Borcherds-Kac-Moody Algebras study group:

Weyl groups & Root systems.

(Wakimoto's book, §2.2 – 2.3)

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7 November 2008

§ 0 – Reminder.

A **BKM matrix** is $A \in M_n(\mathbb{R})$ with:

- $a_{ii} = 2$ or $a_{ii} \leq 0$,
- if $i \neq j$, $a_{ij} \leq 0$,
- $a_{ij} = 0 \iff a_{ji} = 0$,
- if $a_{ii} = 2$ then $a_{ij} \in \mathbb{Z}$ for all j.

Write $I = \{1, \ldots, n\}$. There are **real indices** and **imaginary indices**:

$$I^{re} = \{i \in I \mid a_{ii} = 2\},$$

$$I^{im} = \{i \in I \mid a_{ii} \leq 0\}.$$

Assume A **symmetrisable**, $A = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n)B$.

We have a vector space $\mathfrak{h}$ and:

- **simple roots**, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathfrak{h}^*$,

- **simple coroots** $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subseteq \mathfrak{h}$,

- **real** and **imaginary** simple roots (Π^{re}, Π^{im}),

- a non-degenerate symmetric bilinear form $(.|.)$ on $\mathfrak{h}$, with which we *identify* $\mathfrak{h}$ *and* $\mathfrak{h}^*$.

Properties:

- $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$,

- for all $h \in \mathfrak{h}$, $(\alpha_i^\vee | h) = \varepsilon_i \langle \alpha_i, h \rangle$,

- $\alpha_i^\vee = \varepsilon_i \, \alpha_i$,

- $(\alpha_i | \alpha_j) = b_{ij}$,

- if $a_{ii} \neq 0$:

$$\frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_j)} = \frac{2\langle \alpha_i^\vee, \lambda \rangle}{a_{ii}} \quad (\lambda \in \mathfrak{h}^*).$$

The **BKM algebra** $\mathfrak{g} = \mathfrak{g}(A)$ is defined by

generators: $e_1, \ldots, e_n,\ f_1, \ldots, f_n,\ \mathfrak{h}$

relations:

- $[\mathfrak{h}, \mathfrak{h}] = 0$,
- $[h, e_i] = \alpha_i(h)\, e_i$, for $h \in \mathfrak{h}$,
- $[h, f_i] = -\alpha_i(h)\, f_i$, for $h \in \mathfrak{h}$,
- $[e_i, f_j] = \delta_{i,j}\, \alpha_i^\vee$,
- (for $i \in I^{re}, j \in I, i \neq j$):
$$\begin{cases} (ad\, e_i)^{1-a_{ij}}\, e_j & = & 0, \\ (ad\, f_i)^{1-a_{ij}}\, f_j & = & 0, \end{cases}$$
- (for $i, j \in I, i \neq j$): if $a_{ij} = 0$, then
$$[e_i, e_j] = [f_i, f_j] = 0.$$

Properties:

- root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda,$$

- **positive** and **negative** roots $\Delta_\pm$,

- triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$,

- $(.|.)$ extends to a non-degenerate symmetric *invariant* bilinear form on $\mathfrak{g}$, with

  – if $\alpha + \beta \neq 0$ then $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$,

  – for $\alpha \in \Delta$, $(.|.)$ is non-degenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$,

  – for $\alpha, \beta \in \Delta$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$:

$$[x, y] = (x|y)\,\alpha.$$

§ 1 – Weyl groups.

1.1. Definitions.

Let $\alpha \in \mathfrak{h}^*$ with $(\alpha|\alpha) \neq 0$. Define $r_\alpha \in \mathrm{GL}(\mathfrak{h})$ to be:

$$r_\alpha : \begin{cases} \mathfrak{h} & \longrightarrow & \mathfrak{h} \\ h & \longmapsto & h - 2\frac{(\alpha|h)}{(\alpha|\alpha)}\,\alpha, \end{cases}$$

the **reflection w.r.t. the hyperplane $\alpha^\perp$.**

Exercise:

- $r_\alpha^2 = \mathrm{id}_{\mathfrak{h}}$,
- for $h, h' \in \mathfrak{h}$: $\Big(r_\alpha(h)\Big|r_\alpha(h')\Big) = (h|h')$,
- If both α, β non-isotropic:

$$r_{r_\beta(\alpha)} = r_\beta \circ r_\alpha \circ r_\beta^{-1}.$$

For $i \in I^{re}$, write simply $r_i := r_{\alpha_i}$ for the corresponding **simple reflection**.

Alternative formulas:

$$(\forall\, \lambda \in \mathfrak{h}^*),\ \ r_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda \rangle\, \alpha_i,$$

$$(\forall\, h \in \mathfrak{h}),\ \ r_i(h) = h - \langle \alpha_i, h \rangle\, \alpha_i^\vee.$$

Definition (*Weyl group*):

$$W = \langle r_i \mid i \in I^{re} \rangle \subseteq \mathsf{GL}(\mathfrak{h}).$$

1.2. Properties of the Weyl group.

Immediate properties:

- For non-isotropic $\alpha \in \mathfrak{h}$ and all $w \in W$:

$$r_{w(\alpha)} = w \circ r_\alpha \circ w^{-1}.$$

- The bilinear form $(.|.)$ on $\mathfrak{h}$ is W-invariant.

