrightarrow{\subseteq} & \mathcal{O}_{\bar{X},x} = \mathcal{O}_{X,x}
\end{array}$$

(2) We can assume that all irreducible components of X pass through x and all irreducible components of Y pass through y. Then

$$\mathcal{O}(Y) \subseteq \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x} \supseteq \mathcal{O}(X).$$

Let $h_1, \dots, h_m \in \mathcal{O}(Y)$ be a set of generators and put $f_j := \rho(h_j)$. Then we can find an element $q \in \mathcal{O}(X) \setminus \mathfrak{m}_x$ such that $f_j \in \mathcal{O}(X)_q$ for all j, i.e., $\rho(\mathcal{O}(Y)) \subseteq \mathcal{O}(X)_q$. Now $q = \rho(\frac{r}{s})$ where $r, s \in \mathcal{O}(Y), s \notin \mathfrak{m}_y$ and so $h := \rho(s)q = \rho(r) \notin \mathfrak{m}_x$. But this implies that $\rho(\mathcal{O}(Y)_s) = \mathcal{O}(X)_h$. Hence, there is an isomorphism $\psi\colon X_h \xrightarrow{\sim} Y_s$ with the required properties. $\square$

DEFINITION 2.3. Let X, Y be irreducible affine varieties. A morphism $\varphi\colon X \to Y$ is called *dominant* if the image is dense in Y, i.e. $\overline{\varphi(X)} = Y$. This is equivalent to the condition that $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective (see Proposition 2.2(2)).

It follows that every dominant morphism $\varphi\colon X \to Y$ induces a *finitely generated field extension* $\varphi^*\colon \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$. If this is a finite field extension of degree $d :=$

$[\mathbb{C}(X):\mathbb{C}(Y)]$ we will say that φ is a *morphism of finite degree* d. If $d=1$, i.e. if φ^* induces an isomorphism $\mathbb{C}(Y) \xrightarrow{\sim} \mathbb{C}(X)$ then φ is called a *birational morphism.*

EXERCISE 2.15. Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be a non-constant morphism. Then φ has finite degree d, and there is a non-empty open set $U \subseteq \mathbb{C}$ such that $\#\varphi^{-1}(x) = d$ for all $x \in U$.

There is a similar result as the second part of Proposition 2.3 saying that affine varieties with isomorphic function fields are locally isomorphic. The proof is similar as the proof above and will be left to the reader.

PROPOSITION 2.4. *Let X, Y be irreducible affine varieties and let $\rho\colon \mathbb{C}(Y) \xrightarrow{\sim} \mathbb{C}(X)$ be an isomorphism. Then there exist special open sets $X' \subseteq X$ and $Y' \subseteq Y$ and an isomorphism $\psi\colon X' \xrightarrow{\sim} Y'$ such that $\rho = \psi^*$.*

Products. If f is a function on X and h a function on Y then we denote by $f \cdot h$ the $\mathbb{C}$-valued function on the product $X \times Y$ defined by $(f \cdot h)(x,y) := f(x) \cdot h(y)$.

PROPOSITION 2.5. *The product $X \times Y$ of two affine varieties together with the algebra*

$$\mathcal{O}(X \times Y) := \mathbb{C}[f \cdot h \mid f \in \mathcal{O}(X), h \in \mathcal{O}(Y)]$$

of $\mathbb{C}$-valued functions is an affine variety. Moreover, the canonical homomorphism $\mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$, $f \otimes h \mapsto f \cdot h$, is an isomorphism.

PROOF. Let $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ be closed subvarieties. Then $X \times Y \subseteq \mathbb{C}^{n+m}$ is closed, namely equal to the zero set $\mathcal{V}(I(X) \cup I(Y))$. So it remains to show that $\mathcal{O}(X \times Y) = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_m]/I(X \times Y)$ is generated by the products $f \cdot h$ and that $f \cdot h \in \mathcal{O}(X \times Y)$ for $f \in \mathcal{O}(X)$ and $h \in \mathcal{O}(Y)$. But this is clear since $\bar{x}_i = x_i|_{X \times Y} = x_i|_X \cdot 1$ and $\bar{y}_j = y_j|_{X \times Y} = 1 \cdot y_j|_Y$, and $f|_X \cdot h|_Y = (fh)|_{X \times Y}$ for $f \in \mathbb{C}[x_1,\dots,x_n]$ and $h \in \mathbb{C}[y_1,\dots,y_m]$.

For the last claim, we only have to show that the map $\mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$, $f \otimes h \mapsto f \cdot h$, is injective. For this, let $(f_i \mid i \in I)$ be a basis of $\mathcal{O}(Y)$. Then every element $s \in \mathcal{O}(X) \otimes \mathcal{O}(Y)$ can be uniquely written as $s = \sum_{\text{finite}} s_i \otimes f_i$. If s is in the kernel of the map, then $\sum s_i(x) f_i(y) = 0$ for all $(x,y) \in X \times Y$ and so, for every fixed $x \in X$, $\sum s_i(x) f_i$ is the zero function on Y. This implies that $s_i(x) = 0$ for all $x \in X$ and so $s_i = 0$ for all i. Thus $s = 0$ proving the claim. ☐

EXAMPLE 2.5.

(1) The two projections $\mathrm{pr}_X\colon X \times Y \to X$, $(x,y) \mapsto x$, and $\mathrm{pr}_Y\colon X \times Y \to Y$, $(x,y) \mapsto y$, are morphisms with comorphisms $\mathrm{pr}_X^*(f) = f \cdot 1$ and $\mathrm{pr}_Y^*(h) = 1 \cdot h$.
(2) If $\varphi\colon X \to X'$ and $\psi\colon Y \to Y'$ are morphisms, then so is
$$\varphi \times \psi\colon X \times Y \to X' \times Y', \quad (x,y) \mapsto (\varphi(x), \psi(y)).$$
(3) **Diagonal:** $\Delta\colon X \to X \times X$, $x \mapsto (x,x)$ is a morphism, $\Delta(X) \subseteq X \times X$ is a closed subset defined by the set $\{f \cdot 1 - 1 \cdot f \mid f \in \mathcal{O}(X)\}$, and $X \xrightarrow{\sim} \Delta(X)$ is an isomorphism whose inverse is induced by the projection pr_X.
(4) **Graph:** Let $\varphi\colon X \to Y$ be a morphism. Then
$$\Gamma(\varphi) := \{(x, \varphi(x)) \mid x \in X\} \subseteq X \times Y$$
is a closed subset. Moreover, the projection pr_X induces an isomorphism $p\colon \Gamma(\varphi) \xrightarrow{\sim} X$ and $\varphi = \mathrm{pr}_Y \circ p^{-1}$.

(5) **Matrix multiplication:** The composition of linear maps

$$\mu\colon \operatorname{Hom}(U,V)\times\operatorname{Hom}(V,W)\to\operatorname{Hom}(U,W),\quad (A,B)\mapsto B\circ A$$

is a morphism. Choosing coordinates we find $\mu^*(z_{ij})=\sum_k y_{ik}x_{kj}$.

Exercise 2.16. Show that the ideal of the diagonal $\Delta(X)\subseteq X\times X$ is generated by the function $f\cdot 1-1\cdot f$, $f\in\mathcal{O}(X)$ (see Example 2.5(3)).

Lemma 2.2. *The projection* $\operatorname{pr}_X\colon X\times Y\to X$ *is an open morphism, i.e. the image of an open set under* pr_X *is open.*

Proof. It suffices to show that the image of a special open set $U:=(X\times Y)_g$ is open. Writing $g=\sum f_i\cdot h_i$ with linearly independent h_i one gets $\operatorname{pr}_X(U)=\bigcup_i X_{f_i}$ and the claim follows. □

Proposition 2.6. *If X,Y are irreducible affine varieties then $X\times Y$ is irreducible.*

Proof. Assume that $X\times Y=A\cup B$ with closed subsets A,B. Define

$$X_A:=\{x\in X\mid \{x\}\times Y\subseteq A\}\quad\text{and}\quad X_B:=\{x\in X\mid \{x\}\times Y\subseteq B\}$$

Since Y is irreducible we see that $X=X_A\cup X_B$. Now we claim that X_A and X_B are both closed in X and so one of them equals X, say $X_A=X$. But then $A=X\times Y$ and we are done. To prove the claim we remark that $X\setminus X_A=\operatorname{pr}_X(X\times Y\setminus A)$ which is open by Lemma 2.2 above. □

Corollary 2.1. *If $X=\bigcup_i X_i$ and $Y=\bigcup_j Y_j$ are the irreducible decompositions of X and Y, then $X\times Y=\bigcup_{i,j}X_i\times Y_j$ is the irreducible decomposition of the product.*

3. DIMENSION

Definitions. If k is a field and A a k-algebra then a set $a_1, a_2, \dots, a_n \in A$ of elements from A are called *algebraically independent over k* if they do not satisfy a non-trivial polynomial equation $F(a_1, a_2, \dots, a_n) = 0$ where $F \in k[x_1, \dots, x_n]$. Equivalently, the canonical homomorphism of k-algebras $k[x_1, \dots, x_n] \to A$ defined by $x_i \mapsto a_i$ is injective.

In order to define the dimension of a variety we will need the concept of *transcendence degree* $\operatorname{tdeg}_k K$ of a field extension K/k. It is defined to be the maximal number of algebraically independent elements in K. We refer to [**Ar91**, Chap. 13, Sect. 8] for the basic properties of transcendental extensions.

Definition 3.1. Let X be an irreducible affine variety and $\mathbb{C}(X)$ its field of rational functions. Then the *dimension* of X is defined by

$$\dim X := \operatorname{tdeg}_{\mathbb{C}} \mathbb{C}(X).$$

If X is reducible and $X = \bigcup X_i$ the irreducible decomposition (see 1) then

$$\dim X := \max_i \dim X_i.$$

Finally, we define the *local dimension* of X in a point $x \in X = \bigcup X_i$ to be

$$\dim_x X := \max_{X_i \ni x} \dim X_i.$$

Exercise 3.1. Let X be an affine variety.

(1) $\dim X$ is the maximal number of algebraically independent elements in $\mathcal{O}(X)$.
(2) Assume that $\mathcal{O}(X)$ is generated by r elements. Then $\dim X \le r$, and if $\dim X = r$ then $X \simeq \mathbb{C}^r$.

Exercise 3.2. The function $x \mapsto \dim_x X$ is upper semi-continuous on X. (This means that for all $\alpha \in \mathbb{R}$ the set $\{x \in X \mid \dim_x X < \alpha\}$ is open in X.)

Examples 3.1. (1) We have $\dim \mathbb{C}^n = n$. More generally, if V is a complex vector space of dimension n then $\dim X = n$.
(2) If $U \subseteq X$ is a special open subset which is dense in X, then $\dim U = \dim X$.
(3) For affine varieties X, Y we have $\dim X \times Y = \dim X + \dim Y$.

Lemma 3.1. *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-constant polynomial and $X := \mathcal{V}(f) \subseteq \mathbb{C}^n$ its zero set. Then* $\dim X = n - 1$.

Proof. We can assume that f is irreducible and that the variable x_n occurs in f. Denote by $\bar{x}_i \in \mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]/(f)$ the restrictions of the coordinate functions x_i. Then $\mathbb{C}(X) = \mathbb{C}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$. Since $f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = 0$ we see that $\bar{x}_n \in \mathbb{C}(X)$ is algebraic over the subfield $\mathbb{C}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1})$. Therefore, $\operatorname{tdeg} \mathbb{C}(X) = \operatorname{tdeg} \mathbb{C}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{n-1}) \le n - 1$. On the other hand, the composition

$$\mathbb{C}[x_1, \dots, x_{n-1}] \hookrightarrow \mathbb{C}[x_1, \dots, x_n] \xrightarrow{\text{res}} \mathcal{O}(X)$$

is injective, since the kernel is the intersection $(f) \cap \mathbb{C}[x_1, \dots, x_{n-1}]$ which is zero. Thus, $\operatorname{tdeg} \mathbb{C}(X) \ge n - 1$, and the claim follows. $\square$

The first part of the proof above, namely that $\dim \mathcal{V}(f) < n = \dim \mathbb{C}^n$ has the following generalization.

LEMMA 3.2. *If X is irreducible and $Y \subsetneq X$ a proper closed subset then we have $\dim Y < \dim X$.*

PROOF. We can assume that Y is irreducible. If $h_1, \dots, h_m \in \mathcal{O}(Y)$ are algebraically independent where $m = \dim Y$, and $h_i = \tilde{h}_i|_Y$ for $\tilde{h}_1, \dots, \tilde{h}_m \in \mathcal{O}(X)$ then $\tilde{h}_1, \dots, \tilde{h}_m$ are algebraically independent, too, and so $\dim X \geq \dim Y$. If $\dim Y = \dim X$ then every $f \in \mathcal{O}(X)$ is algebraic over $\mathbb{C}(\tilde{h}_1, \dots, \tilde{h}_m)$. Choose $f \in \mathcal{O}(X)$ in the kernel of the restriction map, i.e. $f|_Y = 0$. Then f satisfies an equation of the form

$$f^k + p_1 f^{k-1} + \cdots + p_{k-1} f + p_k = 0$$

where $p_j \in \mathbb{C}(\tilde{h}_1, \dots, \tilde{h}_m)$ and k is minimal. Multiplying with a suitable $q \in \mathbb{C}[\tilde{h}_1, \dots, \tilde{h}_m]$ we can assume that $p_j \in \mathbb{C}[\tilde{h}_1, \dots, \tilde{h}_m]$. But this implies that $p_k|_Y = 0$. Thus $p_k = 0$ and we end up with a contradiction. □

EXAMPLE 3.2. We have $\dim X = 0$ if and only if X is finite, and this is equivalent to $\dim_{\mathbb{C}} \mathcal{O}(X) < \infty$. (This is clear: If X is irreducible of dimension 0 then $\mathbb{C}(X)$ is algebraic over $\mathbb{C}$ and so $\mathbb{C} = \mathcal{O}(X) = \mathbb{C}(X)$, and the claim follows.)

EXERCISE 3.3. Let $U \subseteq X$ be a dense open set. Then $\dim X \setminus U < \dim X$.

LEMMA 3.3. *Let X be an irreducible affine variety of dimension n. Then there is a special open set $U \subseteq X$ which is isomorphic to a special open set of a hypersurface $\mathcal{V}(h) \subseteq \mathbb{C}^{n+1}$.*

PROOF. The field of rational functions $\mathbb{C}(X)$ has the form

$$\mathbb{C}(X) = \mathbb{C}(x_1, \dots, x_n)[f]$$

where f satisfies a minimal equation: $f^m + p_1 f^{m-1} + \cdots + p_m = 0$, $p_j \in \mathbb{C}(x_1, \dots, x_n)$. Multiplying with a suitable polynomial $\mathbb{C}[x_1, \dots, x_n]$ we can assume that all p_j belong to $\mathbb{C}[x_1, \dots, x_n]$. Then the polynomial $h := y^m + p_1 y^{n-1} + \cdots + p_m \in \mathbb{C}[x_1, \dots, x_n, y]$ is irreducible and defines a hypersurface $H := \mathcal{V}(h) \subseteq \mathbb{C}^{n+1}$ whose field of rational functions $\mathbb{C}(H)$ is isomorphic to $\mathbb{C}(X)$ by construction. Now the claim follows from Proposition 2.4. □

Finite morphisms. Finite morphisms will play an important role in the following. In particular, they will help us to "compare" an arbitrary affine variety X with an affine space $\mathbb{C}^n$ of the same dimension by using the famous Normalization Lemma of Noether.

DEFINITION 3.2. Let $A \subseteq B$ be two rings. We say that B *is finite over* A if B is a finite A-module, i.e.there are $b_1, \dots, b_s \in B$ such that $B = \sum_j A b_j$.

A morphism $\varphi\colon X \to Y$ between two affine varieties is called *finite* if $\mathcal{O}(X)$ is finite over $\varphi^*(\mathcal{O}(Y))$.

If $A \subseteq B \subseteq C$ are rings such that B is finite over A and C is finite over B, then C is finite over A. In particular, if $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are finite morphisms then the composition $\psi \circ \varphi\colon X \to Z$ is finite, too. Another useful remark is the following: If $\varphi\colon X \to Y$ is finite and $X' \subseteq X$, $Y' \subseteq Y$ closed subsets such that $\varphi(X') \subseteq Y'$ then the induced morphism $\varphi'\colon X' \to Y'$ is also finite.

EXAMPLE 3.3. Typical examples of finite morphisms are the ones given in Example 2.2, namely $\varphi\colon \mathbb{C} \to C = \mathcal{V}(y^2-x^3) \subseteq \mathbb{C}^2$ and $\psi\colon \mathbb{C} \to D = \mathcal{V}(y^2-x^2-x^3) \subseteq \mathbb{C}^2$. In both cases, the morphisms are the so-called *normalizations*, a concept which we will discuss later.

On the other hand, the inclusion of a special open set $X_f \hookrightarrow X$ is not finite if f is neither invertible nor zero.

EXERCISE 3.4. Every non-constant morphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ is finite, and the same holds for the non-constant morphisms $\psi\colon \mathbb{C}^* \to \mathbb{C}^*$.

The basic geometric property of a finite morphism is given in the next proposition.

PROPOSITION 3.1. *Let $\varphi\colon X \to Y$ be a finite morphism. Then φ is closed and has finite fibers.*

PROOF. If $y \in Y$ then $\varphi^{-1}(y) = \mathcal{V}_X(\varphi^*(\mathfrak{m}_y))$ (see 2). If $\varphi^{-1}(y) \neq \emptyset$ then the induced morphism $\varphi^{-1}(y) \to \{y\}$ is finite, too, and so $\mathcal{O}(\varphi^{-1}(y))$ is a finite dimensional $\mathbb{C}$-algebra. Thus, the fiber $\varphi^{-1}(y)$ is finite (Example 3.2) proving the second claim.

