GORENSTEIN DERIVED CATEGORIES AND GORENSTEIN SINGULARITY CATEGORIES

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ABSTRACT. The Gorenstein derived category is introduced and studied via Verdier's localization of homotopy category respect to the saturated multiplicative system of Gorenstein quasi-isomorphisms; the relation between the Gorenstein derived category and the derived category is given; for a Gorenstein ring or a finite-dimensional k-algebra, the corresponding bounded Gorenstein derived categories are realized as the homotopy categories of Gorenstein projective objects. This interprets the Gorenstein derived functors as the Hom functor of the corresponding bounded Gorenstein derived category. The Gorensteinness of a ring is measured by its Gorenstein singularity category; and the stable category of a Frobenius category is embedded into the Gorenstein singularity category as a triangulated subcategory.

Key words: Gorenstein projective objects; Gorenstein quasi-isomorphisms; Gorenstein derived and singularity categories

0. Introduction

In the past thirty five years, the theory and the use of derived categories and triangulated categories have enjoyed a vigorous development (see e.g. [Ver2], [I], [BBD], [Hap1], [CPS], [Ric1], [Kel2], [KZ], [Ko], [GM], [N], [RV], [O1], [KS], [Rou]). On the other hand, relative homological algebra, especially Gorenstein homological algebra, has been developed to an advanced level (see e.g. [EC], [AB], [AF], [Y], [EJ2], [Ch], [AM], [Hol3], [CFH], [Vel], [J], [CV]).

In the theory of triangulated categories, one obtains a new triangulated category by Verdier's localization from a saturated multiplicative system of a triangulated category, and what one gets is usually (but not always) algebraic in the sense that it is triangle-equivalent to the stable category of a Frobenius category. One gets in this way the derived category from the saturated multiplicative system of quasi-isomorphisms of homotopy category; by the projective resolutions the bounded derived category $D^b(\mathcal{A})$ is simplified as the homotopy category $K^{-,b}(\mathcal{P})$ of upper bounded complexes of projective objects, with only finitely many non-zero cohomology groups; and the classical right derived functors Ext^n can be interpreted as the Hom functor of $D^b(\mathcal{A})$.

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In homological algebra ([CE]), projective and injective objects play a fundamental role. In Gorenstein homological algebra, they are replaced respectively by Gorenstein projective and Gorenstein injective objects, introduced by E. E. Enochs and O. M. G. Jenda ([EJ1]); and one considers the Gorenstein projective and Gorenstein injective dimensions of objects and of complexes (see e.g. [EJ1], [EJ2], [Ch], [Hol1]-[Hol3], [AF], [AM], [Y], [T]). Note that the existence of *proper* Gorenstein projective (resp. co-proper Gorenstein injective) resolutions is a non-trivial problem (see [J]; also [AM], [Ch], [Hol1], [T], [Vel]). Note that the concept Gorenstein projective objects even goes back to M. Auslander and M. Bridger [AB], where they introduced the G-dimension of finitely generated module M over a two-sided Noetherian ring; and then L. L. Avramov, A. Martisinkovsky and I. Rieten have proved that M is Goreinstein projective if and only if the G-dimension of M is zero (the remark following Theorem (4.2.6) in [Ch]). Also note that the concept has also been generalized to triangulated categories (see [AS], [Bel1], [Bel2]).

It is then natural to have a theory of the so-called Gorenstein derived category, which is the aim of this paper. Note that a chain map $f^\bullet$ is a quasi-isomorphism if and only if it is projectively quasi-isomorphism in the sense that $\operatorname{Hom}_{\mathcal{A}}(P, f^\bullet)$ is a quasi-isomorphism for any projective object P, if and only if it is an injectively quasi-isomorphism. However, a Gorenstein projectively quasi-isomorphism is not a Gorenstein injectively quasi-isomorphism. So, by Verdier's localization of homotopy category respect to the saturated multiplicative system of Gorenstein projectively quasi-isomorphisms we have the Gorenstein projectively derived category $D_{gp}(\mathcal{A})$ and its upper bounded and bounded version, and dually we have the Gorenstein injectively derived category $D_{gi}(\mathcal{A})$, which is not triangle-equivalent to $D_{gp}(\mathcal{A})$.

The relation between the Gorenstein derived category and the derived category is given (Corollary 2.6). Some basic results in derived category have been developed to this setting up. For examples, the homotopy category $K^b(\mathcal{GP})$ of bounded complexes of Gorenstein projective objects is a triangulated subcategory of $D^b_{gp}(\mathcal{A})$ (Theorem 2.11); for a Gorenstein ring, or a finite-dimensional k-algebra, the corresponding bounded Gorenstein derived category is realized as the homotopy category $K^{-,gpb}(\mathcal{GP})$ of upper bounded complexes $X^\bullet$ of Gorenstein projective objects, with only finitely many non-zero cohomology groups $H^n \operatorname{Hom}_{\mathcal{A}}(E, X^\bullet)$ for any Gorenstein projective object E (Theorem 3.5). This permits to interpret the Gorenstein derived functors $\operatorname{Ext}^{\mathrm{n}}_{\mathcal{GP}}$, established in [Hol3] (see also [EJ1], [EJ2], [AM]), as the Hom functor of $D^b_{gp}(\mathcal{A})$ (Theorem 3.8).

As a Verdier's quotient, the singularity category becomes an important topic in algebraic geometry and representation theory of algebras (see e.g. [Buc], [Ric2], [Hap2], [Bel1], [O1], [O2], [Kr], [IK], [CZ]), as they measure the complexity of possible singularities. Also, in Gorenstein homological algebra, one of the basic problems is to recognize Gorenstein rings; and if no, to measure how far it is from the Gorensteinness (see e.g. [M], [J], [CV], [IK], [Hap2], [Bel1]). In the final section we introduce and study the Gorenstein singularity category, and prove that a ring R is Gorenstein if and only if the left and right Gorenstein injectively singularity categories of R are zero (Theorem 4.1); and in general we embed the stable category of a Frobenius category into the Gorenstein injectively singularity category as a triangulated subcategory (Theorem 4.3).

1. **Notations and Preliminaries**

We recall some basic notion and facts, and fix some notations frequently used in this paper.

1.1. Throughout $\mathcal{A}$ is an abelian category with enough projective objects unless stated otherwise, $K(\mathcal{A})$ (resp. $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$) is the homotopy (resp. upper bounded homotopy, and bounded homotopy) category of $\mathcal{A}$; $D(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, and $D^b(\mathcal{A})$) is the derived (resp. upper bounded derived, and bounded derived) category of $\mathcal{A}$. For two complexes $X^\bullet$ and $Y^\bullet$, write $\operatorname{Hom}_\mathcal{A}(X^\bullet, Y^\bullet)$ for the Hom complex. Then we have the well-known formula $\operatorname{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet[n]) = H^n \operatorname{Hom}_\mathcal{A}(X^\bullet, Y^\bullet), \ \forall\ n.$

For basics on triangulated categories and derived categories we refer to [Har], and also [Ver1], [Ver2], [I], [BBD], [Hap1], [Kel2], [N], [GM], [KS]. In particular, by definition, triangulated subcategories are full subcategories; and for a saturated multiplicative system S of a triangulated category $\mathcal{K}$, we refer to [Har] (see also [Ver1] and [I]) for the construction of the quotient triangulated category $S^{-1}\mathcal{K}$ via Verdier's localization, in which each morphism $f: X \longrightarrow Y$ is given by an equivalence class of right fractions $\frac{a}{s}$ presented by $X \overset{s}{\Longleftarrow} Z \overset{a}{\longrightarrow} Y$ with $s \in S, \ Z \in \mathcal{K}$. We emphasize that the definition of a multiplicative system we use (as in [Har], [Ver1], and [I]) is self-dual, it follows that we can also use left fractions, and then get a quotient triangulated category isomorphic to $S^{-1}\mathcal{K}$, and these two isomorphic quotient triangulated categories will be identified (this is needed in 2.7).

1.2. *A complete $\mathcal{A}$-projective resolution* ([EJ1]) is an exact sequence

$$\mathcal{P}^\bullet = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

of projective objects of $\mathcal{A}$, such that $\operatorname{Hom}_\mathcal{A}(\mathcal{P}^\bullet, \mathrm{P})$ is exact for every projective object P of $\mathcal{A}$; and an object E of $\mathcal{A}$ is *Gorenstein projective* if there is a complete $\mathcal{A}$-projective resolution $\mathcal{P}^\bullet$ such that $E \cong \operatorname{Im}(P_0 \longrightarrow P^0)$. It is clear that a projective object of $\mathcal{A}$ is Gorenstein projective; and that in a complete $\mathcal{A}$-projective resolution, all the images and hence all the kernels and cokernels are Gorenstein projective. Denote by $\mathcal{A}$-$\mathcal{GP}$, or simply $\mathcal{GP}$, the full subcategory of $\mathcal{A}$ of Gorenstein projective objects. Note that $\mathcal{GP}$ is closed under extensions, the kernel of an epimorphism, arbitrary coproducts (if $\mathcal{A}$ has arbitrary coproducts) and direct summands. For more facts we refer to [EJ2] and [Hol1].

If $\mathcal{A}$ is the left module category R-Mod of ring R, then we write R-$\mathcal{GP}$ for $\mathcal{A}$-$\mathcal{GP}$; if $\mathcal{A}$ is the finitely generated module category R-mod of R, then we write R-$\mathcal{G}proj$ for $\mathcal{A}$-$\mathcal{GP}$. Note that one can construct the Gorenstein projective modules concretely by the technique of upper triangular matrix artin algebras developed in [ARS] and [Rin]. See [GZ].

If $\mathcal{A}$ has enough injective objects, then one has the dual concept of Gorenstein injective objects, and the dual notations of $\mathcal{GI}$, R-$\mathcal{GI}$, and R-$\mathcal{G}inj$.

Lemma 1.1. (i) *Let E be a Gorenstein projective object of $\mathcal{A}$. Then $\mathrm{Ext}^i_{\mathcal{A}}(E,P) = 0$, $\forall\ i \geq 1$, for any projective object P.*

(ii) *Let $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ be a complete projective resolution such that $E \cong \mathrm{Im}(P_0 \longrightarrow P^0)$. Then the sequences*

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P^1, P) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P^0, P) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, P) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, P) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_0, P) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_1, P) \longrightarrow \cdots$$

are exact for any projective object P.

