STRONGLY GORENSTEIN PROJECTIVE MODULES OVER UPPER TRIANGULAR MATRIX ARTIN ALGEBRAS

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Abstract. We determine all the strongly complete projective resolutions, and all the strongly Gorenstein projective modules, over upper triangular matrix artin algebras. An example is given showing that upper triangular matrix artin algebras really produce new finitely generated strongly Gorenstein projective modules.

Key words: Gorenstein projective modules; strongly Gorenstein projective modules; upper triangle matrix artin algebras.

1. **Introduction and preliminaries**

1.1. Since Eilenberg and Moore [EM], the relative homological algebra, especially the Gorenstein homological algebra, has been developed to an advanced level: the analogues for projective and injective modules are respectively the Gorenstein projective and the Gorenstein injective modules, introduced by Enochs and Jenda ([EJ1]); and one considers the Gorenstein projective and the Gorenstein injective dimensions of modules and complexes (see e.g. [AF], [EJ1], [Y], [EJ2], [C], [AM], [H1], [T], [CFH], [V], [CV]), the existence of proper Gorenstein projective resolutions ([J], [H1], [AM], [T]), the Gorenstein derived functors ([H2], [EJ1], [AM], [EJ2]), the Gorensteinness in triangulated categories ([AS], [B1], [B2]), and the Gorenstein derived categories ([GZ]).

This concept of Gorenstein projective module even goes back to Auslander and Bridger [AB], where they introduced the G-dimension of finitely generated module M over a two-sided Noetherian ring; and then Avramov, Martisinkovsky, and Rieten have proved that M is Goreinstein projective if and only if the G-dimension of M is zero (the remark following Theorem (4.2.6) in [C]).

There is also an analogoue for free module, namely, the strongly Gorenstein projective module ([BM]). As observed by Bennis and Mahdou, a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module ([BM], Theorem 2.7). This also provides a more operating way to obtain the Gorenstein projective modules, if one can construct the strongly Gorenstein projective modules effectively. However, very few is known about this (c.f. [BM], and also [EJ1]).

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The aim of this paper is to give a concrete construction of strongly Gorenstein projective modules, via the existed construction of upper triangular matrix artin algebras ([ARS], [R]). We determine all the strongly complete projective resolutions, and all the strongly Gorenstein projective modules, over upper triangular matrix artin algebras (Theorem 2.2). The feature of this construction is illustrated by Example 2.3, showing that it really produces new finitely generated strongly Gorenstein projective modules. However, one-point extensions do not give new strongly Gorenstein projective modules (Corollary 3.3).

1.2. Let R be a ring. All R-modules considered are left and unital. By R-Mod and R-mod we denote the category of R-modules and the category of finitely generated R-modules, respectively.

Following Enochs and Jenda ([EJ1]), a R-module M is said to be Gorenstein projective (or, G-projective for short) in R-Mod (resp. in R-mod) if there is an exact sequence of projective modules in R-Mod (resp. in R-mod)

$$\mathcal{P}^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow P^2 \longrightarrow \cdots$$

with $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \mathrm{Q})$ exact for any projective module Q in R-Mod (resp. in R-mod), such that $M \cong \ker d^0$. Such a $\mathcal{P}^\bullet$ is called a complete projective resolution of R-Mod (resp. of R-mod). Following Bennis and Mahdou ([BM]), if $\mathcal{P}^\bullet$ in R-Mod (resp. in R-mod) is of the form

$$\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots,$$

then M is said to be a strongly Gorenstein projective module (or, SG-projective for short) in R-Mod (resp. in R-mod). Such a complete projective resolution $\mathcal{P}^\bullet$ of R-Mod (resp. of R-mod) is said to be strong. There hold the following (see [BM])

$$\{\text{projectives in } R\text{-Mod}\} \subsetneqq \{\text{SG-projectives in } R\text{-Mod}\} \subsetneqq \{\text{G-projectives in } R\text{-Mod}\}$$

and

$$\{\text{projectives in } R\text{-mod}\} \subsetneqq \{\text{SG-projectives in } R\text{-mod}\} \subsetneqq \{\text{G-projectives in } R\text{-mod}\}.$$

Denote by R-SGProj (resp. R-SGproj) the full subcategory of SG-projective modules in R-Mod (resp. in R-mod). Note that R-SGProj (resp. R-SGproj) is closed under taking arbitrary direct sums (resp. finite direct sums).