For the first claim it suffices to show that $\overline{\varphi(X)} = \varphi(X)$. Hence we can assume that $\overline{\varphi(X)} = Y$, i.e. that $\varphi^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. If $\varphi^{-1}(y) = \emptyset$ then $\mathcal{O}(X)\mathfrak{m}_y = \mathcal{O}(X)$. (We identify $\mathfrak{m}_y$ with its image $\varphi^*(\mathfrak{m}_y)$.) The Lemma of Nakayama (see the following Lemma 3.4) now implies that $(1+a)\mathcal{O}(X) = 0$ for some $a \in \mathfrak{m}_y$ which is a contradiction since $1 + a \neq 0$. □

LEMMA 3.4 (Lemma of Nakayama). *Let R be a ring, $\mathfrak{a} \subseteq R$ an ideal and M a finitely generated R-module. If $\mathfrak{a}M = M$ then there is an element $a \in \mathfrak{a}$ such that $(1+a)M = 0$. In particular, if M is torsionfree and $\mathfrak{a} \neq R$ then $M = 0$.*

PROOF. Let $M = \sum_{j=1}^k Rm_j$. Then $m_i = \sum_j a_{ij}m_j$ for all i where $a_{ij} \in \mathfrak{a}$. If A denotes the $k \times k$-matrix $(a_{ij})_{i,j}$ and m the column vector $(m_1, \dots, m_k)^t$ this means that $m = A \cdot m$. Thus $(E-A)m = 0$, and so $\det(E-A)m_j = 0$ for all j. But

$$\det(E-A) = \det \begin{bmatrix} 1-a_{11} & -a_{12} & \cdots \\ -a_{21} & 1-a_{22} & \cdots \\ \vdots & & \ddots \end{bmatrix} = 1 + a \quad \text{where } a \in \mathfrak{a}.$$

and the claim follows. □

EXERCISE 3.5. Let X be an affine variety and $x \in X$. Assume that $f_1, \dots, f_r \in \mathfrak{m}_x$ generate the ideal $\mathfrak{m}_x$ modulo $\mathfrak{m}_x^2$, i.e., $\mathfrak{m}_x = (f_1, \dots, f_r) + \mathfrak{m}_x^2$. Then $\{x\}$ is an irreducible component of $\mathcal{V}_X(f_1, \dots, f_r)$.
(Hint: If $C \subseteq \mathcal{V}_X(f_1, \dots, f_r)$ is an irreducible component containing x and $\mathfrak{m} \subseteq \mathcal{O}(C)$ the maximal ideal of x then $\mathfrak{m}^2 = \mathfrak{m}$. Hence $\mathfrak{m} = 0$ by the Lemma of Nakayama above.)

EXERCISE 3.6. Let $\varphi\colon X \to Y$ be a finite surjective morphism. Then $\dim X = \dim Y$.

EXERCISE 3.7. Let X be an affine variety and $X = \bigcup_i X_i$ the irreducible decomposition. A morphism $\varphi\colon X \to Y$ is finite if and only if $\varphi|_{X_i}\colon X_i \to Y$ is finite for all i.

The following easy lemma will be very useful in sequel.

Lemma 3.5. *Let $A \subseteq B$ be rings and $b \in B$. Assume that b satisfies an equation of the form*

$$b^m + a_1 b^{m-1} + a_2 b^{m-2} + \cdots + a_m = 0$$

where $a_1, a_2, \ldots, a_m \in A$. Then the subring $A[b] \subseteq B$ is finite over A.

Proof. It follows from the equation satisfied by b that for $N \geq m$ we have

$$b^N = -a_1 b^{N-1} - a_2 b^{N-2} - \cdots - a_m b^{N-m}, \tag{3}$$

and so, by induction, that $A[b] = \sum_{i=0}^{m-1} Ab^i$. □

The next result is usually called the "Normalization Lemma". It is due to Emmy Noether, but was first formulated, in a special case, by David Hilbert.

Theorem 3.1 (Noether's Normalization Lemma). *Let K be an infinite field and A a finitely generated K-algebra. Then there are algebraically independent elements $a_1, \ldots, a_n \in A$ such that A is finite over $K[a_1, \ldots, a_n]$*

Proof. We proceed by induction on the number m of generators of A as a K-algebra. If $m = 0$ then $A = K$ and there is nothing to prove. If $A = K[b_1, \ldots, b_m]$ and if $b_1, \ldots, b_m$ are algebraically independent, we are done, too. So let's assume that $F(b_1, \ldots, b_m) = 0$ where $F \in K[x_1, \ldots, x_m]$ is a non-zero polynomial. We can also assume that x_m occurs in F. Write

$$F = \sum_{r_1, r_2, \ldots, r_m} \alpha_{r_1, r_2, \ldots, r_m} x_1^{r_1} x_2^{r_2} \cdots x_m^{r_m}$$

and put $r := \max\{r_1 + r_2 + \cdots + r_m \mid \alpha_{r_1, r_2, \ldots, r_m} \neq 0\}$. Substituting $x_j = x'_j + \gamma_j x_m$ for $j = 1, \ldots, m-1$ and we find

$$F = \Big(\sum_{r_1 + r_2 + \cdots + r_m = r} \alpha_{r_1, \ldots, r_m} \gamma_1^{r_1} \cdots \gamma_{m-1}^{r_{m-1}} \Big)\, x_m^r + H(x'_1, \ldots, x'_{m-1}, x_m) \tag{4}$$

where x_m occurs in H with an exponent $< r$. Since K is infinite we can find $\gamma_1, \ldots, \gamma_{m-1} \in K$ such that $\sum_{r_1 + \cdots + r_m = r} \alpha_{r_1, \ldots, r_m} \gamma_1^{r_1} \cdots \gamma_{m-1}^{r_{m-1}} \neq 0$. Setting $b'_j := b_j - \gamma_j b_m$ for $j = 1, \ldots, m-1$ we have $A = K[b'_1, b'_2, \ldots, b'_{m-1}, b_m]$. Now equation (4) implies that b_m satisfies an equation of the form (3), hence A is finite over $K[b'_1, \ldots, b'_{m-1}]$ by Lemma 3.5, and the claim follows by induction. □

Remark 3.1. The proof above shows the following. If $A = K[b_1, \ldots, b_m]$ then there is a number $n \leq m$ and n linear combinations $a_i := \sum_j \gamma_{ij} b_j \in A$ such that $a_1, \ldots, a_n$ are algebraically independent over K and that A is finite over $K[a_1, \ldots, a_m]$.

A first consequence is the following result.

Proposition 3.2. *Let X is an affine variety of dimension n. Then there is a finite surjective morphism $\varphi \colon X \to \mathbb{C}^n$.*

Proof. It follows from the Normalization Lemma (Theorem 3.1) that there exist $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $\mathcal{O}(X)$ is finite over the subring $\mathbb{C}[f_1, \ldots, f_n]$. It follows that $\dim X = n$ (see Exercise 3.1) and that the morphism $\varphi = (f_1, \ldots, f_n) \colon X \to \mathbb{C}^n$ is finite and surjective (Proposition 3.1). □

This result can be improved, using Remark 3.1 above.

PROPOSITION 3.3. *Let $X \subseteq \mathbb{C}^m$ be a closed subvariety of dimension $n \leq m$. Then there is a linear projection $\lambda\colon \mathbb{C}^m \to \mathbb{C}^n$ such that $\lambda|_X\colon X \to \mathbb{C}^n$ is finite and surjective.*

In fact, more is true: *There is an open dense set $U \subseteq \operatorname{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ such that the proposition above holds for any $\lambda \in U$.* We will not give a proof here since it does not follow immediately from our previous results.

EXERCISE 3.8. Let $f_1, f_2, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ be non-constant homogeneous polynomials and put $A := \mathbb{C}[f_1, f_2, \ldots, f_m]$. Then the following statements are equivalent:

(i) $\mathbb{C}[x_1, \ldots, x_n]/(f_1, f_2, \ldots, f_m)$ is a finite dimensional algebra;
(ii) There is a $k \in \mathbb{N}$ such that $(x_1, x_2, \ldots, x_n)^k \subseteq (f_1, f_2, \ldots, f_m)$;
(ii) $\mathbb{C}[x_1, \ldots, x_n]$ is finite over A.

(Remark: The fact, that the f_i's are homogeneous is essential!)

EXERCISE 3.9. Assume that the morphism $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ is given by homogeneous polynomials $f_1, \cdots, f_m$. If $\varphi^{-1}(0)$ is finite then $\varphi^{-1}(0) = \{0\}$ and φ is a finite morphism. (Hint: This follows immediately from the previous exercise.)

EXERCISE 3.10. Let $X \subseteq \mathbb{C}^n$ be cone and $\lambda\colon \mathbb{C}^n \to \mathbb{C}^m$ a linear map. If $X \cap \ker \lambda = \{0\}$ then $\lambda|_X\colon X \to \mathbb{C}^m$ is finite. Moreover, the set of linear maps $\lambda\colon \mathbb{C}^n \to \mathbb{C}^m$ such that $\lambda|_X$ is finite is open in $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$.

Krull's principal ideal theorem. We have seen in Lemma 3.1 that the dimension of a hypersurface $\mathcal{V}(f) \subseteq \mathbb{C}^n$ is equal to $n-1$, i.e. $\operatorname{codim}_{\mathbb{C}^n} \mathcal{V}(f) = 1$ where the *codimension* of a closed subvariety $Y \subseteq X$ is defined by $\operatorname{codim}_X Y := \dim X - \dim Y$. We want to generalize this to arbitrary affine varieties X. First we prove a converse of Lemma 3.5.

LEMMA 3.6. *Let $A \subseteq B$ be rings. Assume that A is Noetherian and that B is finite over A. Then every $b \in B$ satisfies an equation of the form*

$$b^m + a_1 b^{m-1} + a_2 b^{m-2} + \cdots + a_m = 0$$

where $a_1, a_2, \ldots, a_m \in A$.

PROOF. Since A is Noetherian the subalgebra $A[b] \subseteq B$ is finite over A. Therefore, the sequence $A \subseteq A + Ab \subseteq A + Ab + Ab^2 \subseteq \cdots \subseteq A + Ab + \cdots + Ab^k \subseteq \cdots$ becomes stationary. Hence, there is a $m \geq 1$ such that $b^m \in A + Ab + \cdots + Ab^{m-1}$. □

EXERCISE 3.11. Let $r \in \mathbb{C}(x_1, \ldots, x_n)$ satisfy an equation of the form

$$r^m + p_1 r^{m-1} + \cdots + p_m = 0 \text{ where } p_j \in \mathbb{C}[x_1, \ldots, x_n].$$

Then $r \in \mathbb{C}[x_1, \ldots, x_n]$. In particular, if $A \subseteq \mathbb{C}(a_1, \ldots, a_n)$ is a subalgebra which is finite over $\mathbb{C}[a_1, \ldots, a_n]$ then $A = \mathbb{C}[a_1, \ldots, a_n]$.

LEMMA 3.7. *Let A be $\mathbb{C}$-algebra without zero divisors and K its field of fractions. Let $a_1, \ldots, a_n \in A$ be algebraically independent elements such that A is finite over $\mathbb{C}[a_1, \ldots, a_n]$ and denote by $N\colon K \to \mathbb{C}(a_1, \ldots, a_n)$ the norm. Then*

(1) *$N(A) \subseteq \mathbb{C}[a_1, \ldots, a_n]$;*
(2) *For all $a \in A$ we have $\sqrt{Aa \cap \mathbb{C}[a_1, \ldots, a_n]} = \sqrt{\mathbb{C}[a_1, \ldots, a_n] N(a)}$.*

Proof. Let L/K be a finite field extension containing all the conjugates $a^{(1)} := a, a^{(2)}, \dots, a^{(r)}$ of a where $r = [K : \mathbb{C}(a_1, \dots, a_n)]$. Since a belongs to an algebra which is finite over $\mathbb{C}[a_1, \dots, a_n]$, namely A, the same holds for all $a^{(j)}$. Thus, every $a^{(j)}$ satisfies an equation with coefficients in $\mathbb{C}[a_1, \dots, a_n]$ and leading coefficient 1 (Lemma 3.6). This implies by Lemma 3.5 that the subalgebra $\tilde{A} := \mathbb{C}[a_1, \dots, a_n][a^{(j)}] \subseteq L$ is finite over $\mathbb{C}[a_1, \dots, a_n]$ and contains all $a^{(j)}$. Therefore, $N(a) = a^{(1)}a^{(2)} \cdots a^{(r)}$ belongs to $\tilde{A} \cap \mathbb{C}(a_1, \dots, a_n)$ which is equal to $\mathbb{C}[a_1, \dots, a_n]$ by the exercise above. This prove the first claim.

Now we have

$$\prod_j (t - a^{(j)}) = t^r + h_1 t^{r-1} + \cdots + h_{r-1} t + h_r$$

where $h_j \in \tilde{A} \cap \mathbb{C}(a_1, \dots, a_n) = \mathbb{C}[a_1, \dots, a_n]$ and $h_r = (-1)^r N(a)$. It follows that $N(a) = ab$ where $b = (-1)^{r-1}(a^{r-1} + h_1 a^{r-2} + \cdots + h_{r-1}) \in A$ and so $N(a) \in Aa$. Thus, $\mathbb{C}[a_1, \dots, a_n]N(a) \subseteq Aa \cap \mathbb{C}[a_1, \dots, a_n]$.

In order to see that $Aa \cap \mathbb{C}[a_1, \dots, a_n] \subseteq \sqrt{\mathbb{C}[a_1, \dots, a_n]N(a)}$ we choose an element $sa \in Aa \cap \mathbb{C}[a_1, \dots, a_n]$. Then $N(sa) = (sa)^r$, and since $N(sa) = N(s)N(a) \in \mathbb{C}[a_1, \dots, a_n]N(a)$ we finally get $sa \in \sqrt{\mathbb{C}[a_1, \dots, a_n]N(a)}$. □

Theorem 3.2 (Krull's Principal Ideal Theorem). *Let X be an irreducible affine variety and $f \in \mathcal{O}(X)$, $f \neq 0$. Assume that $\mathcal{V}_X(f)$ is non-empty. Then every irreducible component of $\mathcal{V}_X(f)$ has codimension 1 in X. In particular,* $\dim \mathcal{V}_X(f) = \dim X - 1$.

Proof. Let $\mathcal{V}_X(f) = C_1 \cup C_2 \cup \cdots \cup C_r$ be the irreducible decomposition. Choose an $h \in \mathcal{O}(X)$ vanishing on $C_2 \cup C_3 \cup \cdots \cup C_r$ which does not vanish on C_1. Then $\mathcal{V}_{X_h}(f) = C_1 \cap X_h$ is irreducible. Thus, it suffices to consider the case where $\mathcal{V}_X(f) \subseteq X$ is irreducible. By the Normalization Lemma (Theorem 3.1) there is a finite surjective morphism $\varphi \colon X \to \mathbb{C}^n$, $n = \dim X$. By Lemma 3.7(2) we get $\varphi(\mathcal{V}_X(f)) = \mathcal{V}(N(f))$, and so $\dim \mathcal{V}_X(f) = \dim \mathcal{V}(N(f)) = n - 1$ (see Lemma 3.1). □

It is easy to see that this result also holds for *equidimensional* varieties (i.e. varieties X where all irreducible components have the same dimension) if f is a non-zero divisor. For a general X and a non-zero divisor $f \in \mathcal{O}(X)$, we can only say that every irreducible component of $\mathcal{V}_X(f)$ has dimension $\leq \dim X - 1$.

A first consequence is the following result.

Proposition 3.4. *Let X be an irreducible variety and $f_1, f_2, \dots, f_r \in \mathcal{O}(X)$. If $\mathcal{V}_X(f_1, \dots, f_r)$ is non-empty then every irreducible component C of $\mathcal{V}_X(f_1, \dots, f_r)$ has dimension* $\dim C \geq \dim X - r$.

Proof. We proceed by induction on $\dim X$. Define $Y := \mathcal{V}_X(f_1)$, and let $Y = Y_1 \cup \cdots \cup Y_s$ be the decomposition into irreducible components. Then

$$\mathcal{V}_X(f_1, \cdots, f_r) = \bigcup_j \mathcal{V}_{Y_j}(f_2, \dots, f_r)$$

Since $\dim Y_j = \dim X - 1$ for all j we see, by induction, that every irreducible component of $\mathcal{V}_{Y_j}(f_2, \dots, f_r)$ has dimension $\geq (\dim X - 1) - (r - 1) = \dim X - r$, and the claim follows. □

Exercise 3.12. Let X be an affine variety and $f \in \mathcal{O}(X)$ a non-zero divisor. For any $x \in \mathcal{V}_X(f)$ we have $\dim_x \mathcal{V}_X(f) = \dim_x X - 1$.
(Hint: If f is a non-zero divisor, then f is non-zero on every irreducible component X_i of X and so $\mathcal{V}_{X_i}(f)$ is either empty or every irreducible component has codimension 1. Now the claim follows easily.)

Another consequence of Krull's PI-Theorem is the following which gives an alternative definition of the dimension of a variety.

Proposition 3.5. *Let X be an irreducible variety and $Y \subsetneqq X$ a closed irreducible subset. Then there is a strictly decreasing chain of length $n := \dim X$,*

$$X_n = X \supsetneqq X_{n-1} \supsetneqq \cdots \supsetneqq X_d = Y \supsetneqq \cdots \supsetneqq X_1 \supsetneqq X_0$$

of irreducible closed subsets X_j. In particular, $\dim X$ equals the length of a maximal chain of irreducible closed subsets.

Proof. By induction, we only have to show that Y is contained in an irreducible hypersurface $H \subseteq X$. Let $f \in I(Y)$ be a non-zero function. Then $X \supseteq \mathcal{V}_X(f) \supseteq Y$ and so Y is contained in an irreducible component of $\mathcal{V}_X(f)$ which all have codimension 1 by Theorem 3.2. $\square$

Remark 3.2. This result allows to define the *dimension* $\dim A$ *of a* $\mathbb{C}$*-algebra* A as the maximal length of a chain of prime ideal $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_m \subseteq A$. If A is finitely generated then $\dim A$ is finite, and every maximal chain has length $\dim A$. Moreover, $\dim A = \dim A_{\text{red}}$ where $A_{\text{red}} := A/\sqrt{(0)}$, and so $\dim A = \dim X$ where X is an affine variety with coordinate ring isomorphic to A_{red}.

We also see that for a variety X and a point $x \in X$ we have $\dim_x X = \dim \mathcal{O}_{X,x}$.