(iii) *Let R be a ring. Then R-$\mathcal{G}proj \subseteq (R\text{-}\mathcal{GP}) \cap R$-mod. If in addition R is left coherent then R-$\mathcal{G}proj = (R\text{-}\mathcal{GP}) \cap R$-mod.*

Proof. These are easy and well-known (for (iii) see e.g. [Chen]). ■

1.3. Let A be a finite-dimensional k-algebra, and $-^* := \mathrm{Hom}_k(-,k)$. Denote by A-*proj* and A-*inj* the full subcategories of A-mod of the projective A-modules and of the injective A-modules, respectively; and by ν the Nakayama functor $\mathrm{Hom}_A(-,A)^* \cong A^* \otimes_A - :$ A-mod $\longrightarrow A$-mod. Then $\nu : A\text{-}proj \longrightarrow A\text{-}inj$ is an equivalence with a quasi-inverse $\nu^{-1} = \mathrm{Hom}_A(-^*,\ A) \cong \mathrm{Hom}_A(A^*, -) \cong (-^* \otimes_A A^*)^*$; and there is a natural isomorphism $\mathrm{Hom}_A(P, -)^* \cong \mathrm{Hom}_A(-, \nu P)$ for $P \in A$-*proj*. Thus, if A is a self-injective algebra, then ν is an auto-equivalence of A-mod, and ν is the Serre functor of A-*proj*. See [ARS] and [Rin].

Proposition 1.2. (i) *The Nakayama functor ν induces an equivalence A-$\mathcal{G}proj \cong A$-$\mathcal{G}inj$ with a quasi-inverse ν^{-1}.*

(ii) *For $E \in A$-$\mathcal{G}proj$ there is a natural isomorphism $\mathrm{Hom}_A(E, -)^* \cong \mathrm{Hom}_A(-, \nu E)$ on the full subcategory of A-mod of the modules with finite projective dimension.*

Proof. Let $E \in A$-$\mathcal{G}proj$ with complete projective resolution $\mathcal{P}^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ such that $E \cong \mathrm{Im}(P^{-1} \longrightarrow P^0)$, where all P^i are finite-dimensional and projective. Then $\mathrm{Hom}_A(\mathcal{P}^\bullet, A)$, and hence $\nu\mathcal{P}^\bullet$ is acyclic, with $\nu E \cong \mathrm{Im}(\nu P^{-1} \longrightarrow \nu P^0)$. For any finite-dimensional injective module $I \cong \nu P$ with P projective, we have

$$\mathrm{Hom}_A(I, \nu\mathcal{P}^\bullet) \cong \mathrm{Hom}_A(\nu P, \nu\mathcal{P}^\bullet) \cong \mathrm{Hom}_A(P, \mathcal{P}^\bullet),$$

and hence $\mathrm{Hom}_A(I, \nu\mathcal{P}^\bullet)$ is exact. Therefore, $\nu\mathcal{P}^\bullet$ is a complete injective resolution of νE. Similarly one has $\nu^{-1}(A\text{-}\mathcal{G}inj) \subseteq A\text{-}\mathcal{G}proj$.

By Lemma 1.1(ii) $0 \longrightarrow \nu E \longrightarrow \nu P^0 \longrightarrow \nu P^1$ is exact, and hence by the left exactness of ν^{-1} the sequence $0 \longrightarrow \nu^{-1}\nu E \longrightarrow \nu^{-1}\nu P^0 \longrightarrow \nu^{-1}\nu P^1$ is exact. Since $\nu^{-1}\nu P^i \cong P^i$ it follows that $\nu^{-1}\nu E \cong E$ functorially. Similarly $\nu\nu^{-1}|_{A-\mathcal{G}inj} \cong \mathrm{id}$. This proves (i).

By Lemma 1.1(ii) we have the exact sequence $\mathrm{Hom}_A(P^1, P) \longrightarrow \mathrm{Hom}_A(P^0, P) \longrightarrow \mathrm{Hom}_A(E, P) \longrightarrow 0$ for any (finite-dimensional) projective module P, it follows that

$$\mathrm{Hom}_A(P^1, X) \longrightarrow \mathrm{Hom}_A(P^0, X) \longrightarrow \mathrm{Hom}_A(E, X) \longrightarrow 0$$

and

$$\mathrm{Hom}_A(P^1, A) \otimes_A X \longrightarrow \mathrm{Hom}_A(P^0, A) \otimes_A X \longrightarrow \mathrm{Hom}_A(E, A) \otimes_A X \longrightarrow 0$$

are exact for any module X with finite projective dimension. Then by the Five Lemma $\operatorname{Hom}_A(E, A) \otimes_A X \cong \operatorname{Hom}_A(E, X)$. Thus

$$\begin{aligned}\operatorname{Hom}_A(E, X)^* &\cong \operatorname{Hom}_k(\operatorname{Hom}_A(E, A) \otimes_A X, k)\\ &\cong \operatorname{Hom}_A(X, \operatorname{Hom}_k(\operatorname{Hom}_A(E, A), k))\\ &= \operatorname{Hom}_A(X, \nu E). \qquad \blacksquare\end{aligned}$$

Corollary 1.3. (i) *The Nakayama functor* ν *induces a triangle equivalence* $K^b(A\text{-}\mathcal{G}proj) \longrightarrow K^b(A\text{-}\mathcal{G}inj)$.

(ii) *For any* $E^\bullet \in K^b(A\text{-}\mathcal{G}proj)$, *there is a natural isomorphism* $\operatorname{Hom}_A(E^\bullet, -)^* \longrightarrow \operatorname{Hom}_A(-, \nu E^\bullet)$ *on the bounded homotopy category of complexes of modules with finite projective dimension.*

Proof. (i) is clear by Proposition 1.2(i); and (ii) follows an argument in [Hap1], p. 37. We omit the details. $\blacksquare$

2. Gorenstein derived categories

This section is to introduce and study the Gorenstein derived category, as Verdier's quotient of homotopy category respect to the thick triangulated subcategory of Gorenstein projectively acyclic complexes, or equivalently, as Verdier's localization of homotopy category respect to the saturated multiplicative system of Gorenstein projectively quasi-isomorphisms.

Throughout this section $\mathcal{A}$ is an abelian category with enough projective objects, unless stated otherwise.

2.1. A complex $C^\bullet$ is *Gorenstein projectively acyclic*, or simply, *$\mathcal{GP}$-acyclic*, if $\operatorname{Hom}_{\mathcal{A}}(E, C^\bullet)$ is acyclic for all Gorenstein projective objects E.

It follows from (i) below that a $\mathcal{GP}$-acyclic complex is acyclic.

Lemma 2.1. (i) *A complex* $C^\bullet$ *is acyclic if and only if* $\operatorname{Hom}_{\mathcal{A}}(P, C^\bullet)$ *is acyclic for any projective object* P.

(ii) *([CFH]) A complex* $C^\bullet$ *is Gorenstein acyclic if and only if* $\operatorname{Hom}_{\mathcal{A}}(E^\bullet, C^\bullet)$ *is acyclic for any* $E^\bullet \in K^-(\mathcal{GP})$, *or equivalently,* $\operatorname{Hom}_{K(\mathcal{A})}(E^\bullet, C^\bullet[n]) = 0, \ \forall\, n \in \mathbb{Z}$.

Proof. (i) The sufficiency: by the exactness of $\operatorname{Hom}_{\mathcal{A}}(P, C^\bullet)$ we get $\operatorname{Hom}_{\mathcal{A}}(P, \operatorname{Ker} d^i / \operatorname{im} d^{i-1}) = 0$. Since $\mathcal{A}$ has enough projective objects, we have an epimorphism $P \longrightarrow \operatorname{Ker} d^i / \operatorname{im} d^{i-1}$ for some P. Thus $\operatorname{Ker} d^i / \operatorname{Im} d^{i-1} = 0, \ \forall\, i$.

(ii) We refer to Prop. 2.4 in [CFH] for a proof. $\blacksquare$

2.2. A chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ is *a Gorenstein projectively quasi-isomorphism*, or in short, *a $\mathcal{GP}$-quasi-isomorphism*, if $\operatorname{Hom}_{\mathcal{A}}(E, f^\bullet)$ is a quasi-isomorphism, i.e., there are isomorphisms of abelian groups

$$H^n \operatorname{Hom}_{\mathcal{A}}(E, f^\bullet): \ H^n \operatorname{Hom}_{\mathcal{A}}(E, X^\bullet) \cong H^n \operatorname{Hom}_{\mathcal{A}}(E, Y^\bullet), \ \forall\, n \in \mathbb{Z},$$

for all Gorenstein projective objects E.

By (i) below a $\mathcal{GP}$-quasi-isomorphism is a quasi-isomorphism.

Lemma 2.2. (i) *A chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ is a quasi-isomorphism if and only if $\mathrm{Hom}_{\mathcal{A}}(P, f^\bullet)$ is a quasi-isomorphism for all projective objects P.*

(ii) *([CFH]) A chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ is a $\mathcal{GP}$-quasi-isomorphism if and only if there are isomorphisms for any $E^\bullet \in K^-(\mathcal{GP})$:*

$$\mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, f^\bullet[n]): \ \mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, X^\bullet[n]) \cong \mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, Y^\bullet[n]), \ \forall \ n \in \mathbb{Z}.$$

Proof. (i) By applying the cohomological functor $\mathrm{Hom}_{K(\mathcal{A})}(P, -)$ to the distinguished triangle

$$X^\bullet \longrightarrow Y^\bullet \longrightarrow \mathrm{Con}(f^\bullet) \longrightarrow X^\bullet[1]$$

we get the exact sequence

$$\begin{aligned} \cdots &\longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P, \mathrm{Con}(f^\bullet)[n-1]) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P, X^\bullet[n]) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P, Y^\bullet[n]) \\ &\longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P, \mathrm{Con}(f^\bullet)[n]) \longrightarrow \cdots . \end{aligned}$$

By rewriting we have the exact sequence

$$\begin{aligned} \cdots &\longrightarrow H^{n-1}\,\mathrm{Hom}_{\mathcal{A}}(P, \mathrm{Con}(f^\bullet)) \longrightarrow H^n\,\mathrm{Hom}_{\mathcal{A}}(P, X^\bullet) \xrightarrow{H^n\,\mathrm{Hom}_{\mathcal{A}}(P, f^\bullet)} H^n\,\mathrm{Hom}_{\mathcal{A}}(P, Y^\bullet) \\ &\longrightarrow H^n\,\mathrm{Hom}_{\mathcal{A}}(P, \mathrm{Con}(f^\bullet)) \longrightarrow \cdots , \end{aligned}$$

from which and Lemma 2.1(i) the assertion follows.

