A Littelmann Path Model for Borcherds-Kac-Moody algebras

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<u>0 – Review</u>

0.1 Notation. Let $\mathfrak{g}$ be a BKM algebra associated with an $n \times n$ BKM matrix A, and set $I = \{1, 2, \ldots, n\}$.

$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$,

$\mathfrak{n}$ is generated by the e_i, $i \in I$,

$\mathfrak{n}^-$ is generated by the f_i, $i \in I$,

$\mathfrak{h}$ is the Cartan subalgebra.

$Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$, the root lattice of $\mathfrak{g}$.

$Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$, the positive root lattice of $\mathfrak{g}$.

$P := \{h \in \mathfrak{h}^* \,|\, \langle \alpha_i^\vee, h\rangle \in \mathbb{Z},\, i \in I\}$, the weight lattice of $\mathfrak{g}$.

$P^+ := \{h \in P \,|\, \langle \alpha_i^\vee, h\rangle \geq 0,\, i \in I\}$, the dominant weights.

0.2 Representations of $\mathfrak{g}$

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-diagonalizable if
(a) $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where

$$V_\lambda = \{v \in V \mid hu = \lambda(h)v,\, h \in \mathfrak{h}\}.$$

(b) dim $V_\lambda < +\infty$.

Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module.

- V is called integrable, if for all $i \in I^{re}$, f_i and e_i act locally nilpotently on V.

- V is called a highest weight module if there exists $\Lambda \in \mathfrak{h}^*$ and $0 \neq v_\Lambda \in V_\Lambda$, such that
  (a) $e_i v_\Lambda = 0$ for all $i \in I$.
  (b) $V = U(\mathfrak{g})v_\Lambda$.

If V is a highest weight module of highest weight Λ, then
(a) $V = U(\mathfrak{n}^-)v_\Lambda$.
(b) $\dim V_\Lambda = 1$.
(c) If $V_\lambda \neq 0$, then $\lambda \in \Lambda - Q^+$.

Lemma. Let V be a highest weight $\mathfrak{g}$-module of highest weight Λ. If V is integrable, then $\langle \alpha_i^\vee, \Lambda \rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in I^{re}$.

For each $\Lambda \in \mathfrak{h}^*$ there exists a unique irreducible highest weight $\mathfrak{g}$-module of highest weight Λ. We denote in by $V(\Lambda)$.

Lemma Take $\Lambda \in P^+$ and for all $i \in I$, set $m_i = \langle \alpha_i^\vee, \Lambda \rangle \in \mathbb{Z}_{\geq 0}$. Let V be a highest weight $\mathfrak{g}$-module of highest weight Λ.
Then $V \simeq V(\Lambda)$ (and thus irreducible) if and only if:
(a) if $m_i = 0$, then $f_i v_\Lambda = 0$.
(b) for all $i \in I^{re}$ one has $f_i^{m_i+1} v_\Lambda = 0$.

Example. Let $A \in M_1(\mathbb{R})$ and $\Lambda \in P^+$. Set $m = \langle \alpha_1^\vee, \Lambda \rangle \in \mathbb{Z}_{\geq 0}$. A basis for the $\mathfrak{g}$-module $V(\Lambda)$ is :

- $\{v_\Lambda, f_1 v_\Lambda, \ldots, f_1^m v_\Lambda\}$, if $a_{11} = 2$.

- $\{v_\Lambda\}$, if $a_{11} \leq 0$ and $m = 0$.

- $\{f_1^k v_\Lambda \,|\, k \in \mathbb{Z}_{\geq 0}\}$, if $a_{11} \leq 0$ and $m > 0$.

1 – The Path Model

From now on, we assume that the BKM matrix has all its entries in $\mathbb{Z}$.

A path $\pi : [0, 1] \to \mathbb{R}P$ is a continuous function such that $\pi(0) = 0$ and $\pi(1) \in\ P$.

We identify two paths, if they are equal up to a reparametrization.

$\mathbb{P}$:=set of all paths.

We call the weight of a path $\pi \in \mathbb{P}$ its endpoint, and we write wt $\pi = \pi(1)$.

For all $i \in I$ define on $\mathfrak{h}^*$ (or on P):

$$r_i(x) = x - \langle \alpha_i^\vee, x \rangle \alpha_i.$$

If $i \in I^{re}$, then r_i is a simple reflection. Set

$$W := < r_i \,|\, i \in I^{re} > .$$

If $i \in I^{im}$, then $r_i^k \neq 1$, for all $k > 0$. Set

$$T := < r_i \,|\, i \in I > .$$

Note that T is a monoid.