Corollary 3.1. *Let A be a finitely generated $\mathbb{C}$-algebra and let $a \in A$ be a non-zero divisor. Then $\dim A/Aa \le \dim A - 1$, and equality holds if A_{red} is a domain.*

Proof. Put $\bar{A} := A/(a)$ and denote by $a' \in A_{\text{red}}$ the image of a. Then a' is a non-zero divisor in A_{red} and so $\dim A_{\text{red}}/\sqrt{(a')} \le \dim A_{\text{red}} - 1$ by Theorem 3.2. Since $\bar{A}_{\text{red}} \simeq A_{\text{red}}/\sqrt{(a')}$ we finally get $\dim \bar{A} = \dim \bar{A}_{\text{red}} \le \dim A_{\text{red}} - 1 = \dim A - 1$ $\square$

Decomposition Theorem and dimension formula. Let $\varphi\colon X \to Y$ be a dominant morphism where X, Y are both irreducible. We want to show that the dimension of a non-empty fiber $\varphi^{-1}(y)$ is always $\ge \dim X - \dim Y$ and that we have equality on a dense open set of Y. A crucial step is the following *Decomposition Theorem* for a morphism.

Theorem 3.3. *Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. There is a non-empty special open set $U \subseteq Y$ and a factorization of φ of the form*

$$\begin{tikzcd} \varphi^{-1}(U) \arrow[r, "\rho"] \arrow[dr, "\varphi"'] & U \times \mathbb{C}^r \arrow[d, "\mathrm{pr}_U"] \\ & U \end{tikzcd}$$

where ρ is a finite surjective morphism and $r := \dim X - \dim Y$. In particular, the fibers $\varphi^{-1}(y) = \rho^{-1}(\{y\} \times \mathbb{C}^r)$ have the same dimension for all $y \in U$, namely $\dim X - \dim Y$.

REMARK 3.3. We will see later in Proposition 3.6 that the fibers $\varphi^{-1}(y)$ for $y \in U$ are *equidimensional*, i.e., all irreducible components have the same dimension, namely $\dim X - \dim Y$.

PROOF. Since φ is dominant we will regard $\mathcal{O}(Y)$ as a subalgebra of $\mathcal{O}(X)$. Let $K = \mathbb{C}(Y)$ be the quotient field of $\mathcal{O}(Y)$ and put $A := K \cdot \mathcal{O}(X) \subseteq \mathbb{C}(X)$, the K-algebra generated by K and $\mathcal{O}(X)$. Then A is finitely generated over K and so we can find algebraically independent elements $h_1, \ldots, h_r \in A$ such that A is finite over $K[h_1, \ldots, h_r]$ (Theorem 3.1). It follows that $r = \dim X - \dim Y$.

We claim that there is an $f \in \mathcal{O}(Y)$ such that $h_i = \frac{a_i}{f}$ with $a_i \in \mathcal{O}(X)$ for all i and that $\mathcal{O}(X_f) = \mathcal{O}(X)_f$ is finite over $\mathcal{O}(Y_f)[h_1, \ldots, h_r]$. The first statement is clear, and we can therefore assume that $h_1, \ldots, h_r \in \mathcal{O}(X)$.

For the second statement, let $b_1, \ldots, b_s$ be generators of A over $K[h_1, \ldots, h_r]$. Multiplying with a suitable element of $\mathcal{O}(Y) \subseteq K$ we can first assume that $b_j \in \mathcal{O}(X)$ and then, by adding more elements if necessary, that $b_1, \ldots, b_s$ generate $\mathcal{O}(X)$ as a $\mathbb{C}$-algebra. Now $b_i b_j = \sum_k c_k^{(ij)} b_k$ where $c_k^{(ij)} \in K[h_1, \ldots, h_r]$. Thus we can find an $f \in \mathcal{O}(Y)$ such that $f \cdot c_k^{(ij)} \in \mathcal{O}(Y)[h_1, \ldots, h_r]$. It follows that

$$\sum_j \mathcal{O}(Y_f)[h_1, \ldots, h_r]\, b_j \subseteq \mathcal{O}(X)_f = \mathcal{O}(X_f)$$

is a subalgebra containing $\mathcal{O}(X)$, hence is equal to $\mathcal{O}(X_f)$, and the claim follows.

Setting $U := Y_f$ we get $\varphi^{-1}(U) = X_f$ and obtain a morphism

$$\rho = \varphi \times (h_1, \ldots, h_r) \colon X_f \to Y_f \times \mathbb{C}^r, \ x \mapsto (\varphi(x), h_1(x), \ldots, h_r(x))$$

which satisfies the requirements of the proposition.

The last statement is clear (see Exercise 3.6). □

EXERCISE 3.13. Work out the decomposition of Theorem 3.3 in the case of the following morphisms:

(1) $\varphi\colon M_2 \to M_2$, $\varphi(A) := A^2$.

(2) $\varphi\colon \mathrm{SL}_2 \to \mathbb{C}^3$, $\varphi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) := (ab, ad, cd)$ (see Exercise 2.13).

What is the degree of the finite morphism ρ in each case?

COROLLARY 3.2. *If $\varphi\colon X \to Y$ is a morphism, then there is a set $U \subseteq \varphi(X)$ which is open and dense in $\overline{\varphi(X)}$.*

PROOF. If X is irreducible, this is an immediate consequence of Theorem 3.3 above. In general, we apply this proposition to every irreducible component of X, and use Exercise 1.23. □

PROPOSITION 3.6. *Let X and Y be irreducible varieties and $\varphi\colon X \to Y$ a dominant morphism. If $y \in \varphi(X)$ and C is an irreducible component of the fiber $\varphi^{-1}(y)$ then*

$$\dim C \geq \dim X - \dim Y.$$

PROOF. Set $m := \dim Y$ and let $\psi\colon Y \to \mathbb{C}^m$ be a finite surjective morphism (Theorem 3.1). If we denote by $\tilde{\varphi}\colon X \to \mathbb{C}^m$ the composition $\psi \circ \varphi$, then every fiber of $\tilde{\varphi}$ is a finite union of fibers of φ. Hence it suffices to prove the claim for the morphism $\tilde{\varphi} = (f_1, \ldots, f_m)\colon X \to \mathbb{C}^m$. If $a = (a_1, \ldots, a_m) \in \tilde{\varphi}(X)$ then $\tilde{\varphi}^{-1}(a) =$

$\mathcal{V}_X(f_1 - a_1, f_2 - a_2, \dots, f_m - a_m)$, and the claim follows from Proposition 3.4, a consequence of Krull's Principal Ideal Theorem. $\square$

One might believe that the two propositions above imply that for any morphism $\varphi\colon X \to Y$ the function $y \mapsto \dim \varphi^{-1}(y)$ is upper-semicontinuous (see below). This is not true as one can show by examples. However, a famous theorem of Chevalley says that the function $x \mapsto \dim_x \varphi^{-1}(\varphi(x))$ is upper-semicontinuous on X. The proof is quite involved and we will not present it here.

Example 3.4. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by $(x, y) \mapsto (x, xy)$. It easy to see that the image $\varphi(\mathbb{C}^2)$ is not locally closed in $\mathbb{C}^2$ and that the map $a \mapsto \dim \varphi^{-1}(a)$ is not upper-semicontinuous.

Constructible sets. Recall that a subset $A \subseteq X$ of a variety X is called *locally closed* if A is the intersection of an open and a closed subset, or, equivalently, if A is open in its closure $\bar{A}$. We have seen in examples that images of morphisms need not to be locally closed in general. However, we will show that images of morphisms are always "constructible" in the following sense.

Definition 3.3. A subset C of an affine variety X is called *constructible* if it is a finite union of locally closed subsets.

Exercise 3.14. (1) Finite unions, finite intersections and complements of constructible sets are again constructible.
(2) If C is a constructible, then C contains a set U which is open and dense in $\bar{C}$.

Proposition 3.7. *If $\varphi\colon X \to Y$ is a morphism then the image of a constructible subset is again constructible.*

Proof. Since every open set is the union of finitely many special open sets it suffices to show, in view of the exercise above, that the image of a morphism is constructible. By Corollary 3.2 there is a dense open set $U \subseteq \overline{\varphi(X)}$ contained in the image $\varphi(X)$. Then the complement $Y' := \overline{\varphi(X)} \setminus U$ is closed and $\dim Y' < \dim Y$ (Exercise 3.3). By induction on $\dim \overline{\varphi(X)}$, we can assume that the claim holds for the morphism $\varphi'\colon X' := \varphi^{-1}(Y') \to Y'$ induced by φ. But then $\varphi(X) = U \cup \varphi'(X')$ and we are done. $\square$

Degree of a morphism. Recall that a dominant morphism $\varphi\colon X \to Y$ between irreducible varieties is called of *finite degree* d if $\dim X = \dim Y$ and $d = [\mathbb{C}(X) : \mathbb{C}(Y)]$ (see 2). This has the following geometric interpretation.

Proposition 3.8. *Let X, Y be irreducible affine varieties and $\varphi\colon X \to Y$ a dominant morphism of finite degree d. Then there is a dense open set $U \subseteq Y$ such that $\#\varphi^{-1}(y) = d$ for all $y \in U$.*

Proof. We have $\mathbb{C}(X) = \mathbb{C}(Y)[r]$ where r satisfies the equation
$$r^d + a_1 r^{d-1} + \cdots + a_d = 0.$$
Replacing Y and X by suitable special open sets Y_f and X_f ($f \in \mathcal{O}(Y)$) we can assume that

(1) $\mathcal{O}(X)$ is finite over $\mathcal{O}(Y)$ (Theorem 3.3);
(2) $r \in \mathcal{O}(X)$;

(3) $a_i \in \mathcal{O}(Y)$;
(4) $\mathcal{O}(X) = \mathcal{O}(Y)[r]$.

This implies that

$$\mathcal{O}(X) = \bigoplus_{j=0}^{d-1} \mathcal{O}(Y) r^j \xleftarrow{\sim} \mathcal{O}(Y)[t]/(t^d + a_1 t^{d-1} + \cdots + a_d)$$

and so, for every $y \in Y$, we get

$$\mathcal{O}(X)/\mathcal{O}(X)\mathfrak{m}_y \simeq \mathbb{C}[t]/(t^d + a_1(y)t^{d-1} + \cdots + a_d(y))$$

This means that the number of elements in the fiber $\varphi^{-1}(y)$ is equal to the number of different solutions of the equation

$$t^d + a_1(y)t^{d-1} + \cdots + a_d(y) = 0. \tag{5}$$

Now let D be the discriminant of an equation of degree d (see Example 1.2) and define $f(y) := D(a_1(y), \ldots, a_d(y))$. Then $f \in \mathcal{O}(Y)$, and $f(y) \neq 0$ if and only if equation (5) has d different solutions, or, equivalently, the fiber $\varphi^{-1}(y)$ has d points. Thus, the special open set $U := Y_f \subseteq Y$ has the required property. □

REMARK 3.4. One can show that the open set U constructed in the proof has the property that the morphism $\varphi^{-1}(U) \to U$ is an *unramified covering* with respect to the $\mathbb{C}$-topology.

EXERCISE 3.15. What is the degree of the morphism $M_n \to M_n$ given by $A \mapsto A^k$?

EXERCISE 3.16. Let $\varphi\colon X \to Y$ be a dominant morphism where X and Y are both irreducible. If there is an open dense set $U \subseteq X$ such that $\varphi|_U$ is injective, then φ is birational.

4. TANGENT SPACES AND DIFFERENTIALS

Zariski tangent space. A *tangent vector* δ in a point x_0 of an affine variety X is "rule" to differentiate regular functions, i.e., it is a $\mathbb{C}$-linear map $\delta\colon \mathcal{O}(X) \to \mathbb{C}$ satisfying

$$\delta(f \cdot g) = f(x_0)\,\delta(g) + g(x_0)\,\delta(f) \text{ for all } f, g \in \mathcal{O}(X). \tag{6}$$

Such a map is called a *derivation of* $\mathcal{O}(X)$ *in* x_0. It follows that $\delta(f^n) = nf^{n-1}(x_0)\delta(f)$ and so, for any polynomial $F = F(y_1, \dots, y_m)$, we get

$$\delta(F(f_1, \dots, f_m)) = \sum_{j=1}^{m} \frac{\partial F}{\partial y_j}(f_1(x_0), \dots, f_m(x_0))\,\delta(f_j).$$

This implies that a derivation in x_0 is completely determined by its values on a generating set of the algebra $\mathcal{O}(X)$. Moreover, a linear combination of derivations in x_0 is again a derivation in x_0. As a consequence, the derivations in x_0 form a finite dimensional subspace of $\operatorname{Hom}(\mathcal{O}(X), \mathbb{C})$.

Definition 4.1. The *Zariski tangent space* $T_{x_0}X$ of a variety X in a point x_0 is defined to be the set of all tangent vectors in x_0:

$$T_{x_0}X := \operatorname{Der}_{x_0}(\mathcal{O}(X)) := \{\delta\colon \mathcal{O}(X) \to \mathbb{C} \mid \delta \text{ a } \mathbb{C}\text{-linear derivation in } x_0\}.$$

$T_{x_0}X$ is a finite dimensional linear subspace of $\operatorname{Hom}(\mathcal{O}(X), \mathbb{C})$.

Exercise 4.1. Let $\delta \in T_xX$ be a tangent vector in x.

(1) $\delta(c) = 0$ for every constant $c \in \mathcal{O}(X)$.

(2) If $f \in \mathcal{O}(X)$ is invertible, then $\delta(f^{-1}) = -\dfrac{\delta f}{f(x)^2}$.

Example 4.1. If $X = \mathbb{C}^n$ and $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ then

$$T_a\mathbb{C}^n = \bigoplus_i \mathbb{C}\, \frac{\partial}{\partial x_i}\bigg|_a$$

where $\frac{\partial}{\partial x_i}\Big|_a(f) := \frac{\partial f}{\partial x_i}(a)$. Thus we have a canonical isomorphism $T_a\mathbb{C}^n \simeq \mathbb{C}^n$ by identifying $\delta \in \operatorname{Der}_a(\mathbb{C}[x_1, \dots, x_n])$ with $(\delta x_1, \dots, \delta x_n) \in \mathbb{C}^n$.

More generally, if V is a finite dimensional vector space and $x_0 \in V$ we define, for every $v \in V$, the tangent vector $\partial_{v,x_0}\colon \mathcal{O}(V) \to \mathbb{C}$ in x_0 by

$$\partial_{v,x_0}(f) := \frac{f(x_0 + tv) - f(x_0)}{t}\bigg|_{t=0},$$

and thus obtain a canonical isomorphism $V \xrightarrow{\sim} T_{x_0}V$, for every $x_0 \in V$.

Let $\delta \in T_xX$ be a tangent vector. Since $\mathcal{O}(X) = \mathbb{C} \oplus \mathfrak{m}_x$ we see that δ is determined by its restriction to $\mathfrak{m}_x$. Moreover, formula (6) shows that δ vanishes on $\mathfrak{m}_x^2$. Hence, δ induces a linear map $\bar{\delta}\colon \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$.

Lemma 4.1. *Given an affine variety X and a point $x \in X$ there is a canonical isomorphism*

$$T_xX \xrightarrow{\sim} \operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C}).$$

given by $\delta \mapsto \bar{\delta} := \delta|_{\mathfrak{m}_x}$.

Proof. We have already seen that $\delta \mapsto \bar{\delta}$ is injective. On the other hand, let $C \subseteq \mathfrak{m}_x$ be a complement of $\mathfrak{m}_x^2$ so that $\mathcal{O}(X) = \mathbb{C} \oplus C \oplus \mathfrak{m}_x^2$. If $\lambda\colon C \to \mathbb{C}$ is linear then one easily sees that the extension of λ to a linear map δ on $\mathcal{O}(X)$ by putting $\delta|_{\mathbb{C}\oplus\mathfrak{m}_x^2} = 0$ is a derivation in x. □

Exercise 4.2. The canonical homomorphism $\mathcal{O}(X) \to \mathcal{O}_{X,x}$ induces an isomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ is the maximal ideal.

If $U = X_f \subseteq X$ is a special open set and $x \in U$ then $T_xU = T_xX$ in a canonical way. In fact, a derivation δ' of $\mathcal{O}(U)$ induces a derivation δ of $\mathcal{O}(X)$ by restriction: $\delta(h) := \delta'(h|_U)$, and every derivation δ of $\mathcal{O}(X)$ "extends" to a derivation δ' of $\mathcal{O}(U) = \mathcal{O}(X)_f$ by setting $\delta'(\frac{h}{f^m}) = (-m)\frac{\delta h}{f^{m+1}}$ (see Exercise 4.1). The same result follows from Exercise 4.2 using Lemma 4.1.

Exercise 4.3. If $Y \subseteq X$ is a closed subvariety and $x \in Y$ then $\dim T_xY \leq \dim T_xX$. (Hint: The surjection $\mathcal{O}(X) \to \mathcal{O}(Y)$ induces a surjection $\mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \to \mathfrak{m}_{x,Y}/\mathfrak{m}_{x,Y}^2$.)

Proposition 4.1. $\dim T_xX \geq \dim_x X$.

Proof. If $C \subseteq X$ is an irreducible component passing through x we have $\dim T_xC \leq \dim T_xX$ (Exercise 4.3). Thus we can assume that X is irreducible. Choose $f_1, \ldots, f_r \in \mathfrak{m}_x$ such that the residue classes modulo $\mathfrak{m}_x^2$ form a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2$, hence $r = \dim T_xX$, by Lemma 4.1. Since the zero set $\mathcal{V}_X(f_1, \ldots, f_r)$ has $\{x\}$ as an irreducible component (see Exercise 3.5) it follows from Proposition 3.4 that

$$0 = \dim\{x\} \geq \dim X - r = \dim X - \dim T_xX.$$

Hence the claim. □

Proposition 4.2. *There is a canonical isomorphism* $T_{(x,y)}X \times Y \xrightarrow{\sim} T_xX \oplus T_yY$ *where* $x \in X$ *and* $y \in Y$.

Proof. Every derivation δ of $\mathcal{O}(X \times Y)$ in (x, y) induces, by restriction, derivations δ_X of $\mathcal{O}(X)$ in x and δ_Y of $\mathcal{O}(Y)$ in y. This defines a linear map $T_{(x,y)}X \times Y \to T_xX \oplus T_yY$ which is injective, because $\delta(f \cdot h) = \delta_X f \cdot h(y) = f(x) \cdot \delta_Y h$ for $f \in \mathcal{O}(X)$ and $h \in \mathcal{O}(Y)$.