(ii) We refer to Prop. 2.6 in [CFH] for a proof. ■

Lemma 2.3. *(i) Let $E^\bullet \in K^-(\mathcal{GP})$, and $f^\bullet : X^\bullet \longrightarrow E^\bullet$ be a $\mathcal{GP}$-quasi-isomorphism. Then there exists $g^\bullet : E^\bullet \longrightarrow X^\bullet$ such that $f^\bullet g^\bullet$ is homotopic to $\mathrm{Id}_{E^\bullet}$.*

(ii) Let $f^\bullet : E^\bullet \longrightarrow Q^\bullet$ be a $\mathcal{GP}$-quasi-homomorphism with $E^\bullet, Q^\bullet \in K^-(\mathcal{GP})$. Then $f^\bullet$ is a homotopy equivalence.

Proof. (i) By Lemma 2.2(ii) we have an isomorphism

$$\mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, f^\bullet): \ \mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, X^\bullet) \cong \mathrm{Hom}_{K(\mathcal{A})}(E^\bullet, E^\bullet),$$

from which the assertion follows.

(ii) By (i) there exists $g^\bullet : Q^\bullet \longrightarrow E^\bullet$ such that $f^\bullet g^\bullet$ is homotopic to $\mathrm{Id}_{Q^\bullet}$. It is clear that $g^\bullet$ is also a $\mathcal{GP}$-quasi-isomorphism. Again by (i) there exists $h^\bullet : E^\bullet \longrightarrow Q^\bullet$ such that $g^\bullet h^\bullet$ is homotopic to $\mathrm{Id}_{E^\bullet}$. It follows that $f^\bullet$ is a homotopy equivalence. ■

2.3. The relation between $\mathcal{GP}$-quasi-isomorphisms and $\mathcal{GP}$-acyclic complexes is same as the one between quasi-isomorphisms and acyclic complexes.

Lemma 2.4. (i) *A chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ is a $\mathcal{GP}$-quasi-isomorphism if and only if $\mathrm{Con}(f^\bullet)$ is a $\mathcal{GP}$-acyclic complex.*

(ii) *The collection of $\mathcal{GP}$-quasi-isomorphisms is a saturated multiplication system of $K(\mathcal{A})$, compatible with the triangulation.*

Proof. (i) By the similar argument as in the proof of Lemma 2.2(i).

(ii) This follows from (i) and the well-known correspondence between the thick triangulated subcategories and the compatible saturated multiplication systems (see e.g. [Ver1], [Ver2], or [N]). ■

2.4. Denote by $K_{ac}(\mathcal{A})$ (resp. $K^-_{ac}(\mathcal{A})$, and $K^b_{ac}(\mathcal{A})$) the homotopy category of (resp. upper bounded, and bounded) acyclic complexes of objects of $\mathcal{A}$, and by $K_{gpac}(\mathcal{A})$ (resp. $K^-_{gpac}(\mathcal{A})$, and $K^b_{gpac}(\mathcal{A})$) the homotopy category of (resp. upper bounded, and bounded) $\mathcal{GP}$-acyclic complexes. Then $K_{gpac}(\mathcal{A})$, $K^-_{gpac}(\mathcal{A})$, and $K^b_{gpac}(\mathcal{A})$ are thick (èpaisse) triangulated subcategories of $K_{ac}(\mathcal{A})$, $K^-_{ac}(\mathcal{A})$, and $K^b_{ac}(\mathcal{A})$, respectively.

Define

$$D_{gp}(\mathcal{A}) := K(\mathcal{A})/K_{gpac}(\mathcal{A}), \quad D^-_{gp}(\mathcal{A}) := K^-(\mathcal{A})/K^-_{gpac}(\mathcal{A})$$

and

$$D^b_{gp}(\mathcal{A}) := K^b(\mathcal{A})/K^b_{gpac}(\mathcal{A}),$$

which are respectively called *the Gorenstein projectively derived category, the upper bounded Gorenstein projectively derived category*, and *the bounded Gorenstein projectively derived category*, respectively.

Remark. Let $\mathcal{X}$ be an additive full subcategory of $\mathcal{A}$. Then $D_{gp}(\mathcal{X}) := K(\mathcal{X})/K_{gpac}(\mathcal{X})$, $D^-_{gp}(\mathcal{X}) := K^-(\mathcal{X})/K^-_{gpac}(\mathcal{X})$, and $D^b_{gp}(\mathcal{X}) := K^b(\mathcal{X})/K^b_{gpac}(\mathcal{X})$, are also well-defined, where $K_{gpac}(\mathcal{X})$ is the homotopy category of $\mathcal{GP}$-acyclic complexes of objects in $\mathcal{X}$.

Before giving the relation between $D_{gp}(\mathcal{A})$ and the derived category $D(\mathcal{A})$, we first give a general result in the theory of triangulated categories.

2.5. Let $\mathcal{K}_2$ be a triangulated subcategory of triangulated category $\mathcal{K}$. Recall that the quotient $\mathcal{K}/\mathcal{K}_2$ is defined via the Verdier localization $\mathcal{K}/\mathcal{K}_2 := \mathcal{S}^{-1}\mathcal{K}$, where

$$\mathcal{S} := \{\ f : X \longrightarrow Y \text{ in } \mathcal{K} \mid \operatorname{Con}(f) \in \mathcal{K}_2\ \}$$

is the compatible multiplicative system of $\mathcal{K}$ determined by $\mathcal{K}_2$. If $\mathcal{K}_2$ is thick then $\mathcal{S}$ is saturated, and in this case $X \cong 0$ in $\mathcal{K}/\mathcal{K}_2$ implies $X \in \mathcal{K}_2$. Assume $\mathcal{K}_1$ is a triangulated subcategory of $\mathcal{K}$ and $\mathcal{K}_2$ is a full subcategory of $\mathcal{K}_1$. Then $\mathcal{K}_2$ is a triangulated subcategory of $\mathcal{K}_1$ and

$$\{\ f : X \longrightarrow Y \text{ in } \mathcal{K}_1 \mid \operatorname{Con}(f) \in \mathcal{K}_2\ \} = \mathcal{S} \cap \mathcal{K}_1.$$

It follows that $\mathcal{S}\cap\mathcal{K}_1$ is the compatible multiplicative system of $\mathcal{K}_1$ determined by $\mathcal{K}_2$, and hence we have the quotient $\mathcal{K}_1/\mathcal{K}_2 := (\mathcal{S}\cap\mathcal{K}_1)^{-1}\mathcal{K}_1$. It is clear that the canonical functor from $\mathcal{K}_1/\mathcal{K}_2$ to $\mathcal{K}/\mathcal{K}_2$ is fully faithful, and hence $\mathcal{K}_1/\mathcal{K}_2$ can be viewed as a triangulated subcategory of $\mathcal{K}/\mathcal{K}_2$, and if $\mathcal{K}_1$ is saturated in $\mathcal{K}$ then $\mathcal{K}_1/\mathcal{K}_2$ is saturated in $\mathcal{K}/\mathcal{K}_2$.

The following result should be well-known, however there are no references.

Theorem 2.5. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be triangulated subcategories of triangulated category $\mathcal{K}$, and $\mathcal{K}_2$ be a full subcategory of $\mathcal{K}_1$. Then there is an isomorphism of triangulated categories*

$$(\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2) \cong \mathcal{K}/\mathcal{K}_1.$$

Proof. This is a standard application of the universal property. By the universal property of $\mathcal{K}/\mathcal{K}_2$, we get a unique triangle functor F such that the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[rr, "Q_1"] \arrow[dr, "Q_2"'] & & \mathcal{K}/\mathcal{K}_1 \\ & \mathcal{K}/\mathcal{K}_2 \arrow[ur, "F"'] & \end{tikzcd}$$

commutes. Further by the universal property of $(\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2)$ we get a unique triangle functor G such that the diagram

$$\begin{tikzcd} \mathcal{K}/\mathcal{K}_2 \arrow[rr, "F"] \arrow[dr, "\widetilde{Q_1}"'] & & \mathcal{K}/\mathcal{K}_1 \\ & (\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2) \arrow[ur, "G"'] & \end{tikzcd}$$

commutes. Set $F' := \widetilde{Q_1}Q_2$. By the universal property of $\mathcal{K}/\mathcal{K}_1$ we get a unique triangle functor G' such that the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[rr, "F'"] \arrow[dr, "Q_1"'] & & (\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2) \\ & \mathcal{K}/\mathcal{K}_1 \arrow[ur, "G'"'] & \end{tikzcd}$$

commutes.

By an easy verification we have the following commutative diagram

$$\begin{tikzcd} \mathcal{K} \arrow[rr, "F'"] \arrow[dr, "Q_2"'] & & (\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2) \ . \\ & \mathcal{K}/\mathcal{K}_2 \arrow[ur, "G'G\widetilde{Q_1}"'] & \end{tikzcd}$$

On ther other hand we have by definition $F' = \widetilde{Q_1}Q_2$. It follows from the uniqueness we have $\widetilde{Q_1} = G'G\widetilde{Q_1}$, i.e., we have the commutative diagram

$$\begin{tikzcd} \mathcal{K}/\mathcal{K}_2 \arrow[rr, "\widetilde{Q_1}"] \arrow[dr, "\widetilde{Q_1}"'] & & (\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2) \ . \\ & (\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2) \arrow[ur, "G'G"'] & \end{tikzcd}$$

By the uniqueness we obtain $G'G = \mathrm{Id}_{(\mathcal{K}/\mathcal{K}_2)/(\mathcal{K}_1/\mathcal{K}_2)}$.

Similarly (but much easier) we have $GG' = \mathrm{Id}_{\mathcal{K}/\mathcal{K}_1}$. This completes the proof. ■

2.6. We immediately have

Corollary 2.6. *There is an isomorphism of triangulated categories*

$$D(\mathcal{A}) \simeq D_{gp}(\mathcal{A})/(K_{ac}(\mathcal{A})/K_{gpac}(\mathcal{A})); \quad D^-(\mathcal{A}) \simeq D^-_{gp}(\mathcal{A})/(K^-_{ac}(\mathcal{A})/K^-_{gpac}(\mathcal{A}))$$

and

$$D^b(\mathcal{A}) \simeq D^b_{gp}(\mathcal{A})/(K^b_{ac}(\mathcal{A})/K^b_{gpac}(\mathcal{A})).$$

In particular, the dimension of $D^b_{gp}(\mathcal{A})$ is less or equal to the one of $D^b(\mathcal{A})$ as triangulated categories.