1.1 Distance of two weights.

Fix $\Lambda \in P^+$.

Let $\mu, \nu \in T\Lambda$. We write $\mu > \nu$ if there exists :

- a sequence of weights in $T\Lambda$
  $\mu := \lambda_1, \lambda_2, \ldots, \lambda_k := \nu$

- and positive roots in $W\Pi \cap \Delta^+$
  $\beta_1, \ldots, \beta_{k-1}$

such that $\lambda_i = r_{\beta_i}\lambda_{i+1}$ and $\langle \beta_i^\vee, \lambda_{i+1} \rangle > 0$, for all i, with $1 \leq i \leq k-1$.

We call the *distance* of μ and ν and write $\mathrm{dist}\,(\mu, \nu)$ the maximal length of such sequences.

If $\mu = r_\beta \nu > \nu$ and $\mathrm{dist}\,(\mu, \nu) = 1$ we write $\nu \overset{\beta}{\leftarrow} \mu$.

1.2 a-chains. Let $\mu > \nu$ be two weights in $T\Lambda$ and let $0 < a \leq 1$ be a rational number. An a-chain for (μ, ν) is a sequence :

$$\nu := \nu_s \overset{\beta_s}{\leftarrow} \nu_{s-1} \overset{\beta_{s-1}}{\leftarrow} \dots \overset{\beta_2}{\leftarrow} \nu_1 \overset{\beta_1}{\leftarrow} \nu_0 =: \mu,$$

such that for all i with $1 \leq i \leq s$:

(a) $a\langle \beta_i^\vee, \nu_i \rangle \in \mathbb{Z}_{>0}$, if $\beta_i \in W\Pi_{re} \cap \Delta^+$.
(b) $a\langle \beta_i^\vee, \nu_i \rangle = 1$, if $\beta_i \in W\Pi_{im}$.

1.3 Generalized Lakshmibai–Seshadri paths.

Suppose we have :

$\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ a sequence of weights in $T\Lambda$,

$\{0 < a_1 < a_2 < \cdots < a_k = 1\}$ a sequence of rational numbers in $[0, 1]$,

such that for all i, with $1 \leq i \leq k - 1$:

(1) there exists an a_i-chain for $(\lambda_i, \lambda_{i+1})$

(2) if $\lambda_k \neq \Lambda$, there exists an 1-chain for (λ_k, Λ).

To these data we attach a piecewise linear path which for $t \in [a_{j-1}, a_j]$, takes the form

$$\pi(t) = \sum_{i=1}^{j-1} (a_i - a_{i-1})\lambda_i + (t - a_{j-1})\lambda_j.$$

Such a path is called *a Generalized Lakshmibai-Seshadri path of shape* Λ.

We denote the above path by

$$\pi = (\lambda_1, \lambda_2, \ldots, \lambda_k; 0, a_1, \ldots, a_{k-1}, 1).$$

$\mathbb{P}_\Lambda :=$ set of Generalized Lakshmibai-Seshadri paths of shape Λ.

One can check that wt $\pi \in \Lambda - Q^+$.

Then $\mathbb{P}_\Lambda \subset \mathbb{P}$.

There is a unique path in $\mathbb{P}_\Lambda$ of weight Λ, namely the linear path $\pi_\Lambda := (\Lambda; 0, 1)$.

Remark.

For any $\mu \in P$, denote by $\pi_\mu(t) = \mu t$.

For all $i \in I^{re}$, $\pi_{r_i\Lambda}(t) = (r_i\Lambda)t$ is a GLS path.

Similarly, for all $w \in W$, $\pi_{w\Lambda}$ is a GLS path.

For imaginary indices, this is not true in general.

1.4 Examples.

(1) Take $A = (2)$ and $\mathfrak{g}$ the associated Kac-Moody algebra, $\Pi = \{\alpha\}$, $r := r_\alpha$.