Proposition.

(1) $W(\Delta) = \Delta$; in particular $W(\Pi) \subseteq \Delta$.

(2) If $\alpha \in \Delta$ and $w \in W$, then:

$$\mathrm{mult}(w(\alpha)) = \mathrm{mult}(\alpha).$$

1.3. Real and imaginary roots.

Define **real** and **imaginary** roots:

$$\Delta^{re} = W(\Pi^{re}), \quad \Delta^{im} = \Delta \setminus \Delta^{re}.$$

Exercise: $\alpha \in \Delta^{re} \Rightarrow -\alpha \in \Delta^{re}$.

Proposition. For $\alpha \in \Delta^{re}$, we have:

(1) $(\alpha|\alpha) > 0$,
(2) $r_\alpha \in W$,
(3) $\mathrm{mult}(\alpha) = 1$.

Proposition. For $i \in I^{re}$, we have:

$$r_i(\Delta_+ \setminus \{\alpha_i\}) = \Delta_+ \setminus \{\alpha_i\}.$$

1.4. Complement: Presentation of W.

W is a Coxeter group, presented by:

generators: r_i, for $i \in I^{re}$,

relations:

- $r_i^2 = 1$,
- $(r_i r_j)^{m_{i,j}} = 1$, with exponents:

$a_{ij}\, a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

§ 2 – Root systems.

2.1. Definitions.

- $Q = \sum_{i=1}^{n} \mathbb{Z}\,\alpha_i$, the **root lattice**,

- $Q_+ = \sum_{i=1}^{n} \mathbb{N}\,\alpha_i$, the **positive root lattice**.

Support of $\alpha = \sum m_i \alpha_i \in Q$:

$$\mathrm{supp}(\alpha) = \{i \in I \mid m_i \neq 0\}.$$

The **quasi-Dynkin diagram** qDyn(A):

- vertices: 1 2 ... n,
- edges: i — j iff. $a_{ij} \neq 0$.

Say supp(α) is **connected** if it is a connected subset of qDyn(A).

Lemma. $\alpha \in \Delta \Rightarrow$ supp(α) is connected.

2.2. Real roots.

Proposition. Let $\alpha \in \Delta$ and $i \in I^{re}$.

(1) The set $\{j \in \mathbb{Z} \mid \alpha + j\,\alpha_i \in \Delta\}$ is finite:

- it has the form $\{m, m+1, \ldots, M\}$.
- $m + M = -\langle \alpha_i^\vee, \alpha \rangle$.
- the sequence

$$\Big\{\text{mult}(\alpha + j\,\alpha_i) \mid j = m, \ldots, M\Big\}$$

has bilateral symmetry, and the left half is non-decreasing.

(2) $(\alpha|\alpha_i) > 0 \Rightarrow \alpha - \alpha_i \in \Delta_+$
$(\alpha|\alpha_i) < 0 \Rightarrow \alpha + \alpha_i \in \Delta_+$.

Corollary. ($\alpha \in \Delta$ and $i \in I^{re}$)
If $\alpha + \alpha_i \notin \Delta$ and $\alpha - \alpha_i \in \Delta$, then $(\alpha|\alpha_i) > 0$.

2.3. Imaginary roots.

Proposition. Let $i \in I^{im}$ and $\alpha \in \Delta_+ \setminus \{\alpha_i\}$ such that supp$(\alpha + \alpha_i)$ connected. Then:

$$(\forall\, j \in \mathbb{N}), \quad \alpha + j\,\alpha_i \in \Delta_+.$$

Remark. If $\alpha \in \Delta^{re}$, then:

$$\mathbb{Z}\,\alpha \cap \Delta = \{\alpha, -\alpha\}.$$

If $\alpha \in \Delta^{im}$, it may happen that $\mathbb{Z}\alpha \subseteq \Delta$.

Set $\mathfrak{h}'_{\mathbb{R}} = \sum_{i=1}^{n} \mathbb{R}\,\alpha_i$.

Define the **Positive Weyl chamber**:

$$C^\vee = \Big\{ h \in \mathfrak{h}'_{\mathbb{R}} \mid \langle h, \alpha_i^\vee \rangle \geq 0 \ \forall\, i \in I^{re} \Big\}.$$

Lemma.
(1) Δ_+^{im} is W-invariant.
(2) For $\alpha \in \Delta_+^{im}$:

$$\exists\, w \in W \quad \text{st.} \quad w\alpha \in -C^\vee.$$

(3) For $\alpha \in \Delta_+$:

$$\alpha \in \Delta_+^{im} \iff (\alpha|\alpha) \leq 0.$$

Corollary. A root $\alpha \in \Delta$ is real iff. $(\alpha|\alpha) > 0$.