Corollary 2.7. *The following are equivalent*

(i) *$D(\mathcal{A}) \simeq D_{gp}(\mathcal{A})$ (resp. $D^-(\mathcal{A}) \simeq D^-_{gp}(\mathcal{A})$; $D^b(\mathcal{A}) \simeq D^b_{gp}(\mathcal{A})$);*

(ii) *$K_{ac}(\mathcal{A}) = K_{gpac}(\mathcal{A})$ (resp. $K^-_{ac}(\mathcal{A}) = K^-_{gpac}(\mathcal{A})$; $K^b_{ac}(\mathcal{A}) = K^b_{gpac}(\mathcal{A})$);*

(iii) *Any quasi-isomorphism in $K(\mathcal{A})$ (resp. in $K^-(\mathcal{A})$; in $K^b(\mathcal{A})$) is a $\mathcal{GP}$-quasi-isomorphism;*

(iv) *Any Gorenstein projective object is projective.*

(v) *Any Gorenstein projective object is of finite projective dimension.*

Proof. $(i) \Longleftrightarrow (ii)$ follows from Corollary 2.6. $(ii) \Longleftrightarrow (iii)$ follows from Lemma 2.4(i) and Verdier's correspondence between the thick triangulated subcategories and the compatible saturated multiplication systems.

$(iii) \Longrightarrow (iv)$: Let E be a Gorenstein projective object, and $0 \longrightarrow X \longrightarrow Y \stackrel{g}{\longrightarrow} Z \longrightarrow 0$ be an arbitrary short exact sequence. Then g induces a quasi-isomorphism $g^\bullet$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "g"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

By the assumption $g^\bullet$ is a $\mathcal{GP}$-quasi-isomorphism, thus $0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, Z) \longrightarrow 0$ is exact, i.e., E is projective.

$(iv) \Longrightarrow (iii)$ follows from Lemma 2.2(i); and $(v) \Longrightarrow (iv)$ is Prop. 10.2.3 in [EJ2]: the projective dimension of a Gorenstein projective object is either zero or infinite. ■

2.7. The following is well-known (see e.g. ([Kel2], Lem. 10.3; or [KS], Prop. 10.2.6).

Lemma 2.8. *Let $\mathcal{B}$ and $\mathcal{D}$ be two triangulated subcategories of a triangulated category $\mathcal{C}$. If one of the following conditions are satisfied then the canonical triangle functor $\mathcal{D}/\mathcal{D} \cap \mathcal{B} \longrightarrow \mathcal{C}/\mathcal{B}$ is fully faithful.*

(i) *Each morphism $X \longrightarrow B$ with $B \in \mathcal{B}$ and $X \in \mathcal{D}$ admits a factorization $X \longrightarrow B' \longrightarrow B$ with $B' \in \mathcal{D} \cap \mathcal{B}$.*

(ii) *Each morphism $B \longrightarrow Y$ with $B \in \mathcal{B}$ and $Y \in \mathcal{D}$ admits a factorization $B \longrightarrow B' \longrightarrow Y$ with $B' \in \mathcal{D} \cap \mathcal{B}$.*

Theorem 2.9. *$D^b_{gp}(\mathcal{A})$ and $D^-_{gp}(\mathcal{A})$ are triangulated subcategories of $D_{gp}(\mathcal{A})$.*

Proof. We first prove $D^-_{gp}(\mathcal{A}) := K^-(\mathcal{A})/K^-_{gpac}(\mathcal{A})$ is a triangulated subcategory of $D_{gp}(\mathcal{A}) := K(\mathcal{A})/K_{gpac}(\mathcal{A})$, by using Lemma 2.8$(i)$. Let $f^\bullet : X^\bullet \longrightarrow B^\bullet$ be a chain map with $B^\bullet \in K_{gpac}(A)$ and $X^\bullet \in K^-(\mathcal{A})$. We may assume that $X^i = 0$ for $i > 0$. Let $B'^\bullet$ be the complex with $B'^i = B^i$ for $i \leq 0$, $B'^1 = \mathrm{Im}\, d^0$ and $B'^j = 0$ for $j \geq 2$. Then $f^\bullet$

admits the following natural factorization:

$$\begin{tikzcd}
X^\bullet : \arrow[d, "f^\bullet"'] & \cdots \arrow[r] & X^{-1} \arrow[r] \arrow[d] & X^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
B'^\bullet : \arrow[d] & \cdots \arrow[r] & B^{-1} \arrow[r] \arrow[d, equal] & B^0 \arrow[r] \arrow[d, equal] & \operatorname{Im} d^0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
B^\bullet : & \cdots \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

It remains to prove that $B'^\bullet$ is $\mathcal{GP}$-acyclic. Since $B^\bullet$ is $\mathcal{GP}$-acyclic it suffices to prove that

$$\operatorname{Hom}_{\mathcal{A}}(E, B^{-1}) \stackrel{\operatorname{Hom}_{\mathcal{A}}(E,d^{-1})}{\longrightarrow} \operatorname{Hom}_{\mathcal{A}}(E, B^0) \stackrel{\operatorname{Hom}_{\mathcal{A}}(E,\widetilde{d^0})}{\longrightarrow} \operatorname{Hom}_{\mathcal{A}}(E, \operatorname{Im} d_0) \longrightarrow 0 \qquad (*)$$

is exact for any Gorenstein projective object E, where $\widetilde{d^0} : B^0 \longrightarrow \operatorname{Im} d^0$ is induced by d^0.

By Lemma 2.1(i) $B^\bullet$ is acyclic it follows that $0 \longrightarrow \operatorname{Im} d^0 \longrightarrow B^1 \longrightarrow B^2$ is exact, and hence $0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(E, \operatorname{Im} d^0) \stackrel{\sigma}{\longrightarrow} \operatorname{Hom}_{\mathcal{A}}(E, B^1) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(E, B^2)$ is exact. Consider the following commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(E, B^{-1}) \arrow[r, "\operatorname{Hom}(E,d^{-1})"] & \operatorname{Hom}(E, B^0) \arrow[rr, "\operatorname{Hom}(E,d^0)"] \arrow[dr, "\operatorname{Hom}(E,\widetilde{d^0})"'] & & \operatorname{Hom}(E, B^1) \arrow[r, "\operatorname{Hom}(E,d^1)"] & \operatorname{Hom}(E, B^2) \\
& & \operatorname{Hom}(E, \operatorname{Im} d^0) \arrow[ur, hook, "\sigma"'] & &
\end{tikzcd}$$

with the first row exact, by the $\mathcal{GP}$-exactness of $B^\bullet$. Since σ is injective, it follows that

$$\operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(E, \widetilde{d^0}) = \operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(E, d^0).$$

Also

$$\operatorname{Im} \operatorname{Hom}_{\mathcal{A}}(E, \widetilde{d^0}) \cong \operatorname{Im} \operatorname{Hom}_{\mathcal{A}}(E, d^0) = \operatorname{Ker} \operatorname{Hom}_{\mathcal{A}}(E, d^1) \cong \operatorname{Hom}_{\mathcal{A}}(E, \operatorname{Im} d^0),$$

i.e., $\operatorname{Hom}_{\mathcal{A}}(E, \widetilde{d^0})$ is surjective. Thus $(*)$ is exact.

Dually, we can prove $D^b_{gp}(\mathcal{A}) := K^b(\mathcal{A})/K^b_{gpac}(\mathcal{A})$ is a triangulated subcategory of $D^-_{gp}(\mathcal{A}) := K^-(\mathcal{A})/K^-_{gpac}(\mathcal{A})$, by using Lemma 2.8$(ii)$. ■

Remark 2.10. *The proof above also proves that if $(B^\bullet, d)$ is $\mathcal{GP}$-acyclic, so is the truncation $\cdots \longrightarrow B^{i-1} \longrightarrow B^i \longrightarrow \operatorname{Im} d^i \longrightarrow 0$, for each i.*

2.8. The following result makes the morphisms in $D_{gp}(\mathcal{A})$ easier to understand.

Theorem 2.11. *Let $E^\bullet \in K^-(\mathcal{GP})$ and $Y^\bullet$ be an arbitrary complex. Then $Q : f^\bullet \mapsto \frac{f^\bullet}{\mathrm{Id}_{E^\bullet}}$ gives an isomorphism of abelian groups*

$$\operatorname{Hom}_{K(\mathcal{A})}(E^\bullet, Y^\bullet) \cong \operatorname{Hom}_{D_{gp}(\mathcal{A})}(E^\bullet, Y^\bullet).$$

In particular, $K^b(\mathcal{GP})$ and $K^-(\mathcal{GP})$ can be viewed as triangulated subcategories of $D^b_{gp}(\mathcal{A})$ and $D^-_{gp}(\mathcal{A})$, respectively.

Proof. If $\frac{f^\bullet}{\mathrm{Id}_{E^\bullet}} = 0$, then by the calculus of right fractions there exists a $\mathcal{GP}$-quasi-isomorphism $t^\bullet : X^\bullet \longrightarrow E^\bullet$ such that $f^\bullet t^\bullet$ is homotopic to zero. By Lemma 2.3(i) there exists a $\mathcal{GP}$-quasi-isomorphism $g^\bullet : E^\bullet \longrightarrow X^\bullet$ such that $t^\bullet g^\bullet$ is homotopic to $\mathrm{Id}_{E^\bullet}$. Thus $f^\bullet \sim f^\bullet \mathrm{Id}_{E^\bullet} \sim f^\bullet t^\bullet g^\bullet \sim 0$, i.e., Q is injective.

For any $\frac{f^\bullet}{s^\bullet} \in \mathrm{Hom}_{D_{gp}(\mathcal{A})}(E^\bullet, Y^\bullet)$, again by Lemma 2.3($i$) there is a Gorenstein quasi-isomorphism $g^\bullet : E^\bullet \longrightarrow X^\bullet$ such that $s^\bullet g^\bullet$ is homotopic to $\mathrm{Id}_{E^\bullet}$. By the calculus of right fractions this implies that

$$\frac{f^\bullet}{s^\bullet} = \frac{f^\bullet g^\bullet}{\mathrm{Id}_{E^\bullet}} = Q(f^\bullet g^\bullet),$$

i.e., Q is surjective. ∎

Final Remark 2.12. Assume that $\mathcal{A}$ has enough injective objects. Then one has the dual concept: *a $\mathcal{GI}$-acyclic complex* (i.e., a complex $C^\bullet$ such that $\mathrm{Hom}_{\mathcal{A}}(C^\bullet, E)$ is acyclic for all Gorenstein injective objects E); *a $\mathcal{GI}$-quasi-isomorphism* (i.e., a chain map $f^\bullet : X^\bullet \longrightarrow Y^\bullet$ such that $\mathrm{Hom}_{\mathcal{A}}(f^\bullet, E)$ is a quasi-isomorphism for all Gorenstein injective objects E); and *the Gorenstein injectively derived category* $D_{gi}(\mathcal{A})$, *the lower bounded Gorenstein injectively derived category* $D^+_{gi}(\mathcal{A})$, and *the bounded Gorenstein injectively derived category* $D^b_{gi}(\mathcal{A})$. For example, $D^b_{gi}(\mathcal{A})$ is defined as $D^b_{gi}(\mathcal{A}) := K^b(\mathcal{A})/K^b_{giac}(\mathcal{A})$, where $K^b_{giac}(\mathcal{A})$ denotes the homotopy category of bounded $\mathcal{GI}$-acyclic complexes. Then all the dual results hold. We need these in Section 4.