Take $\Lambda \in P^+$ such that $\langle \alpha^\vee, \Lambda \rangle = m > 0$. The only GLS paths of shape Λ are :

$$\pi_0 = (\Lambda; 0, 1),$$

$$\pi_1 = (r\Lambda, \Lambda; 0, \tfrac{1}{m}, 1),$$

$$\pi_2 = (r\Lambda, \Lambda; 0, \tfrac{2}{m}, 1),$$

....

$$\pi_m = (r\Lambda; 0, 1).$$

(2) Take $A = (-k)$ with $k \geq 0$, $\mathfrak{g}$ the associated generalized Kac-Moody algebra, $\Pi = \{\alpha\}$, $r := r_\alpha$.

Take $\Lambda \in P^+$ such that $\langle \alpha^\vee, \Lambda \rangle = m > 0$. The only GLS paths of shape Λ are :

$$
\begin{aligned}
&\pi_0 = (\Lambda; 0, 1), \\
&\pi_1 = (r\Lambda, \Lambda; 0, \tfrac{1}{m}, 1), \\
&\pi_2 = (r^2\Lambda, r\Lambda, \Lambda; 0, \tfrac{1}{m(1+k)}, \tfrac{1}{m}, 1), \\
&\pi_3 = (r^3\Lambda, r^2\Lambda, r\Lambda, \Lambda; 0, \tfrac{1}{m(1+k)^2}, \tfrac{1}{m(1+k)}, \tfrac{1}{m}, 1), \\
&\cdots\cdots \\
&\pi_s = (r^s\Lambda, \ldots, r\Lambda, \Lambda; 0, \tfrac{1}{m(k+1)^{s-1}}, \ldots, \tfrac{1}{m(k+1)}, \tfrac{1}{m}, 1), \\
&\cdots\cdots
\end{aligned}
$$

1.5 Applications of the Littelmann Path Model.

Define the character of $\mathbb{P}_\Lambda$ to be

$$\text{char } \mathbb{P}_\Lambda = \sum_{\pi \in \mathbb{P}_\Lambda} e^{\pi(1)}.$$

(1) The multiplicity of a weight ν in $V(\Lambda)$ equals the number of different paths in $\mathbb{P}_\Lambda$ with endpoint ν. In particular,

$$\text{char } V(\Lambda) = \text{char } \mathbb{P}_\Lambda.$$

(2) By defining the tensor product of two paths to be their concatenation, Littelmann gave a combinatorial proof of the Parthasaraty-Ranga Rao-Varadarajan conjecture for the decomposition of the tensor product of two highest weight modules.

(3) Littelmann defined root operators, $\tilde{e}_i$, $\tilde{f}_i$ on $\mathbb{P}$; these operators are similar to Kashiwara crystal basis operators. In particular, one can endow $\mathbb{P}$ with a crystal structure. As a crystal $\mathbb{P}_{\Lambda} \simeq B(\Lambda)$.

2 – Definition of the Root Operators.

We will define root operators $\tilde{f}_i$, $\tilde{e}_i$ on $\mathbb{P}_\Lambda$, for all $i \in I$.

For any $i \in I$ and $\pi \in \mathbb{P}_\Lambda$, let $h_i^\pi : [0,1] \to \mathbb{R}$ be the continuous function defined by :

$$h_i^\pi(t) = \langle \alpha_i^\vee, \pi(t) \rangle.$$

Lemma. All local minima of h_i^π, $i \in I$, $\pi \in \mathbb{P}_\Lambda$ are integers.

Set $m_i^\pi = \min\{h_i^\pi(t) \,|\, t \in [0,1]\} \in \mathbb{Z}$.

Let $P_i^\pi \in [0,1] \cap \mathbb{Q}$ be maximal such that

$$h_i^\pi(P_i^\pi) = m_i^\pi.$$

If $P_i^\pi = 1$ we set $\tilde{f}_i \pi = 0$.

If not, let $Q_i^\pi \in [P_i^\pi, 1] \cap \mathbb{Q}$ be minimal such that

$$h_i^\pi(Q_i^\pi) = m_i^\pi + 1.$$

We set

$$\tilde{f}_i\pi(t) = \begin{cases} \pi(t), & t \in \ [0, P_i^\pi], \\ r_i\pi(t) + m_i^\pi \alpha_i, & t \in \ [P_i^\pi, Q_i^\pi], \\ \pi(t) - \alpha_i, & t \in \ [Q_i^\pi, 1]. \end{cases}$$

Notice that $\text{wt } \tilde{f}_i\pi = \text{wt } \pi - \alpha_i$.

Lemma. In $[P_i^\pi, Q_i^\pi]$ the function h_i^π is strictly increasing.