In order to see that the map is surjective we first claim that given two derivations $\delta_1 \in T_xX$ and $\delta_2 \in T_yY$ there is a unique linear map $\delta\colon \mathcal{O}(X \times Y) \to \mathbb{C}$ such that $\delta(f \cdot h) = \delta_1 f \cdot h(y) = f(x) \cdot \delta_2 h$. This follows from Proposition 2.5 and the universal property of the tensor product. Now it is easy to see that this map δ is a derivation in (x, y) and that $\delta_X = \delta_1$ and $\delta_Y = \delta_2$. □

Tangent spaces of subvarieties. Let $X \subseteq V$ be closed subvariety of the vector space V and $x_0 \in X$. If $\delta \in T_{x_0}V = V$ is a tangent vector which vanishes on $I(X) = \ker(\mathrm{res}\colon \mathcal{O}(V) \to \mathcal{O}(X))$ then the induced map $\bar{\delta}\colon \mathcal{O}(X) \to \mathbb{C}$ is a derivation in x_0, and vice versa. Thus we have the following result.

Proposition 4.3. *If* $X \subseteq V$ *is a closed subvariety and* $x_0 \in X$ *then*

$$T_{x_0}X = \{\delta \in T_{x_0}V \mid \delta(f) = 0 \text{ for all } f \in I(X)\} \subseteq T_{x_0}V = V.$$

More explicitly, let $V = \mathbb{C}^n$ *and assume that the ideal* $I(X)$ *is generated by* $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$. *Then, for* $x_0 \in X$, *we get*

$$T_{x_0} X = \{a = (a_1, \dots, a_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} a_i \frac{\partial f_j}{\partial x_i}(x_0) = 0 \text{ for } j = 1, \dots, s\}.$$

In particular,

$$\dim T_{x_0} X = n - \operatorname{rk} \left[\frac{\partial f_j}{\partial x_i}(x_0)\right]_{i,j}.$$

The matrix $\left[\frac{\partial f_j}{\partial x_i}(x_0)\right]_{i,j}$ is called the *Jacobian matrix* at the point x_0 and will be denoted by $\operatorname{Jac}(f_1, \dots, f_s)(x_0)$.

Example 4.2. Consider the plane curve $C = \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$. Then $I(C) = (y^2 - x^3)$ and so the tangent space in an arbitrary point $x_0 = (a, b) \in C$ is given by $T_{(a,b)}C = \{(u, v) \in \mathbb{C}^2 \mid -3a^2u + 2bv = 0\}$. Since $(a, b) = (t^2, t^3)$ for some $t \in \mathbb{C}$ we get

$$T_{(t^2,t^3)}C = \begin{cases} \mathbb{C}^2 & \text{for } t = 0, \\ \mathbb{C}\begin{bmatrix} 2 \\ 3t \end{bmatrix} & \text{for } t \neq 0. \end{cases}$$

Exercise 4.4. Calculate the tangent spaces of the plane curves $C_1 := \mathcal{V}(y - x^2)$ and $C_2 = \mathcal{V}(y^2 - x^2 - x^3)$ in arbitrary points (a, b).

Remark 4.1. Consider the $\mathbb{C}$-algebra $\mathbb{C}[\varepsilon] := \mathbb{C}[t]/(t^2)$ called the algebra of *dual numbers*. By definition, we have $\mathbb{C}[\varepsilon] = \mathbb{C} \oplus \mathbb{C}\varepsilon$ and $\varepsilon^2 = 0$. If X is an affine variety and $\rho\colon \mathcal{O}(X) \to \mathbb{C}[\varepsilon]$ an algebra homomorphism, then ρ is of the form $\rho = \operatorname{ev}_x \oplus \delta_x\varepsilon$ for some $x \in X$ where ev_x is the evaluation map at x and δ_x a derivation in x, i.e., $\rho(f) = f(x) + \delta_x(f)\varepsilon$. Conversely, if δ_x is a derivation in x then $\rho := \operatorname{ev}_x \oplus \delta_x\varepsilon$ is an algebra homomorphism. If $X = V$ is a vector space, then the homomorphisms $\rho\colon \mathcal{O}(V) \to \mathbb{C}[\varepsilon]$ are in one-to-one correspondence with the elements of $V \oplus V\varepsilon$. In fact, there are canonical bijections

$$\operatorname{Alg}_{\mathbb{C}}(\mathcal{O}(V), \mathbb{C}[\varepsilon]) \xrightarrow{\sim} \operatorname{Hom}(V, \mathbb{C}[\varepsilon]) \xrightarrow{\sim} V \oplus V\varepsilon,$$

and the inverse map associates to $x + v\varepsilon \in V \oplus V\varepsilon$ the algebra homomorphism $\rho\colon f \mapsto f(x + v\varepsilon)$. Since

$$f(x + v\varepsilon) = f(x) + \partial_{v,x} f\, \varepsilon$$

it follows again from the above that $T_x V$ can be canonically identified with V. This formula is very useful for calculating tangent spaces as we will see below.

Example 4.3. (a) The tangent space of GL_n at E is the space of all $n \times n$-matrices and the tangent space of SL_n at $E \in \mathrm{SL}_n$ is the subspace of traceless matrices:

$$T_E \mathrm{SL}_n = \mathfrak{sl}_n := \{A \in M_n \mid \operatorname{tr} A = 0\} \subseteq T_E \mathrm{GL}_n = \mathfrak{gl}_n := M_n.$$

In fact, $I(\mathrm{SL}_n) = (\det)$ and an easy calculation shows that $\det(E + A\varepsilon) = 1 + \operatorname{tr}(A)\varepsilon$ which implies, by Proposition 4.3, that $A \in M_n$ belongs to $T_E \mathrm{SL}_n$ if and only if $\operatorname{tr} A = 0$.

(b) Next we look at the orthogonal group $\mathrm{O}_n := \{A \in M_n \mid AA^t = E\}$. As a closed subset O_n is defined by $\binom{n+1}{2}$ quadratic equations and so $\dim \mathrm{O}_n \geq n^2 - \binom{n+1}{2} = \binom{n}{2}$. On the other hand, we have

$$(E + X\varepsilon)(E + X\varepsilon)^t = E + (X + X^t)\varepsilon$$

which shows that $T_E \mathrm{O}_n \subseteq \{X \in M_n \mid X \text{ skewsymmetric}\}$. Since this space has dimension $\binom{n}{2}$ and since $\dim_E \mathrm{O}_n = \dim \mathrm{O}_n$ (Exercise 1.22) it follows from Proposition 4.1 that

$$T_E \mathrm{O}_n = T_E \mathrm{SO}_n = \mathfrak{so}_n := \{X \in M_n \mid X \text{ skewsymmetric}\}.$$

Exercise 4.5. If $X, Y \subseteq \mathbb{C}^n$ are closed subvarieties and $z \in X \cap Y$ then $T_z(X \cap Y) \subseteq T_z X \cap T_z Y \subseteq \mathbb{C}^n$.

Nonsingular varieties. We have seen in Proposition 4.1 that for every point x of an affine variety X one has $\dim T_x X \geq \dim_x X$. We will show now that equality holds in an open set and we will characterize these points.

Definition 4.2. The variety X is called *nonsingular* or *smooth* in $x \in X$ if $\dim T_x X = \dim_x X$. Otherwise it is *singular* in x. The variety is called *nonsingular* or *smooth* if it is nonsingular in every point. We denote by X_{sing} the set of singular points of X.

Example 4.4. Let $H := \mathcal{V}(f) \subseteq \mathbb{C}^n$ be a hypersurface where $f \in \mathbb{C}[x_1, \ldots, x_n]$ is square-free and non-constant, and so $I(H) = (f)$. Then the tangent space in a point $x_0 \in H$ is given by

$$T_{x_0} X := \{a = (a_1, \ldots, a_n) \mid \sum_i a_i \frac{\partial f}{\partial x_i}(x_0) = 0\},$$

and so

$$H_{\text{sing}} = \mathcal{V}(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}) \subseteq H.$$

It follows that H_{sing} is a proper closed subset whose complement is dense. (This is clear for irreducible hypersurfaces since a non-zero derivative $\frac{\partial f}{\partial x_i}$ cannot be a multiple of f and so $\mathcal{V}(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$ is a proper closed subset of $\mathcal{V}(f)$. This implies that every irreducible component of H contains a non-empty open set of nonsingular points which does not meet the other components, and the claim follows.)

It is also interesting to remark that a common point of two or more irreducible components of H is always singular. We will see that this true in general (Corollary 4.1).

Proposition 4.4. *Let X be an irreducible affine variety. Then the set X_{sing} of singular points of X is a proper closed subset of X whose complement is dense.*

Proof. We can assume that X is an irreducible closed subvariety of $\mathbb{C}^n$ of dimension d. If $I(X) = (f_1, \ldots, f_s)$, then, by Proposition 4.3,

$$X_{\text{sing}} = \{x \in X \mid \operatorname{rk}\left[\frac{\partial f_j}{\partial x_i}(x)\right] < n - d\}$$

which is the closed subset defined by the vanishing of all $(n-d)\times(n-d)$ minors of the Jacobian matrix $\mathrm{Jac}(f_1, \ldots, f_s)$. In order to see that X_{sing} has a dense complement, we use the fact, that every irreducible variety contains a special open set which is isomorphic to a special open set of an irreducible hypersurface H (see Lemma 3.3). Since H contains a dense open set of non-singular points (see Example 4.4 above) the claim follows. $\square$

EXERCISE 4.6. If X is an affine variety such that all irreducible components have the same dimension. Then X_{sing} is closed and has a dense complement.
(We will see later in Corollary 4.1 that this holds for every affine variety.)

EXERCISE 4.7. The hypersurface $H \subseteq \mathbb{C}^3$ from Exercise 2.13 is nonsingular.

EXERCISE 4.8. Let $q \in \mathbb{C}[x_1, \dots, x_n]$ be a quadratic form and $Q := \mathcal{V}(q) \subseteq \mathbb{C}^n$. Then 0 is a singular point of Q. It is the only singular point if and only if q is nondegenerate.

EXERCISE 4.9. Determine the singular points of the plane curves

$$E_p := \mathcal{V}(y^2 - p(x))$$

where $p(x)$ is an arbitrary polynomial, and deduce a necessary and sufficient condition for E_p to be nonsingular.

EXERCISE 4.10. Let $X \subseteq \mathbb{C}^n$ be a closed cone (see Exercise 1.3). Then X_{sing} is a cone, too. Moreover, $0 \in X$ is a nonsingular point if and only if X is subvector space.

EXERCISE 4.11. Let X be an affine variety such that the group of automorphisms acts transitively on X. Then X is smooth.

Associated graded algebras. Let R be $\mathbb{C}$-algebra and $\mathfrak{a} \subseteq R$ an ideal. The *associated graded algebra* $\operatorname{gr}_{\mathfrak{a}} R$ is defined in the following way. Consider the $\mathbb{C}$-vector space

$$\operatorname{gr}_{\mathfrak{a}} R := \bigoplus_{i \geq 0} \mathfrak{a}^i/\mathfrak{a}^{i+1} = R/\mathfrak{a} \oplus \mathfrak{a}/\mathfrak{a}^2 \oplus \mathfrak{a}^2/\mathfrak{a}^3 \oplus \cdots$$

and define the multiplication of (homogeneous) elements by

$$(f + \mathfrak{a}^{i+1}) \cdot (h + \mathfrak{a}^{j+1}) := fh + \mathfrak{a}^{i+j+1}$$

for $f \in \mathfrak{a}^i, h \in \mathfrak{a}^j$. It is easy to see that this defines a multiplication on $\operatorname{gr}_{\mathfrak{a}} R$. By definition, $R/\mathfrak{a}$ is a subalgebra of $\operatorname{gr}_{\mathfrak{a}} R$, and $\operatorname{gr}_{\mathfrak{a}} R$ is generated by $\mathfrak{a}/\mathfrak{a}^2$ as a $R/\mathfrak{a}$-algebra. In particular, if R is finitely generated as a $\mathbb{C}$-algebra, then so is $\operatorname{gr}_{\mathfrak{a}} R$.

We want to use this construction to give the following characterization of non-singular points.

THEOREM 4.1. *Let X be an affine variety. A point $x \in X$ is nonsingular if and only if the associated graded algebra $\operatorname{gr}_{\mathfrak{m}_x} \mathcal{O}(X)$ is a polynomial ring. In particular, the local ring $\mathcal{O}_{X,x}$ of a nonsingular point x is a domain and so x belongs to a unique irreducible component of X.*

Before we can give the proof we have to explain some technical results from commutative algebra. Let R be a $\mathbb{C}$-algebra and $\mathfrak{m} \subseteq R$ a maximal ideal. Consider the subalgebra $\tilde{R}$ of $R[t, t^{-1}]$ generated as an R-algebra by t and $\mathfrak{m}t^{-1}$:

$$\tilde{R} := R[t, \mathfrak{m}t^{-1}] = \cdots \oplus \mathfrak{m}^2 t^{-2} \oplus \mathfrak{m}t^{-1} \oplus R \oplus Rt \oplus Rt^2 \oplus \cdots \subseteq R[t, t^{-1}].$$

In the following lemma we collect some basic properties of this construction.

LEMMA 4.2. (1) *If R is finitely generated then so is $\tilde{R}$.*

(2) *There is a canonical isomorphism* $\tilde{R}/\tilde{R}t \xrightarrow{\sim} \operatorname{gr}_{\mathfrak{m}} R$.
(3) *If* $\mathfrak{a} \subseteq \mathfrak{m}$ *is an ideal and* $\tilde{\mathfrak{a}} := \mathfrak{a}[t, t^{-1}] \cap \tilde{R}$ *then* $\tilde{R}/\tilde{\mathfrak{a}} \xrightarrow{\sim} \widetilde{R/\mathfrak{a}}$.
(4) *If* $\mathfrak{n} \subseteq R$ *is the nilradical, then* $\tilde{\mathfrak{n}} := \mathfrak{n}[t, t^{-1}] \cap \tilde{R}$ *is the nilradical of* $\tilde{R}$*, and* $\tilde{R}/\tilde{\mathfrak{n}} \xrightarrow{\sim} \widetilde{R/\mathfrak{n}}$.
(5) *Assume that* R *is a finitely generated domain. Then* $\tilde{R}$ *is a domain, and we have*
$$\dim \tilde{R} = \dim R + 1 \quad \textit{and} \quad \dim \tilde{R}/\tilde{R}t = \dim R.$$
(6) *Assume that* R *finitely generated and that the minimal primes* $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ *are all contained in* $\mathfrak{m}$. *Then the* $\tilde{\mathfrak{p}}_1, \ldots, \tilde{\mathfrak{p}}_r$ *are the minimal primes of* $\tilde{R}$.

Proof. (1) If $R = \mathbb{C}[h_1, \cdots, h_m]$ and $\mathfrak{m} = (f_1, \ldots, f_n)$ then
$$\tilde{R} = \mathbb{C}[h_1, \ldots, h_m, t, f_1 t^{-1}, \ldots, f_n t^{-1}],$$
and so $\tilde{R}$ is finitely generated.

(2) By definition, we have
$$\tilde{R}t = \cdots \oplus \mathfrak{m}^3 t^{-2} \oplus \mathfrak{m}^2 t^{-1} \oplus \mathfrak{m} \oplus Rt \oplus Rt^2 \oplus \cdots.$$
Hence
$$\tilde{R}/\tilde{R}t = \cdots \oplus (\mathfrak{m}^2/\mathfrak{m}^3)t^{-2} \oplus (\mathfrak{m}/\mathfrak{m}^2)t^{-1} \oplus R/\mathfrak{m}$$
and the claim follows.

(3) The canonical map $\pi\colon R[t, t^{-1}] \to (R/\mathfrak{a})[t, t^{-1}]$ induces, by our construction, a surjective homomorphism $\tilde{\pi}\colon \tilde{R} \to \widetilde{R/\mathfrak{a}}$ with kernel $\ker \pi \cap \tilde{R} = \mathfrak{a}[t, t^{-1}] \cap \tilde{R}$.

(4) Put $R_{\text{red}} := R/\mathfrak{n}$. Then $R_{\text{red}}[t, t^{-1}]$ is reduced, i.e. without nilpotent elements $\neq 0$, and so is $\widetilde{R_{\text{red}}}$. Since the kernel of the map $R[t, t^{-1}] \to R_{\text{red}}[t, t^{-1}]$ is equal to $\mathfrak{n}[t, t^{-1}]$ and consists of nilpotent elements the claim follows from (3).

(5) The first part is clear since $R[t, t^{-1}]$ is a domain. Since $\tilde{R}_t = R[t, t^{-1}]$ we get $\dim \tilde{R} = \dim R[t, t^{-1}] = \dim R[t] = \dim R + 1$. Moreover, by the Principal Ideal Theorem (Theorem 3.2) we have $\dim \tilde{R}/\tilde{R}t = \dim \tilde{R} - 1$.

(6) It follows from (3) and (5) that the ideals $\tilde{\mathfrak{p}}_i$ are prime. Since $\bigcap_i \mathfrak{p}_i = \mathfrak{n}$ we obtain from (2)
$$\bigcap_i \tilde{\mathfrak{p}}_i = \bigcap_i \mathfrak{p}_i[t, t^{-1}] \cap \tilde{R} = \mathfrak{n}[t, t^{-1}] \cap \tilde{R} = \tilde{\mathfrak{n}}.$$
Since $\tilde{\mathfrak{p}}_i \cap R[t] = \mathfrak{p}_i[t]$ there are no inclusions $\tilde{\mathfrak{p}}_i \subseteq \tilde{\mathfrak{p}}_j$ for $i \neq j$, and the claim follows. (We use here the well-know fact that the minimal primes in a finitely generated $\mathbb{C}$-algebra are characterized by the condition $\bigcap \mathfrak{p}_i = \mathfrak{n}$, cf. Remark 1.5.) $\square$

In the next lemma we give some properties of the associated graded algebra $\operatorname{gr}_{\mathfrak{m}} R$ where $\mathfrak{m}$ is a maximal ideal of R.