3. Bounded Gorenstein derived categories of Gorenstein rings and finite-dimensional algebras

This section is to study the bounded Gorenstein derived categories of the full subcategory of $\mathcal{A}$ of objects with finite Gorenstein projective dimension, and of finite-dimensional k-algebras. As an application we obtain a description of the bounded Gorenstein derived categories of Gorenstein rings. This also permit us to interpret the Gorenstein derived functors, introduced in [Hol3] and [AM], as the Hom functor of the corresponding bounded Gorenstein derived category.

Throughout this section $\mathcal{A}$ is an abelian category with enough projective objects, unless stated otherwise.

3.1. An object M of $\mathcal{A}$ has a *proper Gorenstein projective resolution* if there is an exact sequence $E^\bullet = \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$ such that all E_i are Gorenstein projective, and that $\mathrm{Hom}_R(E, E^\bullet)$ stays exact for any Gorenstein projective object E. The second requirement guarantee the uniqueness of such a resolution in the homotopy category (the Comparison Theorem. See [EJ2], Exercise 8.1.2). The existence of a proper Gorenstein projective resolution is a non-trivial problem (see [J]. Also [AM], [Ch], [Hol1], [T], [Vel]). We need the following two results due to H. Holm, and P. Jϕrgensen, respectively.

Lemma 3.1. *([Hol1], Theorem 2.10) If an object M has finite Gorenstein projective dimension, then M admits a proper Gorenstein projective resolution $G^\bullet \longrightarrow M \longrightarrow 0$ with $G^\bullet \in K^b(\mathcal{GP})$.*

For the definition of a dualizing complex over non-commutative rings we refer to [YZ], Definition 1.1, or [J], Setup 1.4'. Note that if A is a finite-dimensional k-algebra then A has a dualizing complex $A^* = \mathrm{Hom}_k(A, k)$. Combined with Setup 1.4', Theorems 1.10 and 2.11 of Jϕrgensen [J] we have

Lemma 3.2. *([J]) If A is a finite-dimensional k-algebra, then each module M (not necessarily finitely generated) admits a proper Gorenstein projective resolution $G^\bullet \longrightarrow M \longrightarrow 0$ with $G^\bullet \in K^-(\mathcal{GP})$.*

3.2. Denote by $K^{-,gpb}(\mathcal{GP})$ the full subcategory of $K^-(\mathcal{GP})$ of upper bounded complexes $X^\bullet$ with only finitely many non-zero cohomology groups $H^n \mathrm{Hom}_{\mathcal{A}}(E, X^\bullet)$ for any Gorenstein projective object E. Note that $K^{-,gpb}(\mathcal{GP})$ is a triangulated subcategory of $K^-(\mathcal{GP})$. By the same argument as in the proof of Lemma 2.1(i) we know that any object in $K^{-,gpb}(\mathcal{GP})$ has only finitely many non-zero cohomology groups.

Denote by $f\mathcal{GP}$ the full subcategory of $\mathcal{A}$ of objects with finite Gorenstein projective dimension. Then $f\mathcal{GP}$ is an additive category and hence $K^b(f\mathcal{GP})$ is a triangulated category.

Proposition 3.3. (i) *There exists a functor $G : K^b(f\mathcal{GP}) \longrightarrow K^b(\mathcal{GP})$, and a $\mathcal{GP}$-quasi-isomorphism $\phi_{X^\bullet} : GX^\bullet \longrightarrow X^\bullet$ in $K^b(f\mathcal{GP})$ for $X^\bullet \in K^b(f\mathcal{GP})$, which is functorial in $X^\bullet$.*

Moreover, the inclusion $K^b(\mathcal{GP}) \longrightarrow K^b(f\mathcal{GP})$ is a left adjoint of G.

(ii) *Let A be a finite-dimensional k-algebra. Then there exists a functor $G : K^b(A\text{-Mod}) \longrightarrow K^{-,gpb}(A\text{-}\mathcal{GP})$, and a $\mathcal{GP}$-quasi-isomorphism $\phi_{X^\bullet} : GX^\bullet \longrightarrow X^\bullet$ in $K^-(A\text{-Mod})$ for $X^\bullet \in K^b(A\text{-Mod})$, which is functorial in $X^\bullet$.*

Proof. First, construct F on objects by induction on $w(X^\bullet)$ with $X^\bullet \in K^b(\mathcal{X})$, where $\mathcal{X}$ is $f\mathcal{GP}$ for (i), or A-Mod for (ii). If $w(X^\bullet) = 1$, then we have a $\mathcal{GP}$-quasi-isomorphism $\phi_{X^\bullet} : GX^\bullet \longrightarrow X^\bullet$ by assumption, and $GX^\bullet$ is unique by the Comparison Theorem ([EJ2], Exercise 8.1.2). Assume $w(X^\bullet) \geq 2$, with $X^j \neq 0$ and $X^i = 0$ for $i < j$. Then we have the distinguished triangle in $K^b(\mathcal{X})$

$$X_1^\bullet \xrightarrow{u} X_2^\bullet \longrightarrow X^\bullet \longrightarrow X_1^\bullet[1]$$

with $X_1^\bullet := X^{\bullet \leq j}[-1]$, $X_2^\bullet := X^{\bullet > j}$, where $X^{\bullet \leq j}$ denotes the brutal trunction. By induction there exist $\mathcal{GP}$-quasi-isomorphisms

$$\phi_1 : GX_1^\bullet \longrightarrow X_1^\bullet, \quad \phi_2 : GX_2^\bullet \longrightarrow X_2^\bullet$$

with $GX_1^\bullet,\ GX_2^\bullet \in K^{-,gpb}(A\text{-}\mathcal{GP})$. By Lemma 2.2($ii$) we have an isomorphism

$$\mathrm{Hom}_{K^-(\mathcal{X})}(GX_1^\bullet,\ GX_2^\bullet) \longrightarrow \mathrm{Hom}_{K^-(\mathcal{X})}(GX_1^\bullet,\ X_2^\bullet).$$

It follows that there exists $f^\bullet : GX_1^\bullet \longrightarrow GX_2^\bullet$ such that $\phi_2 \circ f^\bullet = u \circ \phi_1$. By embedding $f^\bullet$ (uniquely) into a distinguished triangle in $K^-(\mathcal{GP})$

$$GX_1^\bullet \xrightarrow{f^\bullet} GX_2^\bullet \longrightarrow GX^\bullet \longrightarrow G_1^\bullet[1]$$

we get a unique $GX^\bullet \in K^-(\mathcal{GP})$. By the axiom of a triangulated category, there exists a morphism $\phi_{X^\bullet} : GX^\bullet \longrightarrow X^\bullet$ such that the following diagram commutes

$$\begin{tikzcd}
GX_1^\bullet \arrow[r, "f^\bullet"] \arrow[d, "\phi_1"'] & GX_2^\bullet \arrow[r] \arrow[d, "\phi_2"'] & GX^\bullet \arrow[r] \arrow[d, "\phi_{X^\bullet}"'] & G_1^\bullet[1] \arrow[d, "\phi_1[1]"'] \\
X_1^\bullet \arrow[r, "u"] & X_2^\bullet \arrow[r] & X^\bullet \arrow[r] & X_1^\bullet[1].
\end{tikzcd}$$

For any Gorenstein projective object Q we have the commutative diagram with exact rows

$$\begin{tikzcd}
(Q, GX_1^\bullet) \arrow[r] \arrow[d, "(\phi_1)_*"'] & (Q, GX_2^\bullet) \arrow[r] \arrow[d, "(\phi_2)_*"'] & (Q, GX^\bullet) \arrow[r] \arrow[d, "(\phi_{X^\bullet})_*"'] & (Q, GX_1^\bullet[1]) \arrow[r] \arrow[d, "(\phi_1[1])_*"'] & (Q, GX_2^\bullet[1]) \arrow[d, "(\phi_2[1])_*"'] \\
(Q, X_1^\bullet) \arrow[r] & (Q, X_2^\bullet) \arrow[r] & (Q, X^\bullet) \arrow[r] & (Q, X_1^\bullet[1]) \arrow[r] & (Q, X_1^\bullet[1])
\end{tikzcd}$$

where $(Q, -)$ denotes the functor $\mathrm{Hom}_{K^-(\mathcal{A})}(Q, [n] \circ -)$. Since ϕ_1 and ϕ_2 are $\mathcal{GP}$-quasi-isomorphisms, it follows that $(\phi_1)_*$, $(\phi_2)_*$, $(\phi_1[1])_*, (\phi_2[1])_*$ are isomorphisms, and hence $(\phi_{X^\bullet})_*$ is an isomorphism for each n, i.e., $\phi_{X^\bullet}$ is a $\mathcal{GP}$-quasi-isomorphism. Since $X^\bullet$ is bounded and $\phi_{X^\bullet} : GX^\bullet \longrightarrow X^\bullet$ is a $\mathcal{GP}$-quasi-isomorphism, it follows that $GX^\bullet$ is in $K^b(\mathcal{GP})$ for (i), or in $K^{-,gpb}(\mathcal{GP})$ for (ii).

Secondly, for $f^\bullet : X^\bullet \longrightarrow Y^\bullet$, since $\phi_{Y^\bullet} : GY^\bullet \longrightarrow Y^\bullet$ is a $\mathcal{GP}$-quasi-isomorphism, it follows that from Lemma 2.2(ii) that

$$\mathrm{Hom}_{K^-(\mathcal{X})}(GX^\bullet,\ GY^\bullet) \cong \mathrm{Hom}_{K^-(\mathcal{X})}(GX^\bullet,\ Y^\bullet).$$

Thus, there exists a unique $Gf^\bullet : GX^\bullet \longrightarrow GY^\bullet$ such that $\phi_{Y_\bullet} \circ Gf^\bullet = f^\bullet \circ \phi_{X^\bullet}$. This shows that G is a functor, and also that $\phi_{X^\bullet} :\ GX^\bullet \longrightarrow X^\bullet$ is functorial in $X^\bullet$.