The following fact is useful.

Proposition 1.1. *For any ring R there holds* $(R\text{-SGProj}) \cap R\text{-mod} = R\text{-SGproj}$.

Proof. Let $X \in R$-SGproj. Then by definition we have an exact sequence

$$\mathcal{P}^\bullet := \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that P is finitely generated projective and that $\mathrm{Hom}_R(\mathcal{P}^\bullet, P')$ is exact for any finitely generated projective R-module P'. For any projective R-module Q, there exists a projective R-module Q' such that $Q \oplus Q' = R^{(I)}$ for some index set I (may be infinite). Since P is finitely generated, it follows that

$$\mathrm{Hom}_R(\mathcal{P}^\bullet, Q) \oplus \mathrm{Hom}_R(\mathcal{P}^\bullet, Q') = \mathrm{Hom}_R(\mathcal{P}^\bullet, R^{(I)}) = \mathrm{Hom}_R(\mathcal{P}^\bullet, R)^{(I)}.$$

Since $\mathrm{Hom}_R(\mathcal{P}^\bullet, R)$ is exact, so is $\mathrm{Hom}_R(\mathcal{P}^\bullet, Q)$, i.e., $X \in R\text{-SGProj} \cap R\text{-mod}$.

Let $X \in R\text{-SGProj} \cap R\text{-mod}$. Then we have a strongly complete projective resolution $\mathcal{P}^\bullet := \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ with $X = \mathrm{Ker} f$. By the exact sequence $0 \longrightarrow X \xrightarrow{\sigma} P \xrightarrow{f} X \longrightarrow 0$ and $X \in R\text{-mod}$ we know that P is finitely generated. ■

In the view of this fact, in this paper all modules considered are finitely generated, although the most part of Section 2 also works in general.

1.3. From now on, we consider an artin algebra Λ, i.e., Λ is an algebra over a commutative artin ring R with R in the center of Λ via the canonical embedding, and Λ is finitely generated as an R-module ([ARS], p.26). If R is a field k, then Λ is exactly a finite-dimensional k-algebra. For the representation theory of Λ, we refer to Auslander, Reiten and Smalϕ [ARS], and Ringel [R]. In particular, Λ-mod is a Krull-Schmidt category. Thus, any finitely generated Λ-module has a unique direct decomposition into indecomposable ([ARS], p.33; [R], p.52).

1.4. Let T and U be rings, and M a T-U-bimodule. Consider the upper triangular matrix ring $\Lambda = \begin{pmatrix} T & M \\ 0 & U \end{pmatrix}$, with multiplication given by the one of matrices. We assume that Λ is an artin algebra throughout this paper: this is exactly the case when there is a commutative artin ring R such that T and U are artin R-algebras and M is finitely generated over R which acts centrally on M ([ARS], p.72).