Lemma 4.3. *Let* R *be a* $\mathbb{C}$*-algebra and* $\mathfrak{m} \subseteq R$ *a maximal ideal.*
(1) *Assume that* $\bigcap_j \mathfrak{m}^j = (0)$. *If* $\operatorname{gr}_{\mathfrak{m}} R$ *is a domain, then so is* R.
(2) *Denote by* $\mathfrak{m}R_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$ *the maximal ideal of the localization* $R_{\mathfrak{m}}$. *There is a natural isomorphism* $\operatorname{gr}_{\mathfrak{m}} R \xrightarrow{\sim} \operatorname{gr}_{\mathfrak{m}R_{\mathfrak{m}}} R_{\mathfrak{m}}$ *of graded* $\mathbb{C}$*-algebras.*

Proof. (1) If $ab = 0$ for non-zero elements $a, b \in R$, we can find $i, j \geq 0$ such that $a \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ and $b \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}$. Thus $\bar{a} := a + \mathfrak{m}^{i+1}$ and $\bar{b} := b + \mathfrak{m}^{j+1}$ are non-zero elements in $\operatorname{gr}_{\mathfrak{m}} A$, and $\bar{a}\bar{b} = ab + \mathfrak{m}^{i+j+1} = 0$. This contradiction proves the claim.

(2) Set $\mathfrak{M} := \mathfrak{m}R_{\mathfrak{m}} \subseteq R_{\mathfrak{m}}$. The image of $S := R \setminus \mathfrak{m}$ in $R/\mathfrak{m}^k$ consists of invertible elements and so $R/\mathfrak{m}^k \to R_{\mathfrak{m}}/\mathfrak{M}^k$ is surjective. It is also injective, because $R_{\mathfrak{m}}/\mathfrak{M}^k$ can be identified with the localization of $R/\mathfrak{m}^k$ at S. Thus $R/\mathfrak{m}^k \xrightarrow{\sim} R_{\mathfrak{m}}/\mathfrak{M}^k$ and so $\mathfrak{m}^i/\mathfrak{m}^{i+1} \xrightarrow{\sim} \mathfrak{M}^i/\mathfrak{M}^{i+1}$ for all $i \geq 0$. □

Finally, we need the following result due to Krull. It implies that in a local Noetherian $\mathbb{C}$-algebra R with maximal ideal $\mathfrak{m}$ we have $\bigcap_{j\geq 0} \mathfrak{m}^j = (0)$.

Lemma 4.4 (Krull). *Let R be a Noetherian $\mathbb{C}$-algebra, $\mathfrak{a} \subseteq R$ an ideal and $\mathfrak{b} := \bigcap_{j\geq 0} \mathfrak{a}^j$. Then $\mathfrak{a}\mathfrak{b} = \mathfrak{b}$. In particular, there is an $a \in \mathfrak{a}$ such that $(1 + a)\mathfrak{b} = 0$.*

Proof. The second claim follows from the first and the Lemma of Nakayama (Lemma 3.4). Let $\mathfrak{a} = (a_1, \dots, a_s)$ and put

$$I := \langle f \mid f \in R[x_1, \dots, x_s] \text{ homogeneous and } f(a_1, \dots, a_s) \in \mathfrak{b}\rangle \subseteq R[x_1, \dots, x_s].$$

It is easy to see that I is an ideal of $R[x_1, \dots, x_s]$ and so $I = (f_1, \dots, f_k)$ where the f_j are homogeneous. Choose an $n \in \mathbb{N}$, $n > \deg f_j$ for all j. By definition, $\mathfrak{b} \subseteq \mathfrak{a}^n$ and so, for every $b \in \mathfrak{b}$, there is a homogeneous polynomial $f \in R[x_1, \cdots, x_s]$ of degree n such that $f(a_1, \dots, a_s) = b$. It follows that $f = \sum_j h_j f_j$ where the h_j are homogeneous of degree > 0, and so $b = f(a_1, \dots, a_s) = \sum_j h_j(a_1, \dots, a_s) f_j(a_1, \dots, a_s) \in \mathfrak{a}\mathfrak{b}$. □

The next proposition is a reformulation of our main Theorem 4.1. For later use we will prove it in this slightly more general form.

Proposition 4.5. *Let R be a finitely generated $\mathbb{C}$-algebra and let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then $\dim \operatorname{gr}_{\mathfrak{m}} R = \dim R_{\mathfrak{m}}$. Moreover, $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim R_{\mathfrak{m}}$ if and only if $\operatorname{gr}_{\mathfrak{m}} R$ is a polynomial ring. If this holds, then $R_{\mathfrak{m}}$ is a domain.*

Proof. Inverting an element from $R \setminus \mathfrak{m}$ does not change $\operatorname{gr}_{\mathfrak{m}} R$ (Lemma 4.3(2)). Therefore we can assume that all minimal primes of R are contained in $\mathfrak{m}$. In particular, we have $\dim R_{\mathfrak{m}} = \dim R = \max_i \dim R/\mathfrak{p}_i$ where $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the minimal prime ideals. Moreover, every element from $R \setminus \mathfrak{m}$ is a non-zero divisor.

Now consider the $\mathbb{C}$-algebra $\tilde{R} = R[t, \mathfrak{m}t^{-1}] \subseteq R[t, t^{-1}]$ introduced above. It follows from Lemma 4.2 that $\tilde{R}$ has the following two properties:

(i) $\tilde{R}/\tilde{R}t \xrightarrow{\sim} \operatorname{gr}_{\mathfrak{m}} R$, by (2).
(ii) $\dim \tilde{R}/\tilde{R}t = \dim R$, by (5) and (6).

Hence, $\dim \operatorname{gr}_{\mathfrak{m}} R = \dim R_{\mathfrak{m}}$, proving the first claim.

Assume now that $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = \dim R_{\mathfrak{m}} =: n$. Then we obtain a surjective homomorphism

$$\rho\colon \mathbb{C}[y_1, \dots, y_n] \to \operatorname{gr}_{\mathfrak{m}} R$$

by sending $y_1, \dots, y_n$ to a $\mathbb{C}$-basis of $\mathfrak{m}/\mathfrak{m}^2$. But every proper residue class ring of $\mathbb{C}[y_1, \dots, y_n]$ has dimension $< n$, and so the homomorphism ρ is an isomorphism.

On the other hand, if $\operatorname{gr}_{\mathfrak{m}} R$ is a polynomial ring then $\dim R_{\mathfrak{m}} = \dim \operatorname{gr}_{\mathfrak{m}} R = \dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2$. Moreover, $\bigcap_{j>0} \mathfrak{m}^j = (0)$ by Lemma 4.4, because every element from $R \setminus \mathfrak{m}$ is a non-zero divisor, and so R is a domain by Lemma 4.3(1). $\square$

COROLLARY 4.1. *If X is an affine variety, then $X_{sing} \subseteq X$ is a closed subset whose complement is dense in X.*

PROOF. Let $X = \bigcup_i X_i$ is the decomposition of X into irreducible components. A point $x \in X_i$ is a singular point of X if and only if it is either a singular point of X_i or it belongs to two different irreducible components. Thus

$$X_{\text{sing}} = \bigcup_i (X_i)_{\text{sing}} \cup \bigcup_{j \neq k} X_j \cap X_k,$$

and the claim follows easily. $\square$

Let us denote by $\hat{\mathcal{O}}_{X,x}$ the $\mathfrak{m}_x$-*adic completion* of the local ring $\mathcal{O}_{X,x}$. It is defined to be the *inverse limit*

$$\hat{\mathcal{O}}_{X,x} := \varprojlim \mathcal{O}(X)/\mathfrak{m}_x^k.$$

(We refer to [**Eis95**, I.7.1 and I.7.2] for more details and some basic properties.) Since $\bigcap \mathfrak{m}_x^k = \{0\}$ we have a natural embedding $\mathcal{O}_{X,x} \subseteq \hat{\mathcal{O}}_{X,x}$.

If $X = \mathbb{C}^n$ and $x = 0$ then the completion coincides with the *algebra of formal power series* in n variables:

$$\hat{\mathcal{O}}_{\mathbb{C}^n,0} = \mathbb{C}[\![x_1, \ldots, x_n]\!].$$

The next result is an easy consequence of Theorem 4.1 above.

COROLLARY 4.2. *The point $x \in X$ is non-singular if and only if $\hat{\mathcal{O}}_{X,x}$ is isomorphic to the algebra of formal power series in* $\dim_x X$ *variables.*

REMARK 4.2. A famous result of Auslander-Buchsbaum states that the local ring $\mathcal{O}_{X,x}$ in a nonsingular point of a variety X is always a *unique factorization domain.* For a proof we refer to [**Mat89**, §20, Theorem 20.3].

Vector fields and tangent bundle. Let X be an affine variety. Denote by $TX := \bigcup_{x \in X} T_x X$ the disjoint union of the tangent spaces and by $p\colon TX \to X$ the natural projection, $\delta \in T_x X \mapsto x$. We call TX the *tangent bundle* of X. We will see later that TX has a natural structure of an affine variety and that p is a morphism.

A *section* $\xi\colon X \to TX$ of p, i.e. $p \circ \xi = \mathrm{Id}_X$ or $\xi_x := \xi(x) \in T_x X$ for all $x \in X$, is a collection $(\xi_x)_{x \in X}$ of tangent vectors and thus can be considered as an operator on regular functions $f \in \mathcal{O}(X)$:

$$\xi f(x) := \xi_x f \text{ for } x \in X.$$

DEFINITION 4.3. An *(algebraic) vector field on X* is a section $\xi\colon X \to TX$ with the property that $\xi f \in \mathcal{O}(X)$ for all $f \in \mathcal{O}(X)$. The space of algebraic vector fields is denoted by $\mathrm{Vec}(X)$.
(In the following, we will mostly talk about "vector fields" and omit the term "algebraic" whenever it is clear from the context.)

Thus a vector field ξ can be considered as a linear map $\xi\colon \mathcal{O}(X) \to \mathcal{O}(X)$, and so $\mathrm{Vec}(X)$ is a subvector space of $\mathrm{End}(\mathcal{O}(X))$. More generally, the vector fields form a *module over* $\mathcal{O}(X)$ where the product $f\xi$ for $f \in \mathcal{O}(X)$ is defined in the obvious way: $(f\xi)_x := f(x)\xi_x$.

EXAMPLE 4.5. Let $X = V$ be a $\mathbb{C}$-vector space and fix a vector $v \in V$. Then $\partial_v \in \mathrm{Vec}(V)$ is defined by $x \mapsto \partial_{v,x}$. It follows that

$$\partial_v f := \left.\frac{f(x+tv)-f(x)}{t}\right|_{t=0} \in \mathcal{O}(X)$$

which implies that this vector field is indeed algebraic. We claim that every algebraic vector field on V is of this form. In fact, if $V = \mathbb{C}^n$ then

$$\mathrm{Vec}(\mathbb{C}^n) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1,\dots,x_n]\frac{\partial}{\partial x_i}$$

which means that every algebraic vector field ξ on $\mathbb{C}^n$ is of the form $\xi = \sum_i h_i \frac{\partial}{\partial x_i}$ where $h_i \in \mathbb{C}[x_1,\dots,x_n] = \mathcal{O}(\mathbb{C}^n)$. (This follows from the two facts that every vector field ξ on $\mathbb{C}^n$ is of this form with arbitrary functions h_i and that $\xi(x_i) = h_i$.)

Another observation is that for every vector field ξ on X the corresponding linear map $\xi\colon \mathcal{O}(X) \to \mathcal{O}(X)$ is a derivation, i.e. ξ is a *linear differential operator*:

$$\xi(fh) = f\,\xi h + h\,\xi f \text{ for all } f,h \in \mathcal{O}(X).$$

PROPOSITION 4.6. *The map sending a vector field to the corresponding linear differential operator defines a bijection* $\mathrm{Vec}(X) \xrightarrow{\sim} \mathrm{Der}(\mathcal{O}(X),\mathcal{O}(X)) \subseteq \mathrm{End}(\mathcal{O}(X))$.

PROOF. It remains to show that every derivation $\xi\colon \mathcal{O}(X) \to \mathcal{O}(X)$ is given by an algebraic vector field. For this, define $\xi_x := \mathrm{ev}_x \circ \xi$. Then the vector field $(\xi_x)_{x\in X}$ is algebraic and the corresponding linear map is ξ. $\square$

The Example 4.5 above shows that for $X = V$ we have a canonical isomorphism $TX \simeq X \times V$, using the identifications $T_xX = V \simeq \{x\} \times V$. Then $p\colon TX \to X$ is identified with the projection pr_X and algebraic vector fields correspond to morphism $\xi\colon X \to X \times V$ of the form $\xi(x) = (x, \xi_x)$.

PROPOSITION 4.7. *Let $X \subseteq V$ be a closed subset.*

(1) *If $\xi \in \mathrm{Vec}(V)$ then $\xi|_X$ defines a vector field on X (i.e. $\xi_x \in T_xX$ for all $x \in X$) if and only if $\xi f = 0$ for all $f \in I(X)$. Moreover, it suffices to test a system of generators of the ideal $I(X)$.*
(2) *There is a canonical bijection $TX \xrightarrow{\sim} \{(x,\delta) \mid \delta \in T_xX \subseteq V\}$ where the latter is a closed subset of $X \times V$. Thus TX has the structure of an affine variety. Using coordinates, we get*

$$TX \xrightarrow{\sim} \{(x,a_1,\dots,a_n) \mid \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(x) = 0 \textit{ for all } f \in I(X)\} \subseteq X \times \mathbb{C}^n$$

(3) *A vector field ξ on X is algebraic if and only if $\xi\colon X \to TX$ is a morphism.*

Proof. (1) We have $\xi_x \in T_x X$ for all $x \in X$ if and only if $\xi_x f = 0$ for all x and all $f \in I(X)$ which is equivalent to $\xi f|_X = 0$ for all $f \in I(X)$.

(2) We can assume that $V = \mathbb{C}^n$ and $\mathcal{O}(V) = \mathbb{C}[x_1, \dots, x_n]$. If $I(X) = (f_1, \dots f_m)$ then, by (1),

$$\begin{aligned} T'X &:= \{(x, \delta_x) \in X \times V \mid \delta \in T_x X\} \\ &= \{(x, a_1, \dots, a_n) \mid \sum_{i=1}^n a_i \frac{\partial f_j}{\partial x_i}(x) = 0 \text{ for } j = 1, \dots, m\} \subseteq X \times \mathbb{C}^n \end{aligned}$$

which shows that this is a closed subspace of $X \times \mathbb{C}^n$. Now (2) follows easily.

(3) Using the identification of TX with the closed subvariety $T'X$ above, an arbitrary section $\xi\colon X \to TX$ has the form $\xi_x = \sum h_i(x) \frac{\partial}{\partial x_i}$ with arbitrary functions h_i on X. The vector field ξ is algebraic if and only if $h_i = \xi \bar{x}_i$ is regular on X which is equivalent to the condition that $\xi\colon X \to TX$ is a morphism. $\square$

Example 4.6. Consider the curve $H := \mathcal{V}(xy - 1) \subseteq \mathbb{C}^2$. Then $I(H) = (xy - 1)$. For a vector field $\xi = a(x, y)\partial_x + b(x, y)\partial_y$ on $\mathbb{C}^2$ we get

$$\xi(xy - 1) = a(x, y)y + b(x, y)x.$$

Thus $\xi(xy-1)|_H = 0$ if and only if $ay+bx = 0$ on H. It follows that $x\partial_x - y\partial_y$ defines a vector field ξ_0 on H and that $\operatorname{Vec}(H) = \mathcal{O}(C)\xi_0$. (In fact, setting $h := ay|_H = -bx|_H$ we get $a|_H = h \cdot x|_H$ and $b|_H = -h \cdot y|_H$.)

The tangent bundle $TH \subseteq H \times \mathbb{C}^2$ has the following description (see Proposition 4.7(1)):

$$TH = \{(t, t^{-1}, \alpha, \beta) \mid \alpha t^{-1} + \beta t = 0\} = \{(t, t^{-1}, -\beta t^2, \beta \mid t \in \mathbb{C}^*, \beta \in \mathbb{C}\} \xrightarrow{\sim} H \times \mathbb{C}.$$

Example 4.7. Now consider Neil's parabola $C := \mathcal{V}(y^2 - x^3) \subseteq \mathbb{C}^2$ (see Example 1.8). Then a vector field $a\partial_x + b\partial_y$ defines a vector field on C if and only if

$$-3ax^2 + 2by = 0 \text{ on } C.$$

To find the solutions we use the isomorphism $\mathcal{O}(C) \xrightarrow{\sim} \mathbb{C}[t^2, t^3]$, $x \mapsto t^2, y \mapsto t^3$ (see Example 2.4). Thus we have to solve the equation $3\bar{a}t = 2\bar{b}$ in $\mathbb{C}[t^2, t^3]$. This is easy: Every solution is a linear combination (with coefficients in $\mathbb{C}[t^2, t^3]$) of the two solutions $(2t^2, 3t^3)$ and $(2t^3, 3t^4)$. This shows that

$$\xi_0 := (2x\partial_x + 3y\partial_y)|_D \quad \text{and} \quad \xi_1 := (2y\partial_x + 3x^2\partial_y)|_D$$

are vector fields on C and that $\operatorname{Vec}(C) = \mathcal{O}(C)\xi_0 + \mathcal{O}(C)\xi_1$. Moreover, $\bar{x}^2\xi_0 = \bar{y}\xi_1$.

Our calculation also shows that every vector field on C vanishes in the singular point 0 of the curve. For the tangent bundle we get

$$TC = \{(t^2, t^3, \alpha, \beta) \mid -3\alpha t^4 + 2\beta t^3 = 0\} \subseteq C \times \mathbb{C}^2$$

which has two irreducible components, namely

$$TC = \{(t^2, t^3, 2\alpha, 3\alpha t) \mid t, \alpha \in \mathbb{C}\} \cup \{(0, 0)\} \times \mathbb{C}^2$$

Exercise 4.12. Determine the vector fields on the curve $D := \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{C}^2$. Do they all vanish in the singular point of D?

Exercise 4.13. Determine the vector fields on the curves $D_1 := \{(t, t^2, t^3) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$ and $D_2 := \{(t^3, t^4, t^5) \in \mathbb{C}^3 \mid t \in \mathbb{C}\}$.
(Hint: For D_2 one can use that $\mathcal{O}(D_2) \simeq \mathbb{C}[t^3, t^4, t^5] = \mathbb{C} \oplus \bigoplus_{i\geq 3} \mathbb{C}t^i$.)