Observe that in the case (i), again by the $\mathcal{GP}$-quasi-isomorphism $\phi_{Y^\bullet} : GY^\bullet \longrightarrow Y^\bullet$ and Lemma 2.2(ii) it is clear that

$$\mathrm{Hom}_{K^b(f\mathcal{GP})}(Q^\bullet, Y^\bullet) \cong \mathrm{Hom}_{K^b(\mathcal{GP})}(Q^\bullet, GY^\bullet),$$

which is functorial both in $Q^\bullet \in K^b(\mathcal{GP})$ and $Y^\bullet \in K^b(f\mathcal{GP})$, i.e., G is a right adjoint of the inclusion $K^b(\mathcal{GP}) \longrightarrow K^b(f\mathcal{GP})$. This completes the proof. ∎

3.3. Denote by $K^b_{gpac}(f\mathcal{GP})$ the bounded homotopy category of $\mathcal{GP}$-acyclic complexes of objects in $f\mathcal{GP}$. Then $K^b_{gpac}(f\mathcal{GP})$ is a thick triangulated subcategory of $K^b(f\mathcal{GP})$. Put $D^b_{gp}(f\mathcal{GP}) := K^b(f\mathcal{GP})/K^b_{gpac}(f\mathcal{GP})$ (note that $f\mathcal{GP}$ is *not* an abelian category in general. However $D^b_{gp}(f\mathcal{GP})$ is still well-defined. cf. 2.4 Remark). By Lemma 2.4 the saturated multiplicative system determined by $K^b_{gpac}(f\mathcal{GP})$ is the class of $\mathcal{GP}$-quasi-isomorphisms in $K^b(f\mathcal{GP})$. It follows that $Q(f^\bullet)$ is an isomorphism in $D^b_{gp}(f\mathcal{GP})$ if and only if $f^\bullet : G^\bullet \longrightarrow X^\bullet$ is a $\mathcal{GP}$-quasi-isomorphisms in $K^b(f\mathcal{GP})$, where $Q : K^b(f\mathcal{GP}) \longrightarrow D^b_{gp}(f\mathcal{GP})$ is the canonical localization functor.

Note that in general $D^b_{gp}(f\mathcal{GP})$ is *not* a full subcategory of $D_{gp}(\mathcal{A})$. However, with the similar argument as in the proof of Theorem 2.11 we have

Lemma 3.4. *Let $E^\bullet \in K^b(\mathcal{GP})$ and $Y^\bullet \in K^b(f\mathcal{GP})$. Then $Q: f^\bullet \mapsto \frac{f^\bullet}{\mathrm{Id}_{E^\bullet}}$ gives an isomorphism of abelian groups*

$$\mathrm{Hom}_{K^b(f\mathcal{GP})}(E^\bullet, Y^\bullet) \cong \mathrm{Hom}_{D^b_{gp}(f\mathcal{GP})}(E^\bullet, Y^\bullet).$$

In particular, $K^b(\mathcal{GP})$ can be viewed as a triangulated subcategory of $D^b_{gp}(f\mathcal{GP})$.

3.4. Now we state the main result of this section.

Theorem 3.5. (i) *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then there is a triangle-equivalence*

$$D^b_{gp}(f\mathcal{GP}) \simeq K^b(\mathcal{GP}).$$

(ii) *Let A be a finite-dimensional k-algebra. Then there is a triangle-equivalence*

$$D^b_{gp}(A\text{-}Mod) \simeq K^{-,gb}(A\text{-}\mathcal{GP}).$$

Proof. (i) Let $F: K^b(\mathcal{GP}) \longrightarrow D^b_{gp}(f\mathcal{GP})$ be the composition of the embedding $K^b(\mathcal{GP}) \longrightarrow K^b(f\mathcal{GP})$ and the localization functor $Q: K^b(f\mathcal{GP}) \longrightarrow D^b_{gp}(f\mathcal{GP})$. By Proposition 3.3$(i)$ F is dense; and by Lemma 3.4 F is fully faithful.

(ii) Let $F: K^{-,gpb}(A\text{-}\mathcal{GP}) \longrightarrow D^-_{gp}(A\text{-Mod})$ be the composition of the embedding $K^{-,gpb}(A\text{-}\mathcal{GP}) \longrightarrow K^-(A\text{-Mod})$ and the localization functor $Q: K^-(A\text{-Mod}) \longrightarrow D^-_{gp}(A\text{-Mod})$. For any complex $X^\bullet \in K^{-,gpb}(A\text{-}\mathcal{GP})$, say, with $H^i\,\mathrm{Hom}_{\mathcal{A}}(E, X^\bullet) = 0$ for $i \leq n$, by the left exactness of $\mathrm{Hom}_{\mathcal{A}}(E, -)$ it is clear that the following chain map is a $\mathcal{GP}$-quasi-isomorphism

$$\begin{array}{cccccccccccccc}
X^\bullet: & \cdots & \longrightarrow & X^{n-2} & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \\
{\scriptstyle f^\bullet}\downarrow & & & \downarrow & & \downarrow & & \| & & \| & & \| & & \\
Y^\bullet: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Ker}\, d^n & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots
\end{array}$$

It follows that the image of F falls in $D^b_{gp}(A\text{-Mod})$ (note that $D^b_{gp}(A\text{-Mod})$ is a full subcategory of $D^-_{gp}(A\text{-Mod})$ by Theorem 2.9), and hence F induces a functor, again denoted by $F: K^{-,gpb}(A\text{-}\mathcal{GP}) \longrightarrow D^b_{gp}(A\text{-Mod})$. Then by Proposition 3.3(ii) F is dense; and by Theorems 2.11 and 2.9 F is fully faithful. ■

3.5. Recall that a ring R is Gorenstein if R is two-sided Noetherian and R has finite injective dimension, both as the left and the right R-module. If R is a Gorenstein ring, then each left and right R-module has finite Gorenstein projective dimension ([EJ2], Theorem 11.5.1). It follows from Theorem 3.5(i) we have

Corollary 3.6. *Let R be a Gorenstein ring. Then there is a triangle-equivalence*

$$D^b_{gp}(R\text{-Mod}) \simeq K^b(R\text{-}\mathcal{GP}).$$

For a finite-dimensional Gorenstein k-algebra A, it follows from Theorem 3.5(ii) that the embedding $K^b(A\text{-}\mathcal{GP}) \hookrightarrow K^{-,gb}(A\text{-}\mathcal{GP})$ is an equivalence. This means

Corollary 3.7. *Let A be a finite-dimensional Gorenstein k-algebra. Then any complex in $K^{-,gb}(A\text{-}\mathcal{GP})$ is homotopical equivalent to a compex in $K^b(A\text{-}\mathcal{GP})$.*

3.6. Let R be a ring. If an R-module M has a proper Gorenstein projective resolution $G^\bullet \longrightarrow M \longrightarrow 0$, then for any R-module N *the Gorenstein right derived functor* $\mathrm{Ext}^n_{R\text{-}\mathcal{GP}}(-,N)$ of $\mathrm{Hom}_R(-,N)$ is defined as

$$\mathrm{Ext}^n_{R\text{-}\mathcal{GP}}(M,N) := H^n\mathrm{Hom_R(G^\bullet, N)}.$$

See [EJ1], [EJ2], [Hol3], [AM]. Note that $\mathrm{Ext}^n_{R\text{-}\mathcal{GP}}(-,N)$ is only well-defined on the full subcategory of objects with proper Gorenstein projective resolutions.

Theorem 3.8. *Let R be a Gorenstein ring, or a finite-dimensional k-algebra. Then for R-modules M and N we have*

$$\mathrm{Ext}^n_{R\text{-}\mathcal{GP}}(M,N) = \mathrm{Hom}_{D^b_{gp}(R\text{-Mod})}(M, N[n]), \ \ n \geq 0.$$

Proof. If R is a Gorenstein ring, then each left and right R-module has finite Gorenstein projective dimension ([EJ2], Theorem 11.5.1), and hence by Theorem 2.10 in [Hol1] (see Lemma 3.1) any module M has a finite proper Gorenstein projective resolution, say $G^\bullet \longrightarrow M \longrightarrow 0$. Thus $M \cong G^\bullet$ in $D^b_{gp}(R\text{-Mod})$. It follows from Corollary 3.6 that

$$\begin{aligned}\mathrm{Hom}_{D^b_{gp}(R\text{-Mod})}(M, N[n]) &= \mathrm{Hom}_{D^b_{gp}(R\text{-Mod})}(G^\bullet, N[n]) \\ &\cong \mathrm{Hom}_{K^b(R\text{-Mod})}(G^\bullet, N[n]) \\ &= H^n\mathrm{Hom}_R(G^\bullet, N) \\ &= \mathrm{Ext}^n_{R\text{-}\mathcal{GP}}(M,N).\end{aligned}$$

The case for finite-dimensional k-algebras can be similarly proved. ■

Final Remark 3.9. Assume that $\mathcal{A}$ has enough injective objects. Dully we have the concept of *a co-proper Gorenstein injective resolution* of an object M (i.e., an exact sequence $E^\bullet = 0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ such that all E_i are Gorenstein injective, and that $\mathrm{Hom}_R(E^\bullet, E)$ stays exact for any Gorenstein injective object E). Then all the dual results of this section hold for $D^b_{gi}(\mathcal{A})$ (cf. Final Remark 2.12). Note that Lemma 3.1 has the dual version (Theorem 2.15 in [Hol3]).

For example, the dual of Corollary 3.6 says that if R is a Gorenstein ring, then there is a triangle-equivalence

$$D^b_{gi}(R\text{-Mod}) \simeq K^b(R\text{-}\mathcal{GI}).$$

We need this remark in the next section.

4. Gorenstein singularity categories: a measure of Gorensteinness

A fundamental problem in relative homological algebra is to recognize Gorenstein rings; and if no, to measure how far it is from the Gorensteinness (see e.g. [J], [CV], [IK], [M], [Hap2]). In this section we prove that a ring R is Gorenstein if and only if the left and right Gorenstein injectively singularity categories of R are zero; and in general we embed the stable category of a Frobenius category into the left (or right) Gorenstein injectively singularity category as a triangulated subcategory.

4.1. Let R be a ring. By the dual version of Theorem 2.11 the canonical functor $K^b(R\text{-}\mathcal{GI}) \longrightarrow D^b_{gi}(R\text{-Mod})$, which is the composition of the embedding $K^b(R\text{-}\mathcal{GI}) \hookrightarrow K^b(R\text{-Mod})$ and the localization $K^b(R\text{-Mod}) \longrightarrow D^b_{gi}(R\text{-Mod})$, is fully faithful. Thus $K^b(R\text{-}\mathcal{GI})$ can be viewed as a thick triangulated subcategory of $D^b_{gi}(R\text{-Mod})$. It is natural to consider Verdier's quotients $D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$ and $D^b_{gi}(\text{Mod-}R)/K^b(\mathcal{GI}\text{-}R)$, which are called *the left and right Gorenstein injectively singularity category of* R, respectively, where Mod-R and $\mathcal{GI}$-R are respectively the right R-module category and the category of the right Gorenstein injective R-modules.