The $\tilde{e}_i$ are defined as follows :

Let $\tilde{e}_i\pi \neq 0$. Then $\tilde{e}_i\pi = \pi'$ if and only if $\tilde{f}_i\pi' = \pi$.

Then $\mathrm{wt}\ \tilde{e}_i\pi = \mathrm{wt}\ \pi + \alpha_i$

Lemma. Let $\pi \in \mathbb{P}_\Lambda$ and $i \in I^{im}$. Then h_i^π is increasing, and one of the following is true :

1. $P_i^\pi = 0$ and $\tilde{f}_i\pi \neq 0$.

2. $P_i^\pi = 1$ and $\tilde{f}_i\pi = 0$.

Lemma. Take $i \in I$ and let $\pi \in \mathbb{P}_\Lambda$ be such that $\tilde{f}_i \pi = \pi' \neq 0$.

1. If $i \in I^{im}$, then $h_i^{\pi'}(t) \geq h_i^{\pi}(t)$, so $P_i^{\pi'} = P_i^{\pi} = 0$, whereas $Q_i^{\pi'} \leq Q_i^{\pi}$, with equality if and only if $a_{ii} = 0$. In particular, $\tilde{f}_i^k \pi \neq 0$, for all $k \geq 0$.

2. If $i \in I^{re}$, then $m_i^{\pi'} = m_i^{\pi} - 1$ and $P_i^{\pi'} = Q_i^{\pi}$. In particular, there exists $k \in \mathbb{N}$ such that $\tilde{f}_i^k \pi = 0$.

Proposition. The set $\mathbb{P}_\Lambda$ is stable under the action of the root operators.

Sketch of Proof.
Let $\pi = (\lambda_1, \ldots, \lambda_k; a_0, a_1, \ldots, a_{k-1}, 1)$ be a GLS path of shape Λ such that $\tilde{f}_i \pi \neq 0$.

Since a GLS path π is piecewise linear, $P_i^\pi = a_t$ for some $0 \leq t \leq k-1$. If $i \in I^{im}$, then $t = 0$.

We can assume $a_{p-1} < Q_i^\pi \leq a_p$ for $t+1 \leq p \leq k$.

The path $\tilde{f}_i \pi$ is equal to

$$f_i \pi = (\lambda_1, \ldots, \lambda_t, r_i \lambda_{t+1}, \ldots, r_i \lambda_p, \lambda_p, \ldots \lambda_k; \\ a_0, a_1, \ldots, a_{p-1}, Q_i^\pi, a_p, \ldots, a_k),$$

with $t = 0$ if $i \in I^{im}$.

We know that $\langle \alpha_i^\vee, \lambda_t \rangle \leq 0$ and $\langle \alpha_i^\vee, \lambda_j \rangle > 0$, for all $j \in \{t+1, p\}$.

We need to show that there exists

- an a_t-chain for $(\lambda_t, r_i\lambda_{t+1})$,
- a_j-chains for $(r_i\lambda_j, r_i\lambda_{j+1})$ for all $j \in \{t+1, p-1\}$,
- an Q_i^π-chain for $(r_i\lambda_p, \lambda_p)$.

Theorem. Let $\mathcal{F}$ denote the monoid generated by the $\tilde{f}_i$, $i \in I$ and $\pi_\Lambda(t) = \Lambda t$. Then

$$\mathbb{P}_\Lambda = \mathcal{F}\pi_\Lambda.$$

Sketch of Proof.

- Since $\pi_\Lambda \in \mathbb{P}_\Lambda$ and the $\tilde{f}_i$, $i \in I$ stabilise $\mathbb{P}_\Lambda$, we obtain $\mathcal{F}\pi_\Lambda \subset \mathbb{P}_\Lambda$.

- We show that π_Λ is the only path in $\mathbb{P}_\Lambda$ such that $\tilde{e}_i\pi_\Lambda = 0$ for all $i \in I$.

- Take any path $\pi \neq \pi_\Lambda$ in $\mathbb{P}_\Lambda$. There exists $i \in I$ and $\pi' \in \mathbb{P}_\Lambda$ such that $\tilde{e}_i\pi = \pi'$ and so $\pi = \tilde{f}_i\pi'$. Then

$$\text{wt } \pi' = \text{wt } \pi + \alpha_i.$$

  If $\pi' \neq \pi_\Lambda$ we continue the procedure.