Recall from [ARS] the category $\mathcal{M}_\Lambda$: an object is a pair $(\begin{pmatrix} X \\ Y \end{pmatrix}, \phi)$ where X is a T-module, Y is a U-module, and $\phi : M \otimes_U Y \longrightarrow X$ is a T-morphism; and a morphism of $\mathcal{M}_\Lambda$ is a pair $\begin{pmatrix} f \\ g \end{pmatrix} : (\begin{pmatrix} X \\ Y \end{pmatrix}, \phi) \longrightarrow (\begin{pmatrix} X' \\ Y' \end{pmatrix}, \phi')$, where $f : X \longrightarrow X'$ is a T-morphism, $g : Y \longrightarrow Y'$ is a U-morphism, such that $f\phi = \phi'(\mathrm{id}_M \otimes_U g)$. It is an isomorphism if and only if so are f and g. Note that $F((\begin{pmatrix} X \\ Y \end{pmatrix}, \phi)) = X \oplus Y$ induces an equivalence between $\mathcal{M}_\Lambda$ and Λ-mod, where the Λ-action on $X \oplus Y$ is given by $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}(x, y) := (rx + \phi(m \otimes_U y), sy)$. For convenience we use the convention $\begin{pmatrix} X \\ Y \end{pmatrix} := (\begin{pmatrix} X \\ Y \end{pmatrix}, \phi)$. In the following we identify Λ-mod with $\mathcal{M}_\Lambda$. Under this identification, any Λ-module is written as $\begin{pmatrix} X \\ Y \end{pmatrix}$ with X a T-module, Y a U-module, together with a T-morphism $\phi : M \otimes_U Y \longrightarrow X$. Thus the Λ-action is written as

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} rx + \phi(m \otimes_U y) \\ sy \end{pmatrix}.$$

Then the indecomposable projective Λ-modules are exactly either of the form $\begin{pmatrix} P \\ 0 \end{pmatrix}$ with P an indecomposable projective T-module, or of the form $\begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix}$ with Q an indecomposable projective U-module, together with the identity map; the indecomposable injective Λ-modules are exactly either of the form $\left(\begin{array}{c} I \\ \operatorname{Hom}_T(M, I) \end{array}\right)$ with I an indecomposable injective T-module, together with the canonical map, or of the form $\left(\begin{array}{c} 0 \\ J \end{array}\right)$ with J an indecomposable injective U-module; and the simple Λ-modules are exactly either of the form $\left(\begin{array}{c} S \\ 0 \end{array}\right)$ with S a simple T-module, or of the form $\left(\begin{array}{c} 0 \\ S' \end{array}\right)$ with S' a simple U-module. Compare [ARS], p.77; and [R], p.90.

2. SG-projective modules over upper triangular matrix artin algebras

The aim of this section is to determine the strongly complete projective resolutions, and hence all the SG-projective modules, over an upper triangular matrix artin algebra $\Lambda = \left(\begin{array}{cc} T & M \\ 0 & U \end{array}\right)$.

2.1. Denote by

$$X := \left(\begin{array}{c} P \oplus (M \otimes_U Q) \\ Q \end{array}\right), \quad f := \left(\begin{array}{c} \left(\begin{array}{cc} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{array}\right) \\ g \end{array}\right) : \ X \longrightarrow X \qquad (1)$$

with P a projective T-module, Q a projective U-module, and $\beta: \ P \longrightarrow M \otimes_U Q$ a T-morphism, where the Λ-action on X is given by $\left(\begin{array}{c} 0 \\ \mathrm{id}_{M \otimes_U Q} \end{array}\right): \ M \otimes_U Q \longrightarrow P \oplus (M \otimes_U Q)$. Note that $X = \left(\begin{array}{c} P \\ 0 \end{array}\right) \oplus \left(\begin{array}{c} M \otimes_U Q \\ Q \end{array}\right)$ is a projective Λ-module, and any projective Λ-module is of this form. It is clear that $f: \ X \longrightarrow X$ is a Λ-morphism. Here P and Q could be zero.

Consider the following conditions $(i) - (v)$:

(i) $\cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$ is an exact sequence of projective U-modules;

(ii) $\mathcal{T}^\bullet: \cdots \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} \cdots$ is a complex of projective T-modules, such that $\operatorname{Hom}_T(\mathcal{T}^\bullet, P')$ is exact for any indecomposable projective T-module P';

(iii) $\beta\alpha + (\mathrm{id}_M \otimes g)\beta = 0$;

(iv) If $\alpha(p) = 0, \ \beta(p) + (\mathrm{id}_M \otimes g)(x) = 0$, then there exists $(p', x') \in P \oplus (M \otimes_U Q)$ such that $p = \alpha(p'), \ \ x = \beta(p') + (\mathrm{id}_M \otimes g)(x')$;