Theorem 4.1. *A two-sided Noetherian ring R is Gorenstein if and only if*

$$D^b_{gi}(R\text{-}Mod)/K^b(R\text{-}\mathcal{GI}) = 0, \ \textit{and} \ \ D^b_{gi}(Mod\text{-}R)/K^b(\mathcal{GI}\text{-}R) = 0.$$

Proof. By Final Remark 3.9 and its right module version it remains to prove the sufficiency. Assume that $D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI}) = 0$. Then for any left module M there is a $\mathcal{GI}$-quasi-isomorphism (see Final Remark 2.12) $f^\bullet : M \longrightarrow G^\bullet$ with $G^\bullet \in K^b(R\text{-}\mathcal{GI})$, where M is viewed as the complex concentrated at component 0. Suppose $G^i = 0$ for $i < t$. Then $t \leq 0$. We may assume $t = 0$: in fact, if $t < 0$ then by the exactness of $\operatorname{Hom}_R(G^\bullet, G^t)$ we see that $G^{t-1} \longrightarrow G^t$ splits, and hence $G^\bullet$ can be shorten. In this way we have a co-proper Gorenstein injective resolution $0 \longrightarrow M \longrightarrow G^\bullet$, thus M has finite Gorenstein injective dimension. In particular, the left R-module R has finite Gorenstein injective dimension. It follows from Theorem 2.1 in [Hol2], which claims that if M has finite projective dimension then the injective dimension of M is equal to the Gorenstein injective dimension of M, that the left R-module R has finite injective dimension.

The same argument shows that the right R-module R has finite injective dimension. That is, R is Gorenstein. ■

Remark The reason of not using *the Gorenstein projectively singularity categories* of R in Theorem 4.1 is caused by the definition of a Gorenstein ring: although there is also an equivalent definition of Gorensteinness in terms of the projective dimensions of injective modules ([EJ2], Theorem 9.1.11), but it needs the projective dimensions of injective modules are *bounded*.

However, if A is a finite-dimensional k-algebra, then by the same argument we have: A is Gorenstein if and only if

$$D^b_{gp}(A\text{-Mod})/K^b(A\text{-}\mathcal{GP}) = 0, \ \text{and} \ \ D^b_{gp}(\text{Mod-}A)/K^b(\mathcal{GP}\text{-}A) = 0.$$

4.2. By Theorem 4.1, the (left, or right) Gorenstein injectively singularity category could be taken as a measure of how far a ring is from the Gorensteinness. In this and next subsections we measure how big $D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$ is by embedding canonically the stable category of a Frobenius category into it.

Fix some notations. Let R be a ring and $\widetilde{\mathcal{GI}}$ be the full subcategory of R-Mod consisting of R-modules with co-proper Gorenstein injective resolutions. Define

$$\mathcal{GI}^{\perp_{\mathcal{GI}}} := \{ \ M \in \widetilde{\mathcal{GI}} \mid \operatorname{Ext}^i_{R\text{-}\mathcal{GI}}(R\text{-}\mathcal{GI}, M) = 0, \ \forall \ i \geq 1\}.$$

Recall that if an R-module M has a co-proper Gorenstein injective resolution $0 \longrightarrow M \longrightarrow G^\bullet$, then for any R-module N *the Gorenstein right derived functor* $\mathrm{Ext}^n_{R\text{-}\mathcal{GI}}(N,-)$ of $\mathrm{Hom}_R(N,-)$ is defined as

$$\mathrm{Ext}^n_{R\text{-}\mathcal{GI}}(N,M) := H^n\mathrm{Hom_R(N, G^\bullet)}.$$

See [Hol3] and [EJ2]. Note that $\mathrm{Ext}^n_{R\text{-}\mathcal{GI}}(N,-)$ is only well-defined on $\widetilde{\mathcal{GI}}$.

Denote by $R\text{-}\mathcal{GI}(M,N)$ the subgroup of $\mathrm{Hom}_R(M,N)$ of R-maps from M to N which factor through the Gorenstein injective modules, and by $\underline{\mathcal{GI}^{\perp_{\mathcal{GI}}}}$ the stable category of $\mathcal{GI}^{\perp_{\mathcal{GI}}}$ modulo $R\text{-}\mathcal{GI}$, i.e., the objects of $\underline{\mathcal{GI}^{\perp_{\mathcal{GI}}}}$ are same as those of $\mathcal{GI}^{\perp_{\mathcal{GI}}}$, and the morphism space from M to N of $\underline{\mathcal{GI}^{\perp_{\mathcal{GI}}}}$ is the quotient group $\mathrm{Hom}_R(M,N)/R\text{-}\mathcal{GI}(M,N)$.

Lemma 4.2. *Let R be a ring. If $M \in \widetilde{\mathcal{GI}}$ and $N \in \mathcal{GI}^{\perp_{\mathcal{GI}}}$, then there is a canonical isomorphism*

$$\mathrm{Hom}_R(M,N)/R\text{-}\mathcal{GI}(M,N) \simeq \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})}(M,N).$$

Consequently, $\underline{\mathcal{GI}^{\perp_{\mathcal{GI}}}}$ is a full subcategory of $D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$.

In particular, if R is Gorenstein then $\mathcal{GI}^{\perp_{\mathcal{GI}}} = R\text{-}\mathcal{GI}$.

Proof. In what follows, a doubled arrow means a morphism belonging to the saturated multiplicative system, determined by the thick triangulated subcategory $K^b(R\text{-}\mathcal{GI})$ of $D^b_{gi}(R\text{-Mod})$ (see [Har], [Ver1], or [I]). A morphism from M to N in $D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$ is denoted by right fraction $\frac{a}{s}:\ M \stackrel{s}{\Longleftarrow} Z^\bullet \stackrel{a}{\longrightarrow} N$, where $Z^\bullet \in D^b_{gi}(R\text{-Mod})$. Note that the mapping cone $\mathrm{Con}(s)$ lies in $K^b(R\text{-}\mathcal{GI})$. We have a distinguished triangle in $D^b_{gi}(R\text{-Mod})$

$$Z^\bullet \stackrel{s}{\Longrightarrow} M \longrightarrow \mathrm{Con}(s) \longrightarrow Z^\bullet[1]. \qquad (*)$$

Consider the map $\theta : \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})}(M,N)$, given by $\theta(f) = \frac{f}{\mathrm{Id}_M}$. First, we prove that θ is surjective. By $M \in \widetilde{\mathcal{GI}}$ we can take a co-proper Gorenstein injective resolution

$$0 \longrightarrow M \stackrel{\varepsilon}{\longrightarrow} G^0 \stackrel{d^0}{\longrightarrow} G^1 \stackrel{d^1}{\longrightarrow} \cdots \longrightarrow G^n \stackrel{d^n}{\longrightarrow} \cdots .$$

Since M is $\mathcal{GI}$-quasi-isomorphic to the complex $G^\bullet := 0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$, and $G^\bullet$ is $\mathcal{GI}$-quasi-isomorphic to the complex $0 \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^{l-1} \twoheadrightarrow \mathrm{Ker}d^l \longrightarrow 0$ for each $l \geq 2$, it follows that M is isomorphic in $D^b_{gi}(R\text{-Mod})$ to $0 \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^{l-1} \longrightarrow \mathrm{Ker}d^l \longrightarrow 0$ for each $l \geq 2$. This complex induces a distinguished triangle in $D^b_{gi}(R\text{-Mod})$

$$G^{\bullet<l}[-1] \longrightarrow \mathrm{Ker}d^l[-l] \stackrel{s'}{\Longrightarrow} M \stackrel{\varepsilon}{\longrightarrow} G^{\bullet<l}. \qquad (**)$$

Note that s' is in the saturated multiplicative system since the mapping cone $G^{\bullet<l}$ of s' lies in $K^b(\mathcal{GI})$. Since $\mathrm{Ext}^i_{R\text{-}\mathcal{GI}}(\mathrm{Ker}d^l, R\text{-}\mathcal{GI}) = 0,\ \forall\ i \geq 1$, and $\mathrm{Con}(s) \in K^b(R\text{-}\mathcal{GI})$, it follows that there exists $l_0 \gg 0$ such that for each $l \geq l_0$

$$\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(\mathrm{Ker}d^l[-l], \mathrm{Con}(s)) = 0.$$

(To see this, let $\mathrm{Con}(s)$ be of the form $0 \longrightarrow W^{-t'} \longrightarrow \cdots \longrightarrow W^t \longrightarrow 0$ with $t',\ t \geq 0$, and each $W^i \in R\text{-}\mathcal{GI}$. Consider the distinguished triangle in $D^b_{gi}(R\text{-Mod})$

$$\mathrm{Con}(s)^{<t}[-1] \longrightarrow W^t[-t] \longrightarrow \mathrm{Con}(s) \longrightarrow \mathrm{Con}(s)^{<t}.$$

Take l_0 to be $t+1$, and apply the cohomological functor $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(\mathrm{Ker}d^l[-l], -)$ to this distinguished triangle. Then by the dual of Theorem 3.8 the assertion follows from induction and $\mathrm{Ext}^i_{R\text{-}\mathcal{GI}}(\mathrm{Ker}d^l, R\text{-}\mathcal{GI}) = 0, \ \forall\, i \geq 1$.)

Write $E = \mathrm{Ker}d^{l_0}$, and take $l = l_0$ in $(**)$. By applying $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(E[-l_0], -)$ to $(*)$ we get $h : E[-l_0] \longrightarrow Z^\bullet$ such that $s' = s \circ h$. So we have $\frac{a}{s} = \frac{(a \circ h)}{s'}$.

Apply $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(-, N)$ to $(**)$, we get an exact sequence

$$\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(M, N) \longrightarrow \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(E[-l_0], N) \longrightarrow \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(G^{\bullet < l_0}[-1], N).$$

We claim that $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(G^{\bullet < l_0}[-1], N) = \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(G^{\bullet < l_0}, N[1]) = 0$.

(In fact, apply $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(-, N[1])$ to the distinguished triangle

$$G^{\bullet < l_0 - 1}[-1] \longrightarrow G^{l_0 - 1}[1 - l_0] \longrightarrow G^{\bullet < l_0} \longrightarrow G^{\bullet < l_0 - 1}$$

in $D^b_{gi}(R\text{-Mod})$. Then the assertion follows from induction and the assumption $N \in \mathcal{GI}^{\perp_{\mathcal{GI}}}$.)

Thus, there exists $f : M \longrightarrow N$ such that $f \circ s' = a \circ h$. So we have $\frac{a}{s} = \frac{(a \circ h)}{s'} = \frac{(f \circ s')}{s'} = \frac{f}{\mathrm{Id}_M}$. This shows that θ is surjective.