Proposition 4.8. *The vector fields* $\mathrm{Vec}(X)$ *on* X *form a Lie algebra with Lie bracket*

$$[\xi, \eta] := \xi \circ \eta - \eta \circ \xi.$$

Proof. By Proposition 4.6 it suffices to show that for any two derivations ξ, η of $\mathcal{O}(X)$ the commutator $\xi \circ \eta - \eta \circ \xi$ is again a derivation. But this is a general fact and holds for any associative algebra, see the following Exercise 4.15. □

Exercise 4.14. Let A be an arbitrary associative $\mathbb{C}$-algebra. Then A is a Lie algebra with Lie bracket $[a, b] := ab - ba$, i.e., the bracket $[\ ,\ ]$ satisfies the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \text{ for all } a, b, c \in A.$$

Exercise 4.15. Let R be an associative $\mathbb{C}$-algebra. If $\xi, \eta \colon R \to R$ are both $\mathbb{C}$-derivations, then so is the commutator $\xi \circ \eta - \eta \circ \xi$. This means that the derivations $\mathrm{Der}(R)$ form a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(R)$.

Exercise 4.16. Let $X \subseteq \mathbb{C}^n$ be a closed and irreducible. Then $\dim TX \geq 2 \dim X$. If X is smooth then TX is irreducible and smooth of dimension $\dim TX = 2 \dim X$.
(Hint: If $I(X) = (f_1, \dots f_m)$ then $TX \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is defined by the equations

$$f_j = 0 \text{ and } \sum_{i=1}^{n} y_i \frac{\partial f_j}{\partial x_i}(x) = 0 \text{ for } j = 1, \dots, m.$$

The Jacobian matrix of this system of $2m$ equations in $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$ has the following block form

$$\begin{bmatrix} \mathrm{Jac}(f_1, \dots, f_m) & 0 \\ * & \mathrm{Jac}(f_1, \dots, f_m) \end{bmatrix}$$

and thus has rank $\geq 2 \cdot \mathrm{rk}\,\mathrm{Jac}(f_1, \dots, f_m) = 2(n - \dim X)$.)

Differential of a morphism. Let $\varphi \colon X \to Y$ be a morphism of affine varieties and let $x \in X$.

Definition 4.4. The *differential of* φ *in* x is the linear map

$$d\varphi_x \colon T_x X \to T_{\varphi(x)} Y$$

defined by $\delta \mapsto d\varphi_x(\delta) := \delta \circ \varphi^*$.

If $Z \subseteq X$ is a closed subvariety and $z \in Z$, then we get for the induced morphism $\varphi|_Z \colon Z \to Y$ that $d(\varphi|_Z)_z = d\varphi_z|_{T_z Z}$. Another obvious remark is that the differential of a constant morphism is the zero map.

Remark 4.3. Set $y := \varphi(x)$. The comorphism $\varphi^* \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ defines a homomorphism $\mathfrak{m}_y \to \mathfrak{m}_x$ and thus a linear map $\bar{\varphi}^* \colon \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$. It is easy to see that the differential $d\varphi_x$ corresponds to the dual map of $\bar{\varphi}^*$ under the isomorphisms $T_x X \simeq \mathrm{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C})$ and $T_y Y \simeq \mathrm{Hom}(\mathfrak{m}_y/\mathfrak{m}_y^2, \mathbb{C})$ (see Lemma 4.1).

Example 4.8. Using the identification $T_{(x,y)} X \times Y = T_x X \oplus T_y Y$ (see Proposition 4.2) we easily see that the differential $d(\mathrm{pr}_X)_x \colon T_{(x,y)} X \times Y \to T_x X$ coincides with the linear projection $\mathrm{pr}_{T_x X}$.

PROPOSITION 4.9. *Let* $\varphi = (f_1, \dots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$, $f_j \in \mathcal{O}(\mathbb{C}^n) = \mathbb{C}[x_1, \dots, x_n]$. *Then the differential*

$$d\varphi_x\colon T_x\mathbb{C}^n = \mathbb{C}^n \to T_{\varphi(x)}\mathbb{C}^m = \mathbb{C}^m$$

of φ *in* $x \in \mathbb{C}^n$ *is given by the Jacobi matrix*

$$\operatorname{Jac}(f_1, \dots, f_m)(x) = \left(\frac{\partial f_j}{\partial x_i}(x)\right)_{i,j}.$$

PROOF. The identification of the tangent space $T_x\mathbb{C}^n = \operatorname{Der}_x(\mathcal{O}(\mathbb{C}^n))$ with $\mathbb{C}^n$ is given by $\delta \mapsto (\delta x_1, \dots, \delta x_n)$ (see Example 4.1). This implies that

$$d\varphi_x(\delta) = ((\delta \circ \varphi^*)(y_1), \dots, (\delta \circ \varphi^*)(y_m)) = (\delta f_1, \dots, \delta f_m).$$

Now the claim follows since

$$\delta f_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot \delta x_i.$$

$\square$

EXERCISE 4.17. Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be morphisms of affine varieties and let $x \in X$. Then

$$d(\psi \circ \varphi)_x = d\psi_y \circ d\varphi_x$$

where $y := \varphi(x) \in Y$.

In order to calculate explicitly differentials of morphisms we will again use the algebra $\mathbb{C}[\varepsilon]$ of dual numbers (Remark 4.1). Recall that for $\delta \in T_xX$ the map $\rho := \operatorname{ev}_x \oplus \delta\varepsilon\colon \mathcal{O}(X) \to \mathbb{C}[\varepsilon]$ is a homomorphism of algebras and vice versa. If $\varphi\colon X \to Y$ is a morphism and $x \in X$, $y := \varphi(x) \in Y$, then we obtain, by definition, the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\operatorname{ev}_x \oplus \delta\varepsilon"] & \mathbb{C}[\varepsilon] \\
\mathcal{O}(Y) \arrow[u, "\varphi^*"] \arrow[ur, "\operatorname{ev}_y \oplus d\varphi_x(\delta)\varepsilon"'] &
\end{tikzcd}$$

If $X := V$ and $Y := W$ are vector spaces then a homomorphism $\rho\colon \mathcal{O}(V) \to \mathbb{C}[\varepsilon]$ corresponds to an element $x \oplus v\varepsilon \in V \oplus V\varepsilon$ where $\rho(f) = f(x + v\varepsilon)$, and so $\rho \circ \varphi^*$ corresponds to the element $\varphi(x+v\varepsilon) \in W \oplus W\varepsilon$. Thus we obtain the following result which is very useful for calculating differentials of morphisms.

LEMMA 4.5. *Let* $\varphi\colon V \to W$ *be a morphism between vector spaces, and let* $x \in V$ *and* $v \in T_xV = V$. *Then we have*

$$\varphi(x + \varepsilon v) = \varphi(x) + d\varphi_x(v)\,\varepsilon$$

where both sides are considered as elements of $W \oplus W\varepsilon$.

EXAMPLE 4.9. The differential of the morphism $?^m\colon M_n \to M_n$, $A \mapsto A^m$, in E is $m \cdot \operatorname{Id}$. In fact, $(E + X\varepsilon)^m = E + mX\varepsilon$.

The differential of $\varphi\colon M_2 \to M_2$, $\varphi(A) := A^2$, in an arbitrary matrix A is given by $d\varphi_A(X) = AX + XA$, because $(A + X\varepsilon)^2 = A^2 + (AX + XA)\varepsilon$.

The differential of the matrix multiplication $\mu\colon M_n \times M_n \to M_n$ in (E, E) is the addition: $(E + X\varepsilon)(E + Y\varepsilon) = E + (X + Y)\varepsilon$.

EXERCISE 4.18. Consider the multiplication $\mu\colon M_2 \times M_2 \to M_2$ and show:

(1) $d\mu_{(A,B)}$ is surjective, if A or B is invertible.
(2) If $\operatorname{rk} A = \operatorname{rk} B = 1$, then $d\mu_{(A,B)}$ has rank 3.
(3) We have $\operatorname{rk} d\mu_{(A,0)} = \operatorname{rk} d\mu_{(0,A)} = 2 \operatorname{rk} A$.

EXERCISE 4.19. Calculate the differential of the morphism $\varphi\colon \operatorname{End}(V) \times V \to V$ given by $(\rho, v) \mapsto \rho(v)$, and determine the pairs (ρ, v) where $d\varphi_{(\rho,v)}$ is surjective.

Tangent spaces of fibers. Let $\varphi\colon X \to Y$ be a morphism, $x \in X$ and $F := \varphi^{-1}(\varphi(x))$ the fiber through x. Since $\varphi|_F$ is the constant map, its differential in any point is zero and so $T_x F \subseteq \ker d\varphi_x$. This proves the first part of the following result.

PROPOSITION 4.10. *Let $\varphi\colon X \to Y$ be a morphism, $x \in X$ and $F := \varphi^{-1}(\varphi(x))$ the fiber through x.*

(1) *$T_x F \subseteq \ker d\varphi_x$.*
(2) *If the fiber F is reduced in x, then $T_x F = \ker d\varphi_x$.*

PROOF. Put $y := \varphi(x) \in Y$. By definition the fiber is reduced in x if and only if the ideal in the local ring $\mathcal{O}_{X,x}$ generated by $\varphi^*(\mathfrak{m}_y)$ is perfect which means that $\mathcal{O}_{F,x} = \mathcal{O}_{X,x}/\varphi^*(\mathfrak{m}_y)\mathcal{O}_{X,x}$ (see Definition 2.2).

Now let $\delta \in T_x X$ be a derivation of $\mathcal{O}(X)$ in x. If $\delta \in \ker d\varphi_x$ then $\delta \circ \varphi^* = 0$. Hence δ, regarded as a derivation of $\mathcal{O}_{X,x}$, vanishes on $\varphi^*(\mathfrak{m}_y)\mathcal{O}_{X,x}$ and thus induces a derivation of $\mathcal{O}_{F,x}$ in x, i.e., $\delta \in T_x F$. $\square$

EXAMPLE 4.10. Let $X \subseteq \mathbb{C}^n$ be a closed subset and $I(X) = (f_1, \ldots, f_m)$. Consider the morphism $\varphi = (f_1, \ldots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$. Then $X = \varphi^{-1}(0)$, and this fiber is reduced in every point. Thus, for every $x \in X$,

$$T_x X = \ker d\varphi_x = \ker \operatorname{Jac}(f_1, \ldots, f_m)(x)$$

as we have already seen in Proposition 4.3.

EXERCISE 4.20. For every point $(x, y) \in X \times Y$ we have $T_x X = \ker d(\mathrm{pr}_Y)_{(x,y)}$ and $T_y X = \ker d(\mathrm{pr}_X)_{(x,y)}$ where $\mathrm{pr}_X, \mathrm{pr}_Y$ are the canonical projections (see Proposition 4.2).

EXERCISE 4.21. For the closed subset $N \subseteq M_2$ of nilpotent 2×2-matrices we have $I(N) = (\operatorname{tr}, \det)$.

Morphisms of maximal rank. The main result of this section is the following theorem.

THEOREM 4.2. *Let $\varphi\colon X \to Y$ be a dominant morphism between two irreducible varieties X and Y. Then there is a dense open set $U \subseteq X$ such that $d\varphi_x\colon T_x X \to T_{\varphi(x)} Y$ is surjective for all $x \in U$.*

We first work out an important example which will be used in the proof of the proposition above.

EXAMPLE 4.11. Let Y be an irreducible affine variety and $X \subseteq Y \times \mathbb{C}$ an irreducible hypersurface. Assume that $I(X) = (f)$ where $f = \sum_{i=0}^n f_i t^i \in \mathcal{O}(Y)[t] =$

$\mathcal{O}(Y \times \mathbb{C})$ and $f_n = 1$. Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "\subseteq"] \arrow[dr, "p"'] & Y \times \mathbb{C} \arrow[d, "\mathrm{pr}_Y"] \\ & Y \end{tikzcd}$$

Then the differential $dp_{(y,a)} \colon T_{(y,a)}X \to T_yY$ is surjective if $\dfrac{\partial f}{\partial t}(y,a) \neq 0$, and this holds on a dense open set of X.

Proof. We have $T_{(y,a)}X \subseteq T_{(y,a)}Y \times \mathbb{C} = T_yY \oplus \mathbb{C}$, and this subspace is given by $T_{(y,a)}X = \{(\delta, \lambda) \mid (\delta,\lambda)f = 0\}$, because $I(X) = (f)$. Now we have

$$(\delta,\lambda)f = \sum_{i=0}^{n} (\delta f_i \cdot a^i + f_i(y) \cdot i \cdot a^{i-1} \cdot \lambda) = \sum_{i=0}^{n} \delta f_i \cdot a^i + \frac{\partial f}{\partial t}(y,a) \cdot \lambda$$

Since $dp_{(y,a)}(\delta,\lambda) = \delta$ we see that $dp_{(y,a)}$ is surjective if $\frac{\partial f}{\partial t}(y,a) \neq 0$ which proves the first claim. But $\frac{\partial f}{\partial t}$ cannot be a multiple of f and thus does not vanish on X, proving the second claim. $\square$

The next lemma shows that the situation described in the example above always holds on an open set for every morphism of finite degree.

Lemma 4.6. *Let X, Y be irreducible affine varieties and $\varphi \colon X \to Y$ a morphism of finite degree. Then there is a special open set $U \subseteq Y$ and a closed embedding $\gamma \colon \varphi^{-1}(U) \hookrightarrow U \times \mathbb{C}$ with the following properties:*

(i) *$I(\gamma(U)) = (f)$ where $f = \sum_{i=0}^{n} f_i t^i \in \mathcal{O}(U)[t]$;*
(ii) *$\mathrm{pr}_U \circ \gamma = \varphi|_{\varphi^{-1}(U)}$.*

$$\begin{tikzcd} \varphi^{-1}(U) \arrow[r, "\simeq"] \arrow[rr, bend left, "\gamma"] \arrow[dr, "\varphi"'] & \mathcal{V}_{U\times\mathbb{C}}(f) \arrow[r, "\subseteq"] \arrow[d, "p"] & U \times \mathbb{C} \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

Proof. We have to show that there is a non-zero $s \in \mathcal{O}(Y)$ such that $\mathcal{O}(X)_s \simeq \mathcal{O}(Y)_s[t]/(f)$ with a polynomial $f \in \mathcal{O}(Y)_s[t]$. Then the claim follows by setting $U := Y_s$.

By assumption, the field $\mathbb{C}(X)$ is a finite extension of $\mathbb{C}(Y)$ of degree n, say,

$$\mathbb{C}(X) = \mathbb{C}(Y)[h] \simeq \mathbb{C}(Y)[t]/(f)$$

where $f = \sum_{i=0}^{n} f_i t^i$, $f_i \in \mathbb{C}(Y)$ and $f_n = 1$. There is an non-zero element $s \in \mathcal{O}(Y)$ such that

(a) $f_i \in \mathcal{O}(Y)_s$ for all i,
(b) $h \in \mathcal{O}(X)_s$ and
(c) $\mathcal{O}(X)_s = \mathcal{O}(Y)_s[h] = \bigoplus_{i=0}^{n-1} \mathcal{O}(Y)_s h^i$.

In fact, (a) and (b) are clear. For (c) we first remark that $\mathcal{O}(Y)_s[h] = \bigoplus_{i=0}^{n-1} \mathcal{O}(Y)_s h^i \subseteq \mathcal{O}(X)_s$, because of (a) and (b). If $h_1, \ldots, h_m$ is a set of generators of $\mathcal{O}(X)$ we can find a non-zero $s \in \mathcal{O}(Y)$ such that $h_i \in \mathcal{O}(Y)_s[h]$, proving (c).

Setting $U := Y_s$ we get $\varphi^{-1}(U) = X_s$ and $\mathcal{O}(X_s) = \mathcal{O}(Y_s)[t]/(f)$, by (c), and the claim follows. $\square$

Proof of Theorem 4.2. By the Decomposition Theorem (Theorem 3.3) we can assume that φ is the composition of a finite surjective morphism and a projection of the form $Y \times \mathbb{C}^r \to Y$. Since the differential of the second morphism is surjective in any point we are reduced to the case of a finite morphism. Now the claim follows from Lemma 4.6 above and the Example 4.11. $\square$

Lemma 4.7. *Let $\varphi\colon X \to Y$ be a morphism, $x \in X$ and $y := \varphi(x) \in Y$. Assume that X is smooth in x and $d\varphi_x$ is surjective.*

(1) *Y is smooth in y.*
(2) *The fiber $\varphi^{-1}(y)$ is reduced and smooth in x, and $\dim_x F = \dim_x X - \dim_y Y$.*

Proof. By assumption,

$$\dim T_x F \leq \dim \ker d\varphi_x = \dim T_x X - \dim T_y Y \leq \dim X - \dim Y \leq \dim_x F$$

which implies that we have equality everywhere. In particular, X and F are both smooth in x.

If we denote by $\bar{\mathfrak{m}} \subseteq \mathcal{O}(X)/\mathfrak{m}_y\mathcal{O}(X)$ the maximal ideal corresponding to $x \in F$ one easily sees that $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$ is the cokernel of the natural map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ induced by φ^*. The duality between $\mathfrak{m}_x/\mathfrak{m}_x^2$ and $T_x X$ (see Lemma 4.1 and Remark 4.3) implies that $\dim \ker d\varphi_x = \dim_{\mathbb{C}} \bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$. Since $\dim \ker d\varphi_x = \dim_x F = \dim \mathcal{O}(X)_x/\mathfrak{m}_y\mathcal{O}(X)_x$ it follows that $\mathcal{O}(X)_x/\mathfrak{m}_y\mathcal{O}(X)_x$ is a domain (Proposition 4.5), and so F is reduced in x. $\square$

Corollary 4.3. *For every morphism $\varphi\colon X \to Y$ there is a dense special open set $U \subseteq X$ such that all fibers of the morphism $\varphi|_U\colon U \to Y$ are reduced and smooth.*

Proof. One easily reduces to the case where X is irreducible. Then there is a special open set $U \subseteq X$ which is smooth (Corollary 4.1) and such that $d\varphi_x$ is surjective for all $x \in U$ (Theorem 4.2). Now the claim follows from the previous Lemma 4.7. $\square$

Corollary 4.4 (Lemma of Sard). *Let $\varphi\colon \mathbb{C}^n \to \mathbb{C}^m$ be a dominant morphism and set $S := \{x \in \mathbb{C}^n \mid d\varphi_x$ is not surjective$\}$. Then S is closed and $\overline{\varphi(S)}$ is a proper closed subset of $\mathbb{C}^m$. In particular, there is a dense open set $U \subseteq \mathbb{C}^m$ such that all fibers $\varphi^{-1}(y)$ for $y \in U$ are reduced and smooth of dimension $n-m$.*

Proof. If $\varphi = (f_1, \ldots, f_m)$ then $S = \{x \in \mathbb{C}^n \mid \operatorname{rk} \operatorname{Jac}(f_1, \ldots, f_m)(x) < m\}$ and so S is closed in $\mathbb{C}^n$. Moreover, the differential of $\varphi|_S\colon S \to \mathbb{C}^m$ at any point of S is not surjective. Therefore, by Theorem 4.2, the closure of the image $\varphi(S)$ has dimension strictly less than m. $\square$

Exercise 4.22. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial. Then $\mathcal{V}(f - \lambda)$ is a smooth hypersurface for almost all $\lambda \in \mathbb{C}$.