(v) For an arbitrary indecomposable projective U-module Q', if $(s, t) \in \operatorname{Hom}_T(P, M \otimes_U Q') \oplus \operatorname{Hom}_U(Q, Q')$ with $s\alpha + (\mathrm{id}_M \otimes t)\beta = 0$ and $tg = 0$, then there exists $(s', t') \in \operatorname{Hom}_T(P, M \otimes_U Q') \oplus \operatorname{Hom}_U(Q, Q')$, such that $s = s'\alpha + (\mathrm{id}_M \otimes t')\beta$ and $t = t'g$.

Lemma 2.1. *With the notations above, if the conditions $(i)-(v)$ are satisfied, then*

$$\mathcal{P}^\bullet: \ \cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots \tag{2}$$

is a strongly complete Λ-projective resolution; conversely, any strongly complete Λ-projective resolution is the form (2), where X and f are given in (1), satisfying the conditions $(i)-(v)$.

Proof. First, we prove the sufficiency. It follows from $(i)-(iii)$ that

$$\begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix} = \begin{pmatrix} \alpha^2 & 0 \\ \beta\alpha + (\mathrm{id}_M \otimes g)\beta & \mathrm{id}_M \otimes g^2 \end{pmatrix} = 0.$$

Note that (iv) implies that

$$\mathrm{Ker}\begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix} \subseteq \mathrm{Im}\begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix};$$

it follows that $\mathrm{Ker}\begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix}$, and hence by (i) the sequence $\mathcal{P}^\bullet$ is exact.

For any projective T-module P', $\mathrm{Hom}_\Lambda(\begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix}, \begin{pmatrix} P' \\ 0 \end{pmatrix}) = 0$. Since by assumption $\mathrm{Hom}_T(\mathcal{T}^\bullet, P')$ is exact, it follows that $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} P' \\ 0 \end{pmatrix})$ is exact.

For any projective U-module Q' (we may assume that Q' is indecomposable), we have

$$\mathrm{Hom}_\Lambda(\begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix}, \begin{pmatrix} M \otimes_U Q' \\ Q' \end{pmatrix}) \cong \mathrm{Hom}_T(P, M \otimes_U Q') \oplus \mathrm{Hom}_U(Q, Q').$$

Put $Y := \mathrm{Hom}_T(P, M \otimes_U Q') \oplus \mathrm{Hom}_U(Q, Q')$. By $(i)-(iii)$ we see that the sequence

$$\cdots \xrightarrow{\phi} Y \xrightarrow{\phi} Y \xrightarrow{\phi} Y \xrightarrow{\phi} \cdots$$

is a complex, where

$$\phi(s,t) := (s\alpha + (\mathrm{id}_M \otimes t)\beta, \ tg)$$

for $(s,t) \in \mathrm{Hom}_T(P, M \otimes_U Q') \oplus \mathrm{Hom}_U(Q, Q')$; and by (v) it is exact. This means that $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} M \otimes_U Q' \\ Q' \end{pmatrix})$ is exact. It follows that $\mathcal{P}^\bullet$ is a strongly complete Λ-projective resolution.

Secondly, we prove the necessity. Let $\mathcal{P}^\bullet: \ \cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots$ be an arbitrary strongly complete Λ-projective resolution. Then X is of the form $\begin{pmatrix} P \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix} = \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix}$ with P a projective T-module and Q a projective U-module. Write f as

$$\begin{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix} \\ g \end{pmatrix} : \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix} \longrightarrow \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix}$$

with T-morphisms $\beta:\ P\longrightarrow M\otimes_U Q$ and $\gamma: M\otimes_U Q\longrightarrow P$. Since $f:\ X\longrightarrow X$ is a Λ-morphism it follows that we have the commutative diagram