On the other hand, if $f : M \longrightarrow N$ with $\theta(f) = \frac{f}{\mathrm{Id}_M} = 0$ in $D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$, then there exists $s : Z^\bullet \Longrightarrow M$ with $\mathrm{Con}(s) \in K^b(R\text{-}\mathcal{GI})$ such that $f \circ s = 0$. Use the same notation as in $(*)$ and $(**)$. By the argument above we have $s' = s \circ h$, and hence $f \circ s' = 0$. Therefore, by applying $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(-, N)$ to $(**)$ we see that there exists $f' : G^{\bullet < l_0} \longrightarrow N$ such that $f' \circ \varepsilon = f$.

Consider the following distinguished triangle in $D^b_{gi}(R\text{-Mod})$

$$G^0[-1] \longrightarrow (G^{\bullet < l_0})^{>0} \longrightarrow G^{\bullet < l_0} \xrightarrow{\pi} G^0,$$

where π is the natural morphism. Again since $N \in \mathcal{GI}^{\perp_{\mathcal{GI}}}$, it follows from induction and the dual of Theorem 3.8 that $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}((G^{\bullet < l_0})^{>0}, N) = 0$. By applying $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(-, N)$ to the above triangle we obtain an exact sequence

$$\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(G^0, N) \longrightarrow \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(G^{\bullet < l_0}, N) \longrightarrow 0.$$

It follows that there exists $g : G^0 \longrightarrow N$ such that $g \circ \pi = f'$. Hence $f = g \circ (\pi \circ \varepsilon)$, it follows that f factors through G^0 in R-Mod. This proves that the kernel of θ is $R\text{-}\mathcal{GI}(M, N)$, which completes the proof. ■

A very special case of this lemma is that if $D^b_{gi}(R\text{-Mod}) = D^b(R\text{-Mod})$ (i.e., the Gorenstein injective modules are exactly the injective modules), then $\underline{\mathcal{I}^\perp}$ is a full subcategory of $D^b(R\text{-Mod})/K^b(\mathcal{I})$, which is well-known (see e.g. [Hap2]).

4.3. We refer to Appendix A in [Kel1] (also [Q], and p.10 in [Hap1]) for the definition of an exact category. An exact category $(\mathcal{A}, \mathcal{E})$ is a Frobenius category, if $(\mathcal{A}, \mathcal{E})$ has enough injective objects and enough projective objects, such that the injective objects coincide with the projective objects. Denote by $\underline{\mathcal{A}}$ its stable category. For a morphism $u : X \longrightarrow Y$ in $\mathcal{A}$, denote its image in $\underline{\mathcal{A}}$ by $\underline{u}$.

Let R be a ring. Denote by $\mathfrak{a}(\mathcal{GI})$ the full subcategory of R-Mod of modules $M \cong \operatorname{Ker} d^0$, where

$$E^\bullet = \cdots \longrightarrow G^{-1} \longrightarrow G^0 \xrightarrow{d^0} G^1 \longrightarrow \cdots$$

is exact with all $G^i \in R\text{-}\mathcal{GI}$, satisfying the following conditions:

(*i*) $\operatorname{Hom}_R(E^\bullet, G)$ is exact for any $G \in R\text{-}\mathcal{GI}$;

(*ii*) $\operatorname{Ext}^j_{R\text{-}\mathcal{GI}}(R\text{-}\mathcal{GI}, \operatorname{Ker} d^i) = 0$ for $j \geq 1,\ i \leq 0$.

Since $\operatorname{Ker} d^i \in \widetilde{\mathcal{GI}}$, it follows that $\operatorname{Ext}^j_{R\text{-}\mathcal{GI}}(R\text{-}\mathcal{GI}, \operatorname{Ker} d^i)$ is well-defined. It is clear that $\mathcal{GI} \subseteq \mathfrak{a}(\mathcal{GI}) \subseteq \mathcal{GI}^{\perp_{\mathcal{GI}}}$ (cf. 4.2).

Let $\mathcal{E}$ be the class of all the $\mathcal{GI}$-acyclic complexes $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathfrak{a}(\mathcal{GI})$, i.e., $0 \longrightarrow \operatorname{Hom}_R(Z,G) \longrightarrow \operatorname{Hom}_R(Y,G) \longrightarrow \operatorname{Hom}_R(X,G) \longrightarrow 0$ is exact for any $G \in R\text{-}\mathcal{GI}$.

Theorem 4.3. *Let R be a ring. Then $(\mathfrak{a}(\mathcal{GI}), \mathcal{E})$ is a Frobenius category with $R\text{-}\mathcal{GI}$ as the projective-injective objects; and the natural functor $\underline{\mathfrak{a}(\mathcal{GI})} \longrightarrow D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$ is a fully faithful triangle functor.*

Proof. By the definition of $\mathcal{E}$ any Gorenstein injective module is injective in $\mathfrak{a}(\mathcal{GI})$; and it is also projective in $\mathfrak{a}(\mathcal{GI})$: this follows from the following claim, which follows from Theorem 8.2.5 in [EJ2].

Claim: Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an element in $\mathcal{E}$, and $G \in R\text{-}\mathcal{GI}$. Then there is the long exact sequence of abelian groups

$$\begin{aligned}
0 &\longrightarrow \operatorname{Hom}_R(G,X) \longrightarrow \operatorname{Hom}_R(G,Y) \longrightarrow \operatorname{Hom}_R(G,Z) \longrightarrow \operatorname{Ext}^1_{R\text{-}\mathcal{GI}}(G,X) \\
&\longrightarrow \operatorname{Ext}^1_{R\text{-}\mathcal{GI}}(G,Y) \longrightarrow \operatorname{Ext}^1_{R\text{-}\mathcal{GI}}(G,Z) \longrightarrow \cdots \longrightarrow \operatorname{Ext}^n_{R\text{-}\mathcal{GI}}(G,X) \longrightarrow \cdots.
\end{aligned}$$

For any $X \in \mathfrak{a}(\mathcal{GI})$, by an easy argument we have $0 \longrightarrow X \longrightarrow G(X) \longrightarrow S(X) \longrightarrow 0$ in $\mathcal{E}$ with $G(X) \in R\text{-}\mathcal{GI}$. This implies that $\mathfrak{a}(\mathcal{GI})$ has enough injectives. If X is injective in $\mathfrak{a}(\mathcal{GI})$, then by applying $\operatorname{Hom}_R(-,X)$ we see that the short exact sequence splits, and hence $X \in R\text{-}\mathcal{GI}$.

Similarly $\mathfrak{a}(\mathcal{GI})$ has enough projectives, and any projective object in $\mathfrak{a}(\mathcal{GI})$ is a Gorenstein injective module. That is, $(\mathfrak{a}(\mathcal{GI}), \mathcal{E})$ is a Frobenius category with $R\text{-}\mathcal{GI}$ as the projective-injective objects.

Since $\mathfrak{a}(\mathcal{GI}) \subseteq \mathcal{GI}^{\perp_{\mathcal{GI}}}$, it follows from Lemma 4.2 that there is a canonical embedding $\sigma:\ \underline{\mathfrak{a}(\mathcal{GI})} \longrightarrow D^b_{gi}(R\text{-Mod})/K^b(R\text{-}\mathcal{GI})$. It remains to show that σ is a triangle functor.

Let $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ be an element in $\mathcal{E}$, and $0 \longrightarrow X \xrightarrow{i_X} G(X) \xrightarrow{\pi_X} S(X) \longrightarrow 0$ be an element in $\mathcal{E}$ with $G(X) \in R\text{-}\mathcal{GI}$. Then $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} Z \xrightarrow{\underline{-w}} S(X)$ is a distinguished triangle in $\underline{\mathfrak{a}(\mathcal{GI})}$, where w is an R-map such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "\rho"] & Z \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"] & G(X) \arrow[r, "\pi_X"] & S(X) \arrow[r] & 0;
\end{tikzcd} \tag{4.1}$$

and any distinguished triangle in $\underline{\mathfrak{a}(\mathcal{GI})}$ is given in this way.

On the other hand, we have a distinguished triangle in $D^b_{gi}(R\text{-Mod})$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w'} X[1] \tag{4.2}$$

with right fraction

$$w' = \frac{p_X}{v'} \in \mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(Z, X[1]), \tag{4.3}$$

where $p_X : \mathrm{Con}(u) \longrightarrow X[1]$ is the natural chain map, and $v' : \mathrm{Con}(u) \longrightarrow Z$ is the $\mathcal{GI}$-quasi-isomorphism induced by v. Denote by $p'_X : \mathrm{Con}(i_X) \longrightarrow X[1]$ the natural morphism of complexes, and $\pi'_X : \mathrm{Con}(i_X) \longrightarrow S(X)$ the $\mathcal{GI}$-quasi-isomorphism induced by π_X. Then right fraction $\beta_X := -\frac{p'_X}{\pi'_X}$ is in $\mathrm{Hom}_{D^b_{gi}(R\text{-Mod})}(S(X), X[1])$. We claim that $w' = -\beta_X w$ in $D^b_{gi}(R\text{-Mod})$, and hence by (4.2), $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-\beta_X w} X[1]$ is a distinguished triangle in $D^b_{gi}(R\text{-Mod})$, and then a distinguished triangle in $D^b_{gi}(R\text{-Mod})/K^b(\mathcal{R}\text{-}\mathcal{GI})$.

In fact, by (4.3) the claim is equivalent to $p_X = -\beta_X(wv')$ in $D^b_{gi}(R\text{-Mod})$. Denote by ρ' the chain map $\mathrm{Con}(u) \longrightarrow \mathrm{Con}(i_X)$ induced by ρ. Then $-\beta_X(wv') = (\frac{p'_X}{\pi'_X})(wv') = p'_X\rho' = p_X$, where the second equality follows from the calculus of right fractions and $wv = \pi_X\rho$ in (4.1).

By the distinguished triangle $X \xrightarrow{i_X} G(X) \longrightarrow \mathrm{Con}(i_X) \xrightarrow{p'_X} X[1]$ in $D^b_{gi}(R\text{-Mod})$ we get the corresponding one in $D^b_{gi}(R\text{-Mod})/K^b(\mathcal{R}\text{-}\mathcal{GI})$ with $G(X) = 0$, it follows that p'_X, and hence β_X, is an isomorphism in $D^b_{gi}(R\text{-Mod})/K^b(\mathcal{R}\text{-}\mathcal{GI})$. This shows that $\beta : \sigma \circ S \longrightarrow [1] \circ \sigma$ is a natural isomorphism. This completes the proof. ■

Remark 4.4. *It seems to be interesting to know when* $\underline{\mathfrak{a}(\mathcal{GI})} = D^b_{gi}(R\text{-}Mod)/K^b(R\text{-}\mathcal{GI})$.

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