Corollary 4.5. *If $\varphi\colon X \to Y$ is a morphism such that $d\varphi_x = 0$ for all $x \in X$, then the image $\varphi(X)$ is finite. In particular, if X is connected then φ is constant.*

PROOF. If $X' \subseteq X$ is an irreducible component and $Y' := \overline{\varphi(X')}$, then the induced morphism $\varphi' \colon X' \to Y'$ has the same property, namely $d\varphi'_x = 0$ for all $x \in X'$. It follows now from Theorem 4.2 that $\dim Y' = 0$. Hence φ is constant on X'. □

EXAMPLE 4.12. Let V be a vector space and $W \subseteq V$ a subspace. If $X \subseteq V$ is a closed irreducible subvariety such that $T_x X \subseteq W$ for all $x \in X$ then $X \subseteq x + W$ for any $x \in X$.
(This follows from the previous corollary applied to the morphism $\varphi \colon X \to V/W$ induced by the linear projection $V \to V/W$.)

5. NORMAL VARIETIES AND DIVISORS

Normality.

DEFINITION 5.1. Let $A \subseteq B$ be rings. An element $b \in B$ is *integral over* A if b satisfies an equation of the form

$$b^n = \sum_{i=0}^{n-1} a_i b^i \quad \text{where } a_i \in A.$$

Equivalently, $b \in B$ is integral over A if and only if the subring $A[b] \subseteq B$ is a finite A-module.

If every element from B is integral over A we say that B *is integral over* A.

EXERCISE 5.1. Let $A \subseteq B$ be rings. If A is Noetherian and B finite over A, then B is integral over A.

LEMMA 5.1. *Let $A \subseteq B \subseteq C$ be rings and assume that A is Noetherian.*

(1) *If B is integral over A and C integral over B, then C is integral over A.*
(2) *The set*
$$B' := \{b \in B \mid b \text{ is integral over } A\}$$
is a subring of B.

PROOF. (1) Let $c \in C$. Then we have an equation $c^m = \sum_{j=0}^{m-1} b_j c^j$ with $b_j \in B$. In particular, the coefficients b_j are integral over A and so, by induction, $A_1 := A[b_0, b_1, \ldots, b_{m-1}]$ is a finitely generated A-module. Moreover, $A_1[c]$ is a finitely generated A_1-module, hence a finitely generated A-module. But then $A[c] \subseteq A_1[c]$ is also finitely generated.

(2) Let $b_1, b_2 \in B'$. Then $A[b_1]$ is integral over A and b_2 is integral over A, hence integral over $A[b_1]$, and so $A[b_1, b_2]$ is integral over $A[b_1]$. Thus, by (1), $A[b_1, b_2]$ is integral over A which implies that $b_1 + b_2$ and $b_1 b_2$ are both integral over A, hence belong to B'. □

EXERCISE 5.2. Let $f \in \mathbb{C}[x]$ be a non-constant polynomial. Then $\mathbb{C}[x]$ is integral over the subalgebra $\mathbb{C}[f]$.

DEFINITION 5.2. Let A be a domain with field of fraction K. We call A *integrally closed* if the following holds:

If $x \in K$ is integral over A then $x \in A$.

An affine variety X is *normal* if X is irreducible and $\mathcal{O}(X)$ is integrally closed. We say that X *is normal in* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is integrally closed.

EXAMPLE 5.1. A unique factorization domain A is integrally closed. In particular, $\mathbb{C}^n$ is a normal variety.
(Let K be the field of fractions of A and $x \in K$ integral over A: $x^n = \sum_{i=0}^{n-1} a_i x^i$ where $a_i \in A$. Write $x = \frac{a}{b}$ where $a, b \in A$ have no common divisor. Then $a^n = b(\sum_{i=0}^{n-1} a_i b^{n-i-1} a^i)$ which implies that b is a unit in A and so $x \in A$.)

EXERCISE 5.3. If the domain A is integrally closed, then so is every ring of fraction A_S where $1 \in S \subseteq A$ is multiplicatively closed.

LEMMA 5.2. *Let X be an irreducible variety. Then X is normal if and only if all local rings $\mathcal{O}_{X,x}$ are integrally closed.*

PROOF. If X is normal then $\mathcal{O}_{X,x} = \mathcal{O}(X)_{\mathfrak{m}_x}$ is integrally closed (see the Exercise above), and the reverse implication follows from $\mathcal{O}(X) = \bigcap_{x\in X} \mathcal{O}_{X,x}$ (Exercise 1.26). □

Integral closure and normalization.

PROPOSITION 5.1. *Let A be a finitely generated $\mathbb{C}$-algebra with no zero-divisors $\neq 0$ and with field of fractions K, and let L/K be a finite field extension. Then*

$$A' := \{x \in L \mid x \text{ is integral over } A\} \supseteq A$$

is a finitely generated $\mathbb{C}$-algebra which is finite over A.

PROOF. We already know that A' is a $\mathbb{C}$-algebra (Lemma 5.1(2)).

(a) We first assume that $A = \mathbb{C}[z_1, \ldots, z_m]$ is a polynomial ring and $K = \mathbb{C}(z_1, \ldots, z_m)$. Let $L = K[x]$ where x is integral over A and $[L : K] =: n$. Denote by $x_1 := x, x_2, \ldots, x_n$ the conjugates of x in some Galois extension L' of K. Clearly, all x_j are integral over A, because they satisfy the same equation as x.

If $y = \sum_{i=0}^{n-1} c_i x^i$ ($c_i \in K$) is an arbitrary element of L we obtain the "conjugates" of y in L' in the form

$$y_j = \sum_{i=0}^{n-1} c_i x_j^i \quad \text{for } j = 1, \ldots, n.$$

The $n \times n$-matrix $X := (x_j^i)$ has determinant $d = \prod_{j<k}(x_j - x_k)$ which is integral over A. Obviously, d^2 is symmetric, hence fixed under the Galois group of L'/K, and so $d^2 \in K$. Since d^2 is also integral over A we finally get $d^2 \in A$. From Cramer's rule we obtain

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = X^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \frac{1}{d} \operatorname{Adj}(X) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

This shows that if y is integral over A then so is dc_i for all i, hence $d^2 c_i \in A$ for all i. This implies that $d^2 A' \subseteq \sum_{i=0}^{n-1} A x^i$, and so A' is a finitely generated A-module.

(b) For the general case we use Noether's Normalization Lemma (Theorem 3.1) which states that A contains a polynomial ring $A_0 = \mathbb{C}[x_1, \ldots, x_m]$ such that A is finite over A_0. Thus A is integral over A_0 and therefore, by Lemma 5.1(1)

$$A' = \{x \in L \mid x \text{ is integral over } A_0\}.$$

It follows from part (a) that A' is a finitely generated A_0-module, hence also a finitely generated A-module. □

DEFINITION 5.3. Let A be a finitely generated $\mathbb{C}$-algebra with no zero-divisors $\neq 0$. If L is a finite field extension of the field of fractions of A, then

$$A' := \{x \in L \mid x \text{ is integral over } A\} \supseteq A$$

is called the *integral closure of A in L*. Clearly, A' is integrally closed.

Let X be an irreducible affine variety and denote by $\mathcal{O}(X)' \subseteq \mathbb{C}(X)$ the integral closure of $\mathcal{O}(X)$ in its field of fractions $\mathbb{C}(X)$. By Proposition 5.1 there is a normal variety $\tilde{X}$ and a finite birational morphism $\eta\colon \tilde{X} \to X$ such that $\mathcal{O}(\tilde{X}) \simeq \mathcal{O}(X)'$. More precisely, we have the following result.

LEMMA 5.3. *Let X be an irreducible variety and $\eta\colon \tilde{X} \to X$ a morphism with the following two properties:*

(1) *$\tilde{X}$ is normal;*
(2) *η is finite and birational.*

Then $\mathcal{O}(\tilde{X})$ is the integral closure of $\eta^*(\mathcal{O}(X))$ in $\mathbb{C}(\tilde{X}) = \eta^*(\mathbb{C}(X))$, and we have the following universal property:

(P) *If Y is a normal affine variety then every dominant morphism $\varphi\colon Y \to X$ factors through η: There is a uniquely determined $\tilde{\varphi}\colon Y \to \tilde{X}$ such that $\varphi = \eta \circ \tilde{\varphi}$:*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\eta"] \\ Y \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & X \end{tikzcd}$$

PROOF. Since η is birational we have $\eta^*(\mathcal{O}(X)) \subseteq \mathcal{O}(\tilde{X}) \subseteq \mathbb{C}(\tilde{X}) = \eta^*(\mathbb{C}(X))$. By (2) $\mathcal{O}(\tilde{X})$ is finite, hence integral over $\eta^*(\mathcal{O}(X))$, and by (1) it is the integral closure of $\eta^*(\mathcal{O}(X))$.

If Y is normal affine variety and $\varphi\colon Y \to X$ a dominant morphism then

$$\mathcal{O}(X) \xrightarrow{\sim} \varphi^*(\mathcal{O}(X)) \subseteq \mathcal{O}(Y) \subseteq \mathbb{C}(Y).$$

Denote by $\mathcal{O}(X)'$ the integral closure of $\mathcal{O}(X)$ in $\mathbb{C}(X)$. Since $\mathcal{O}(Y)$ is integrally closed it follows that $\varphi^*(\mathcal{O}(X)') \subseteq \mathbb{C}(Y)$ is contained in $\mathcal{O}(Y)$. Since η^* induces an isomorphism $\mathcal{O}(X)' \xrightarrow{\sim} \mathcal{O}(\tilde{X})$ there is a uniquely determined homomorphism $\rho\colon \mathcal{O}(\tilde{X}) \to \mathcal{O}(Y)$ which makes the following diagram commutative:

$$\begin{tikzcd} & \mathcal{O}(\tilde{X}) \arrow[dl, "\rho"'] \\ & \mathcal{O}(X)' \arrow[u, "\simeq"] \arrow[dl] \\ \mathcal{O}(Y) & \mathcal{O}(X) \arrow[l, "\varphi^*"] \arrow[u, hook, "\subseteq"] \arrow[uu, bend right, "\eta^*"'] \end{tikzcd}$$

Clearly, the corresponding morphism $\tilde{\varphi}\colon Y \to \tilde{X}$ is the unique morphism such that $\varphi = \eta \circ \tilde{\varphi}$. $\square$

DEFINITION 5.4. The morphism $\eta\colon \tilde{X} \to X$ constructed above is called *normalization of X*. It follows from Lemma 5.3 that it is unique up to a uniquely determined isomorphism.

EXERCISE 5.4. If $\varphi\colon X \to Y$ is a finite surjective morphism where X is irreducible and Y is normal, then $\#\varphi^{-1}(y) \leq \deg \varphi$ for all $y \in Y$. (See Proposition 3.8 and its proof.)

PROPOSITION 5.2. *Let X be an irreducible variety. Then the set*

$$X_{norm} := \{x \in X \mid X \text{ is normal in } x\}$$

is open and dense in X.

PROOF. Let $\mathcal{O}(X)' \subseteq \mathbb{C}(X)$ be the integral closure of $\mathcal{O}(X)$ and define

$$\mathfrak{a} := \{f \in \mathcal{O}(X) \mid f\mathcal{O}(X)' \subseteq \mathcal{O}(X)\}.$$

Then $\mathfrak{a}$ is a non-zero ideal of $\mathcal{O}(X)$, because $\mathcal{O}(X)'$ is finite over $\mathcal{O}(X)$, and $X_{\text{norm}} = X \setminus \mathcal{V}_X(\mathfrak{a})$. In fact, for $S := \mathcal{O}(X) \setminus \mathfrak{m}_x$ we have

$$\mathcal{O}_{X,x} = \mathcal{O}(X)_S \subseteq \mathcal{O}(X)'_S$$

and the latter is the integral closure of $\mathcal{O}_{X,x}$. On the other hand, $\mathcal{O}(X)_S = \mathcal{O}(X)'_S$ if and only if $S \cap \mathfrak{a} \neq \emptyset$ which is equivalent to $x \notin \mathcal{V}_X(\mathfrak{a})$. □

EXERCISE 5.5. Consider the morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^4$, $(x,y) \mapsto (x, xy, y^2, y^3)$.

(1) φ is finite and $\varphi\colon \mathbb{C}^2 \to Y := \varphi(\mathbb{C}^2)$ is the normalization.
(2) $0 \in Y$ is the only non-normal and the only singular point of Y.
(3) Find defining equations for $Y \subseteq \mathbb{C}^4$ and generators of the ideal $I(Y)$.

EXERCISE 5.6. If X is a normal variety then so is $X \times \mathbb{C}^n$.

Discrete valuation rings and smoothness. Let K be a field.

DEFINITION 5.5. A *discrete valuation of the field K* is a surjective map $\nu\colon K^* := K \setminus \{0\} \to \mathbb{Z}$ with the following properties:

(a) $\nu(xy) = \nu(x) + \nu(y)$;
(b) $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

To simplify the notation one usually defines $\nu(0) := \infty$.

EXAMPLE 5.2. Let $K = \mathbb{Q}$ and $p \in \mathbb{N}$ a prime number. Define $\nu_p(x) := r \in \mathbb{Z}$ if p occurs with exponent r in the rational number $x \neq 0$. Then $\nu_p\colon \mathbb{Q}^* \to \mathbb{Z}$ is a discrete valuation of $\mathbb{Q}$.

The following lemma collects some facts about discrete valuations. The easy proofs are left to the reader.

LEMMA 5.4. *Let K be a field and $\nu\colon K^* \to \mathbb{Z}$ a discrete valuation.*

(1) *$A := \{x \in K \mid \nu(x) \geq 0\}$ is a subring of K.*
(2) *$\mathfrak{m} := \{x \in K \mid \nu(x) > 0\} \subseteq A$ is a maximal ideal of A.*
(3) *$\{x \in K \mid \nu(x) = 0\}$ are the units of A.*
(4) *For every non-zero $x \in K$ we have $x \in A$ or $x^{-1} \in A$.*
(5) *$\mathfrak{m} = (x)$ for every $x \in K$ with $\nu(x) = 1$.*
(6) *$\mathfrak{m}^k = \{x \in K \mid \nu(x) \geq k\}$ and these are all non-zero ideals of A.*
(7) *If $\mathfrak{m} = (x)$ then every $z \in K$ has a unique expression of the form $z = tx^k$ where $k \in \mathbb{Z}$ and t is a unit of A.*

DEFINITION 5.6. A domain A is called a *discrete valuation ring* if there is a discrete valuation ν of its field of fractions K such that $A = \{x \in K \mid \nu(x) \geq 0\}$. In particular, A has all the properties listed in Lemma 5.4 above.

EXERCISE 5.7. Let A be a discrete valuation ring with field of fraction K. If $B \subseteq K$ is a subring containing A then either $B = A$ or $B = K$.

In the sequel we will use the following characterization of a discrete valuation rings (see [**AtM69**, Proposition 9.2]).

PROPOSITION 5.3. *Let A be a Noetherian local domain of dimension 1, i.e. the maximal ideal $\mathfrak{m} \neq (0)$ and (0) are the only prime ideals in A. Then the following statements are equivalent:*

(i) *A is a discrete valuation ring.*
(ii) *A is integrally closed.*
(iii) *The maximal ideal $\mathfrak{m}$ is principal.*
(iv) $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.
(v) *Every non-zero ideal of A is a power of $\mathfrak{m}$.*
(vi) *There is an $x \in A$ such that every non-zero ideal of A is of the form (x^k).*

PROOF. (i)⇒(ii): If $x \in K$ and $x \notin A$ then $A[x] = K$ which is not finite over A.

(ii)⇒(iii): Let $a \in \mathfrak{m}$, $a \neq 0$. Then $\mathfrak{m}^k \subseteq (a)$ and $\mathfrak{m}^{k-1} \not\subseteq (a)$ for some $k > 0$. Choose an element $b \in \mathfrak{m}^{k-1} \setminus (a)$ and put $x := \frac{a}{b}$. Then $x^{-1}\mathfrak{m} = \frac{1}{a} b\mathfrak{m} \subseteq \frac{1}{a}\mathfrak{m}^k \subseteq A$. If $x^{-1}\mathfrak{m} \subseteq \mathfrak{m}$ then x^{-1} would be integral over A and so $x^{-1} \in A$, contradicting the construction. Thus $x^{-1}\mathfrak{m} = A$ and so $\mathfrak{m} = (x)$.

(iii)⇒(iv): If $\mathfrak{m} = (x)$ then $\mathfrak{m}/\mathfrak{m}^2 = A/\mathfrak{m} \cdot (x + \mathfrak{m}^2)$, and $\mathfrak{m}^2 \neq \mathfrak{m}$.

(iv)⇒(v): Let $\mathfrak{a} \subseteq A$ be a non-zero ideal. Then $\sqrt{\mathfrak{a}} = \mathfrak{m}$ and so $\mathfrak{m}^k \subseteq \mathfrak{a}$ for some $k \in \mathbb{N}$. Put $\bar{A} := A/\mathfrak{m}^k$ and denote by $\bar{\mathfrak{m}} \subseteq \bar{A}$ the image of $\mathfrak{m}$. Since $\mathfrak{m} = (x) + \mathfrak{m}^2$ we get $\mathfrak{m} = (x) + \mathfrak{m}^k$ for all $k \in \mathbb{N}$ and so $\bar{\mathfrak{m}} = (\bar{x}) \subseteq \bar{A}$. Now it is easy to see that $\bar{\mathfrak{a}} = \bar{\mathfrak{m}}^r$ for some $r \leq k$, and so $\mathfrak{a} = \mathfrak{m}^r$.

(v)⇒(vi): We have $\mathfrak{m} \neq \mathfrak{m}^2$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then, by assumption, $(x) = \mathfrak{m}^k$ for some $k \geq 1$, and so $\mathfrak{m} = (x)$.

