Generalized $q$-Schur algebras

At the very outset of representation theory roughly 100 years ago, Issai Schur studied polynomial representations of general linear groups, relating them to representations of symmetric groups. This relation was eventually encapsulated in a double centralizer property often referred to as “Schur–Weyl duality.” In 1980 J.A. Green published a monograph [Springer Lecture Notes 830] in which Schur’s approach was reversed, showing how results of Gordon James on modular representations of symmetric groups can be understood in terms of Schur’s original framework. Green emphasized the importance of certain finite dimensional algebras, implicit in Schur’s early work, which he named “Schur algebras.” Later it was observed that the module category for a Schur algebra is a highest weight category in the sense of Cline–Parshall–Scott; thus Schur algebras are examples of quasihereditary algebras enjoying striking homological properties. There have been many further recent results in modular representation theory coming out of this connection. One is a result of Erdmann (exploiting Ringel’s theory of tilting modules) that shows the equivalence of knowing modular decomposition numbers of symmetric and general linear groups. Another recent striking application of Schur algebras was their use by Friedlander and Suslin in proving finite generation of cohomology of finite group schemes.

In 1986 all this was generalized by S. Donkin, who established a theory of “generalized Schur algebras” associated to any reductive algebraic group $G$. A generalized Schur algebra may be defined by truncating the category of rational $G$-modules at a finite saturated set $\pi$ of dominant weights. This was inspired and motivated by earlier work of Jantzen, who had studied truncation to the category of $p^n$-bounded representations. Generalized Schur algebras are quasihereditary, and the original Schur algebras in Type $A$ defined by Green are examples of generalized Schur algebras. Thus, one gets for example “symplectic” and “orthogonal” Schur algebras associated with symplectic and orthogonal groups. Such generalized Schur algebras have deep connections with the Brauer centralizer algebras introduced by Brauer [Annals of Math, 1937], similar to the connection between general linear groups and symmetric groups. Detailed study of this connection in the modular case has recently been undertaken by a number of authors.

Also in the eighties it was noticed by Jimbo, and Dipper and James, that Schur–Weyl duality admits a $q$-deformation in which the symmetric group is replaced by an Iwahori–Hecke algebra and the general linear group by an appropriate quantum group. This led to the Dipper–James theory of $q$-Schur algebras, which had unexpected applications to modular representations of
finite general linear groups in the non-defining characteristic. The $q$-Schur algebras were studied from a completely different perspective in a 1990 paper of Beilinson, Lusztig, and MacPherson where they were used to give a beautiful geometric construction of the quantized enveloping algebra of $\mathfrak{gl}_n$. Further developments in this direction include the spectacular Lascaux–Leclerc–Thibon conjecture relating decomposition numbers for $q$-Schur algebras to canonical bases of the quantized enveloping algebra $U_q(\hat{\mathfrak{sl}}_n)$ (corresponding to the affine Kac–Moody Lie algebra $\hat{\mathfrak{sl}}_n$) and its subsequent proof by Ariki along lines suggested by work of Grojnowski.

Again, the construction of $q$-Schur algebras may be generalized from Type $A$ to any type. One can create a theory of “generalized $q$-Schur algebras” by analogy with Donkin’s theory of generalized Schur algebras for algebraic groups. These algebras are again quasihereditary, so their module category is again a highest weight category.

These lectures focus on the theory of generalized Schur algebras and generalized $q$-Schur algebras. It turns out that one can give a simple unified definition of all of them by means of generators and relations, in terms of the underlying root datum. In a sense, this generalizes Serre’s work from the sixties on generators and relations for semisimple Lie algebras. Turning the viewpoint around, one can reconstruct the corresponding quantized enveloping algebra from an inverse system of $q$-Schur algebras.