- Since for any path $\pi \in \mathbb{P}_\Lambda$, wt $\pi \leq \Lambda$ this procedure will stop.

2–The Character Formula.

Recall that for an $\mathfrak{h}$-diagonalisable $\mathfrak{g}$-module V, its character is defined to be :

$$\text{char } V := \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda.$$

We have computed the character of the irreducible integrable highest weight $\mathfrak{g}$-module $V(\Lambda)$.

$$\text{char } V(\Lambda) = \frac{\sum_{w\in W}\sum_{F\in \mathcal{P}(\Pi_{im})^{\Lambda}} (-1)^{\ell(w)+|F|} e^{w(\Lambda+\rho-s(F))}}{\sum_{w\in W}\sum_{F\in \mathcal{P}(\Pi_{im})} (-1)^{\ell(w)+|F|} e^{w(\rho-s(F))}}$$

- $\mathcal{P}(\Pi_{im})$ is the set of all subsets of Π_{im} of pairwise orthogonal roots.

- $\mathcal{P}(\Pi_{im})^{\Lambda}$ is the set of all sets F in $\mathcal{P}(\Pi_{im})$, such that $\langle \alpha_i^{\vee}, \Lambda\rangle = 0$ for all $\alpha_i \in F$.

- $|F| = \text{Card } F$.

- $s(F) = \sum_{\alpha_i\in F} \alpha_i$.

- $\rho \in \mathfrak{h}^*$ such that $\langle \alpha_i^{\vee}, \rho\rangle = \frac{1}{2}a_{ii}$ for all $i \in I$.

Theorem. $\operatorname{char} \mathbb{P}_\Lambda = \operatorname{char} V(\Lambda)$.

A crucial fact for the proof of the above is the following :

Lemma. Let $i, j \in I^{im}$ and $\pi_1, \pi_2 \in \mathbb{P}_\Lambda$ be such that $\tilde{f}_i \pi_1 = \tilde{f}_j \pi_2$.
Then $\tilde{f}_i$, $\tilde{f}_j$ commute and $\pi_1 = \tilde{f}_j \pi_1'$, for some $\pi_1' \in \mathbb{P}_\Lambda$.

Sketch of Proof of Theorem in case $\Pi = \Pi_{im}$.

We need to show that :

$$\sum_{\pi \in \mathbb{P}_\Lambda} \sum_{F \in \mathcal{P}(\Pi)} (-1)^{|F|} e^{-s(F)+\pi(1)} =$$

$$\sum_{F \in \mathcal{P}(\Pi)^\Lambda} (-1)^{|F|} e^{\Lambda - s(F)}$$

Let Ω be the set of all pairs $(F_0, \pi_0) \in \mathcal{P}(\Pi) \times \mathbb{P}_\Lambda$ such that :

for all $\alpha_i \notin F_0$ with $\langle \alpha_i^\vee, s(F_0) \rangle = 0$ we have $\tilde{e}_i \pi_0 = 0$.

For $(F_0, \pi_0) \in \Omega$ define

$\Omega(F_0, \pi_0) := \{(F_0 \setminus \{\alpha_{i_1}, \ldots \alpha_{i_k}\}, \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \pi_0)\} \subset \mathcal{P}(\Pi) \times \mathbb{P}_\Lambda$.

Lemma. $\mathcal{P}(\Pi) \times \mathbb{P}_\Lambda = \bigsqcup_{(F_0, \pi_0) \in \Omega} \Omega(F_0, \pi_0)$.

What we need to show becomes :

$$\sum_{(F_0,\pi_0)\in\Omega} \quad \sum_{(F,\pi)\in\Omega(F_0,\pi_0)} (-1)^{|F|} e^{-s(F)+\pi(1)} =$$

$$\sum_{F\in\ \mathcal{P}(\Pi)^{\Lambda}} (-1)^{|F|} e^{\Lambda - s(F)}$$

Set

$$\Sigma := \sum_{(F,\pi)\in\Omega(F_0,\pi_0)} (-1)^{|F|} e^{-s(F)+\pi(1)}$$

Lemma.

1. If $|\Omega(F_0,\pi_0)| > 1$, then $\Sigma = 0$.

2. If $|\Omega(F_0,\pi_0)| = 1$, then $\pi_0 = \pi_\Lambda$ and $F_0 \in \mathcal{P}(\Pi)^{\Lambda}$.