(vi)⇒(i): By assumption, every element $a \in A$ has a unique expression of the form $a = tx^k$ where $k \in \mathbb{N}$ and t a unit of A. Define $\nu(a) := k$. This has a well-defined extension to K^* by setting $\nu(\frac{a}{b}) := \nu(a) - \nu(b)$ for $a, b \in A$, $b \neq 0$. One easily verifies that ν is a discrete valuation of K and that A is the corresponding valuation ring. □

If Y be an irreducible curve, i.e. $\dim Y = 1$, then the local rings $\mathcal{O}_{Y,y}$ satisfy the assumptions of the proposition above. The equivalence of (i), (ii) and (iv) then gives the following result. (In fact, we do not need to assume that Y is irreducible; cf. Theorem 4.1.)

PROPOSITION 5.4. *Let Y be an affine variety and $y \in Y$ such that* $\dim_y Y = 1$. *Then the following statements are equivalent:*

(i) *The local ring $\mathcal{O}_{Y,y}$ is a discrete valuation ring.*
(ii) *Y is normal in y.*
(iii) *Y is smooth in y.*

In particular, a normal curve is smooth and an irreducible smooth curve is normal.

REMARK 5.1. Let X be an irreducible variety and $H \subseteq X$ an irreducible hypersurface, i.e. $\operatorname{codim}_X H = 1$. The ideal $\mathfrak{p} := I(H)$ of H is a minimal prime ideal $\neq (0)$ and thus the localization $\mathcal{O}_{X,H} := \mathcal{O}(X)_\mathfrak{p}$ is a local Noetherian domain of dimension 1. If X is normal it follows from Proposition 5.3 that $\mathcal{O}_{X,H}$ is a discrete valuation ring which corresponds to a discrete valuation $\nu_H \colon \mathbb{C}(X)^* \to \mathbb{Z}$.

E.g. if $f \in \mathbb{C}[x_1, \dots, x_n]$ is a non-constant irreducible polynomial and $H := \mathcal{V}(f)$, then the valuation ν_H has the following description: *For a rational function $r \in \mathbb{C}(x_1, \dots, x_n)$ we have $\nu_H(r) = m$ if f occurs with exponent m in a primary decomposition of r.*

Normal varieties. We start with the following generalization of the previous result saying that normal curves are smooth (Proposition 5.4). Recall that the singular points X_{sing} of an affine variety form a closed subset with a dense complement (Proposition 4.1).

PROPOSITION 5.5. *Let X be a normal affine variety. Then* $\operatorname{codim}_X X_{\text{sing}} \geq 2$.

PROOF. (a) Let $H \subseteq X$ be an irreducible hypersurface and assume that $I(H) = (f)$. We claim that if $x \in H$ is a singular point of X then x is a singular point of H, too. In fact, $\mathcal{O}(H) = \mathcal{O}(X)/(f)$ and $\mathfrak{m}_{H,x} = \mathfrak{m}_x / f\mathcal{O}(X)$. Thus $\mathfrak{m}_{H,x}/\mathfrak{m}_{H,x}^2 = (\mathfrak{m}_x/\mathfrak{m}_x^2)/\mathbb{C} \cdot \bar{f}$ and so $\dim T_x H \geq \dim T_x X - 1 > \dim X - 1 = \dim H$.

(b) Now assume that $\operatorname{codim}_X X_{\text{sing}} = 1$, and let $H \subseteq X_{\text{sing}}$ be an irreducible hypersurface of X. If $\mathfrak{p} := I(H)$ is a principal ideal it follows from (a) that H consists of singular points. But this contradicts the fact that the smooth points of an irreducible variety form a dense open set.

In general, the localization $\mathcal{O}_{X,H}$ is a discrete valuation ring (Remark 5.1) and therefore its maximal ideal $\mathfrak{p}\mathcal{O}_{X,H}$ is principal (Proposition 5.3). This implies that we can find an element $s \in \mathcal{O}(X) \setminus \mathfrak{p}$ such that the ideal $\mathfrak{p}\mathcal{O}(X)_s \subseteq \mathcal{O}(X)_s = \mathcal{O}(X_s)$ is principal. Since $\mathfrak{p}\mathcal{O}(X)_s = I(H \cap X_s)$ we arrive again at a contradiction, namely that all points of $H \cap X_s$ are singular. $\square$

Another important property of normal varieties is that regular functions can be extended over closed subset of codimension ≥ 2.

PROPOSITION 5.6. *Let X be a normal affine variety and $r \in \mathbb{C}(X)$ a rational function which is defined on an open set $U \subseteq X$. If* $\operatorname{codim}_X X \setminus U \geq 2$ *then r is a regular function on X.*

PROOF. Define the "ideal of denominators" $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid q \cdot r \in \mathcal{O}(X)\}$. By definition $U \subseteq V \setminus \mathcal{V}_X(\mathfrak{a})$ and so, by assumption, $\operatorname{codim}_X \mathcal{V}_X(\mathfrak{a}) \geq 2$.

Using NOETHER'S Normalization Lemma (Theorem 3.1) we can find a finite surjective morphism $\varphi \colon X \to \mathbb{C}^n$. We have $\varphi(\mathcal{V}_X(\mathfrak{a})) = \mathcal{V}(\mathfrak{a} \cap \mathbb{C}[x_1, \dots, x_n])$ and $\dim \varphi(\mathcal{V}_X(\mathfrak{a})) = \dim \mathcal{V}(\mathfrak{a} \cap \mathbb{C}[x_1, \dots, x_n]) \leq n - 2$. This implies that we can find two polynomials $q_1, q_2 \in \mathfrak{a} \cap \mathbb{C}[x_1, \dots, x_n]$ with no common divisor (see the following Exercise 5.8). As a consequence, we can find $p_1, p_2 \in \mathcal{O}(X)$ such that $r = \dfrac{p_1}{q_1} = \dfrac{p_2}{q_2}$.

If $r = r^{(1)}, r^{(2)}, \dots, r^{(d)}$ are the conjugates of r in some finite field extension $L/\mathbb{C}(x_1, \dots, x_n)$ of degree d containing $\mathbb{C}(X)$ we have

$$r^{(i)} = \frac{p_1^i}{q_1} = \frac{p_2^i}{q_2} \quad \text{for } i = 1, \dots, d$$

where the $p_1^{(i)}$ are the conjugates of p_1 and the $p_2^{(i)}$ the conjugates of p_2. The element $r \in \mathbb{C}(X)$ satisfies the equation

$$\prod_{i=1}^{d}(t - r^{(i)}) = t^d + \sum_{j=1}^{d} b_j t^{n-j} = 0$$

where the coefficients $b_j \in \mathbb{C}(x_1, \dots, x_n)$ are given by the elementary symmetric functions s_j in the following form:

$$b_j = \pm s_j(r^{(1)}, \dots, r^{(d)}) = \pm \frac{1}{q_1^j} s_j(p_1^{(1)}, \dots, p_1^{(d)}) = \pm \frac{1}{q_2^j} s_j(p_1^{(1)}, \dots, p_2^{(d)}).$$

Since $p_1, p_2 \in \mathcal{O}(X)$ are integral over $\mathbb{C}[x_1, \dots, x_n]$ we see that both $s_j(p_1^{(1)}, \dots, p_1^{(d)})$ and $s_j(p_2^{(1)}, \dots, p_2^{(d)})$ belong to $\mathbb{C}[x_1, \dots, x_n]$. Since q_1 and q_2 have no common factor this implies that $b_j \in \mathbb{C}[x_1, \dots, x_n]$. As a consequence, r is integral over $\mathbb{C}[x_1, \dots, x_n]$ and thus belongs to $\mathcal{O}(X)$. □

EXERCISE 5.8. Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal with the property that any two elements $f_1, f_2 \in \mathfrak{a}$ have a non-trivial common divisor. Then there is a non-constant h which divides every element of $\mathfrak{a}$.

COROLLARY 5.1. *If X is a normal variety then $\mathcal{O}(X) = \bigcap_{\mathfrak{p}} \mathcal{O}(X)_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through the minimal prime ideals $\neq (0)$.*

PROOF. Let $r \in \bigcap_{\mathfrak{p}} \mathcal{O}(X)_{\mathfrak{p}}$ and define $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid q \cdot r \in \mathcal{O}(X)\}$. It follows that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for all minimal primes $\mathfrak{p} \neq 0$, and so $\mathcal{V}_X(\mathfrak{a})$ does not contain an irreducible hypersurface. This implies that $\operatorname{codim}_X \mathcal{V}_X(\mathfrak{a}) \geq 2$ and so r is regular by the Proposition 5.6 above. □

We thus have the following characterization of normal varieties. *An irreducible variety X is normal if and only if the following two condition hold:*

(a) *For every minimal prime $\mathfrak{p} \neq (0)$ the local ring $\mathcal{O}(X)_{\mathfrak{p}}$ is a discrete valuation ring;*
(b) *$\mathcal{O}(X) = \bigcap_{\mathfrak{p}} \mathcal{O}(X)_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through the minimal prime ideals $\neq (0)$.*

We have seen in examples that there are bijective morphisms which are not isomorphisms. This cannot happen if the target variety is normal.

PROPOSITION 5.7. *Let X be an irreducible and Y a normal affine variety and let $\varphi\colon X \to Y$ be a dominant morphism. Assume*

(a) $\operatorname{codim}_Y \overline{Y \setminus \varphi(X)} \geq 2$, *and*
(b) $\deg \varphi = 1$.

Then φ is an isomorphism.

PROOF. By assumption (b), we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O}(Y) & \xrightarrow{\subseteq} & \mathcal{O}(X) \\
{\scriptstyle\subseteq}\big\downarrow & & {\scriptstyle\subseteq}\big\downarrow \\
\mathbb{C}(Y) & = & \mathbb{C}(X)
\end{array}$$

If $H \subseteq Y$ is an irreducible hypersurface then, by assumption (a), H meets the image $\varphi(X)$ in a dense set and so $\overline{\varphi(\varphi^{-1}(H))} = H$. This implies that there is an irreducible hypersurface $H' \subseteq X$ such that $\overline{\varphi(H')} = H$. If we denote by $\mathfrak{p} := I(H) \subseteq \mathcal{O}(Y)$ and $\mathfrak{p}' := I(H') \subseteq \mathcal{O}(X)$ the corresponding minimal prime ideals we get $\mathfrak{p}' \cap \mathcal{O}(Y) = \mathfrak{p}$. Thus

$$\mathcal{O}(Y)_{\mathfrak{p}} \subseteq \mathcal{O}(X)_{\mathfrak{p}'} \subsetneqq \mathbb{C}(Y) = \mathbb{C}(X).$$

Since $\mathcal{O}(Y)_{\mathfrak{p}}$ is a discrete valuation ring this implies $\mathcal{O}(Y)_{\mathfrak{p}} = \mathcal{O}(X)_{\mathfrak{p}'}$ (see Exercise 5.7). Thus, by Corollary 5.1,

$$\mathcal{O}(X) \subseteq \bigcap_{\mathfrak{p}'} \mathcal{O}(X)_{\mathfrak{p}'} = \bigcap_{\mathfrak{p}} \mathcal{O}(Y)_{\mathfrak{p}} = \mathcal{O}(Y),$$

and the claim follows. □

There is a partial converse of Proposition 5.5 which is a special case of the so-called Serre Criterion for Normality which we will explain below without giving a proof.

Proposition 5.8. *Let $H \subseteq \mathbb{C}^n$ be an irreducible hypersurface. If the singular points H_{sing} have codimension ≥ 2 in H, then H is normal.*

Example 5.3. Let $Q_n := \mathcal{V}(x_1^2 + x_2^2 + \cdots + x_n^2) \subseteq \mathbb{C}^n$. Then $\dim Q_n = n - 1$ and $0 \in Q_n$ is the only singular point. Thus Q_n is normal for $n \geq 3$.

Exercise 5.9. Show that the nilpotent cone $N := \{A \in M_2 \mid A \text{ nilpotent}\}$ is a normal variety.

Proposition 5.9 (Serre's Criterion). *Let $X \subseteq \mathbb{C}^n$ be the zero set of $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$: $X := \mathcal{V}(f_1, \ldots, f_r)$. Define*

$$X' := \{x \in X \mid \operatorname{rk} \operatorname{Jac}(f_1, \ldots, f_r)(x) < r\}.$$

(1) *If $X \setminus X'$ is dense in X then $I(X) = (f_1, \ldots, f_r)$ and $X' = X_{\text{sing}}$.*
(2) *If $\operatorname{codim}_X X \setminus X' \geq 2$ then X is normal.*

Example 5.4. let $N := \{A \in M_n \mid A \text{ nilpotent}\}$ the nilpotent cone in M_n. We claim that N is a normal variety.

Proof. Consider the morphism $\pi\colon M_n \to \mathbb{C}^n$, $\pi(A) := (\operatorname{tr} A, \operatorname{tr} A^2, \ldots, \operatorname{tr} A^n)$. Then $N = \pi^{-1}(0)$. If $P \in N$ is a nilpotent element of rank $n-1$ then $\operatorname{rk} d\pi_P = n$. In fact, $\operatorname{tr}(P + \varepsilon X)^k = \operatorname{tr}(P^k + \varepsilon k P^{k-1} X) = \varepsilon k \operatorname{tr}(P^{k-1} X)$. Taking P in Jordan normal form one easily sees that $d\pi_P\colon X \mapsto (\operatorname{tr} X, \operatorname{tr} PX, \operatorname{tr} P^2 X, \ldots, \operatorname{tr} P^{n-1} X)$ is surjective. It follows that $\operatorname{rk} \operatorname{Jac}(f_1, \ldots, f_n)(P) = n$ for the functions $f_j(A) := \operatorname{tr} A^j$ and for $P \in N' := \{\text{nilpotent matrices of rank } n-1\}$. Now one shows that $\operatorname{codim}_N N \setminus N' = 2$. □

Divisors. Let X be a normal affine variety. Define

$$\mathcal{H} := \{H \subseteq X \mid H \text{ irreducible hypersurface}\}.$$

Definition 5.7. A *divisor on X* is a finite formal linear combination

$$D = \sum_{H \in \mathcal{H}} n_H \cdot H \quad \text{where } n_H \in \mathbb{Z}.$$

We write $D \geq 0$ if $n_H \geq 0$ for all $H \in \mathcal{H}$. The set of divisors forms the *divisor group*

$$\operatorname{Div} X = \bigoplus_{H \in \mathcal{H}} \mathbb{Z} \cdot H.$$

Recall that for any irreducible hypersurface $H \in \mathcal{H}$ we have defined a discrete valuation $\nu_H \colon \mathbb{C}(X)^* \to \mathbb{Z}$ whose discrete valuation ring is the local ring $\mathcal{O}_{X,H}$ (see Remark 5.1).

Definition 5.8. For $f \in \mathbb{C}(X)^*$ we define the *divisor of* (f) by

$$(f) := \sum_{H \in \mathcal{H}} \nu_H(f) \cdot H.$$

Such a divisors is called a *principal divisor.*

Remarks 5.2. (1) (f) is indeed a divisor, i.e. $\nu_H(f) \neq 0$ only for finitely many $H \in \mathcal{H}$.
(This is clear for $f \in \mathcal{O}(X) \setminus \{0\}$, because $\nu_H(f) > 0$ if and only if $H \subseteq \mathcal{V}(f)$, and follows for a general $f = \frac{p}{q}$ because $(f) = (p) - (q)$, by definition.)
(2) $(f \cdot h) = (f) + (h)$ for all $f, h \in \mathbb{C}(X)$.
(3) $(f) \geq 0$ if and only if $f \in \mathcal{O}(X)$.
(We have $\nu_H(f) \geq 0$ if and only if $f \in \mathcal{O}_{X,H}$. Since $\bigcap_{H \in \mathcal{H}} \mathcal{O}_{X,H} = \mathcal{O}(X)$ the claim follows.)
(4) $(f) = 0$ if and only if f is a unit in $\mathcal{O}(X)$.
(If $(f) = 0$ then, by (3), $f \in \mathcal{O}(X)$ and $f^{-1} \in \mathcal{O}(X)$.)

Definition 5.9. Two divisors $D, D' \in \operatorname{Div} X$ are called *linearly equivalent*, written $D \sim D'$, if $D - D'$ is a principal divisor. The set of equivalence classes is the *divisor class group of* X:

$$\operatorname{Cl} X := \operatorname{Div} X / \{\text{principal divisors}\}$$

It follows that we have an exact sequence of commutative groups

$$1 \to \mathcal{O}(X)^* \to \mathbb{C}(X)^* \to \operatorname{Div} X \to \operatorname{Cl} X \to 0$$

Remark 5.3. We have $\operatorname{Cl} X = 0$ if and only if $\mathcal{O}(X)$ is a unique factorization domain. In fact, a unique factorization domain is characterized by the condition that all minimal prime ideals $\mathfrak{p} \neq (0)$ are principal.

Example 5.5. Let $C \subseteq \mathbb{C}^2$ be a smooth curve. If $f \in \mathcal{O}(C)$ and $\tilde{f} \in \mathbb{C}[x, y]$ a representative of f, then

$$(f) = \sum_{P \in C \cap \mathcal{V}(\tilde{f})} m_P \cdot P,$$

and the integers $m_P > 0$ can be understood as the *intersection multiplicity* of C and $\mathcal{V}(\tilde{f})$ in P. E.g. if the intersection is transversal, i.e., $T_P C \cap T_P \mathcal{V}(\tilde{f}) = (0)$ then $m_P = 1$ (see the following Exercise 5.10).

Exercise 5.10. Let $C, E \subseteq \mathbb{C}^2$ be two irreducible curves, $I(C) = (f)$ and $I(E) = (h)$. If $P \in C \cap E$ define $m_P := \dim_{\mathbb{C}} \mathbb{C}[x, y]/(f, h)$. Show that

(1) If C is smooth and $\bar{h} = h|_C \in \mathcal{O}(C)$, then $(\bar{h}) = \sum_{P \in C \cap E} m_P \cdot P$
(2) If $P \in C \cap E$ and $T_P C \cap T_P E = (0)$ then $m_P = 1$.

EXERCISE 5.11.

(1) For the parabola $C = \mathcal{V}(y - x^2)$ we have $\operatorname{Cl} C = (0)$.
(2) For an elliptic curve $E = \mathcal{V}(y^2 - x(x^2 - 1))$ every divisor D is linearly equivalent to 0 or to P for a suitable point $P \in E$.

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