Let:

$$K = \left\{ \alpha \in Q_+ \text{ st. } \left\{ \begin{array}{l} \langle \alpha, \alpha_i^\vee \rangle \leq 0 \ \forall\, i \in I^{re}, \\ \text{supp}(\alpha) \ \ \text{connected.} \end{array} \right. \right\}$$

$$\overset{\circ}{K} = \{\alpha \in K \text{ st. } |\text{supp}(\alpha)| \geq 2\}.$$

Imaginary roots can be **domestic** or **alien**:

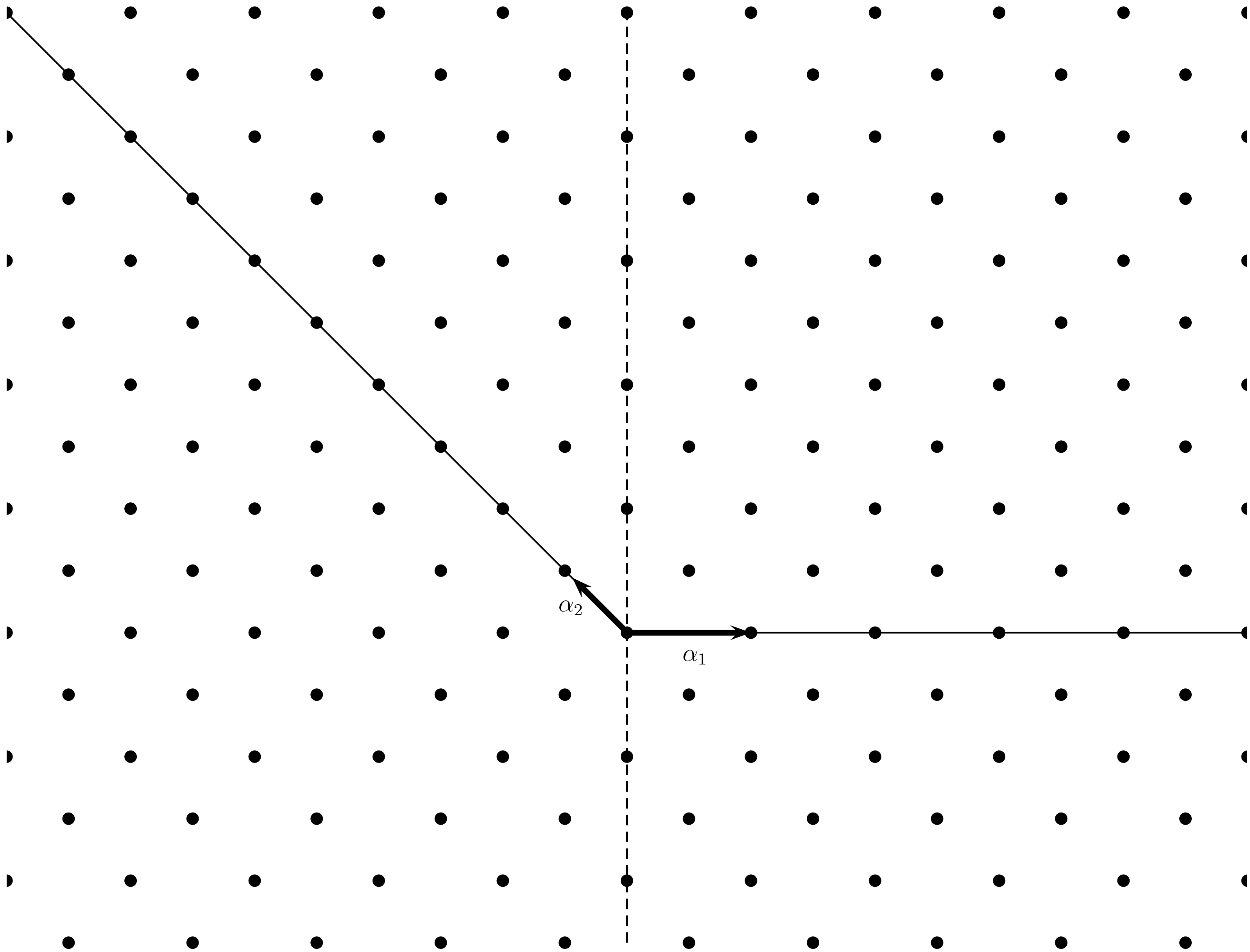
- $\Delta^{dom} = W(\Pi^{im}) \cup W(-\Pi^{im}) \subseteq \Delta^{im}$,
- $\Delta^{ali} = \Delta^{im} \setminus \Delta^{dom}$.

Theorem (imaginary roots of symmetrisable BKM matrix).

(1) $\Delta_+^{ali} = W(\overset{\circ}{K})$.

(2) $\Delta_+^{im} = W(\overset{\circ}{K}) \cup W(\Pi^{im})$.

Exercise: On the grid of next page (thanks to Daiva for drawing it!), represent Δ_+ in the case $A = \begin{bmatrix} 2 & -1 \\ -c & -d \end{bmatrix}$, where $c > 0$ and $d \geq 0$.



α_2
α_1