$$\begin{array}{ccc} M\otimes_U Q & \xrightarrow{\begin{pmatrix}0\\ \mathrm{id}_{M\otimes Q}\end{pmatrix}} & P\oplus(M\otimes_U Q)\\ \Big\downarrow{\scriptstyle \mathrm{id}_M\otimes g} & & \Big\downarrow{\scriptstyle\begin{pmatrix}\alpha & \gamma\\ \beta & \mathrm{id}_M\otimes g\end{pmatrix}}\\ M\otimes_U Q & \xrightarrow{\begin{pmatrix}0\\ \mathrm{id}_{M\otimes Q}\end{pmatrix}} & P\oplus(M\otimes_U Q)\end{array}$$

and hence $\gamma=0$, i.e., f is given as in (1). By $\mathrm{Ker}f=\mathrm{Im}f$ we have $\mathrm{Ker}g=\mathrm{Im}g$, and

$$\mathrm{Ker}\begin{pmatrix}\alpha & 0\\ \beta & \mathrm{id}_M\otimes g\end{pmatrix}=\mathrm{Im}\begin{pmatrix}\alpha & 0\\ \beta & \mathrm{id}_M\otimes g\end{pmatrix}.$$

These imply that (i), (iii), (iv) are satisfied, and $\alpha^2=0$.

By the exactness of $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix}P'\\ 0\end{pmatrix})$ and of $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix}M\otimes_U Q'\\ Q'\end{pmatrix})$ we see that $\mathrm{Hom}_T(\mathcal{T}^\bullet, P')$ is exact, and that (v) is satisfied. This completes the proof. ■

2.2. Keep the notations T, U, M, Λ. Put

$$^{\perp}M:=\{\ L\in T\text{-mod}\ |\ \mathrm{Ext}^i_T(L,M)=0,\ \forall\ i\geq 1\};$$

$$\begin{pmatrix}(T\text{-SGproj})\cap\ ^{\perp}M\\ 0\end{pmatrix}:=\{\begin{pmatrix}P\\ 0\end{pmatrix}\in\Lambda\text{-mod}\ |\ P\in(T\text{-SGproj})\cap\ ^{\perp}M\ \};$$

$$\begin{pmatrix}M\otimes_U(U\text{-SGproj})\\ U\text{- SGproj}\end{pmatrix}:=\{\begin{pmatrix}M\otimes_U Q\\ Q\end{pmatrix}\in\Lambda\text{-mod}\ |\ Q\in U\text{-SGproj}\ \}$$

where the Λ-module structure of $\begin{pmatrix}M\otimes_U Q\\ Q\end{pmatrix}$ is given by the identity map $M\otimes_U Q\longrightarrow M\otimes_U Q$.

Theorem 2.2. *Let $\Lambda=\begin{pmatrix}T & M\\ 0 & U\end{pmatrix}$ be an artin algebra, and N a Λ-module. Then N is a SG-projective Λ-module if and only if one of the following holds:*

1) $N\in\begin{pmatrix}(T\text{-SGproj})\cap\ ^{\perp}M\\ 0\end{pmatrix}$;

2) $N=\begin{pmatrix}M\otimes_U L\\ L\end{pmatrix}\in\begin{pmatrix}M\otimes_U(U\text{-SGproj})\\ U\text{-SGproj}\end{pmatrix}$, *where* $L=\mathrm{Ker}\,g$ *with strongly complete U-projective resolution* $\cdots\xrightarrow{g}Q\xrightarrow{g}Q\xrightarrow{g}Q\xrightarrow{g}\cdots$ *such that* $\mathrm{Ker}(\mathrm{id}_M\otimes g)=M\otimes_U\mathrm{Ker}\,g$.

3) $N = \mathrm{Ker} f = \begin{pmatrix} \{(p,x) \in P \oplus (M \otimes_U Q) \mid \alpha(p) = 0,\ \beta(p) + (\mathrm{id}_M \otimes g)(x) = 0\} \\ \mathrm{Ker} g \end{pmatrix}$,

where

$$f = \begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix} \\ g \end{pmatrix} : \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix} \longrightarrow \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix},$$

P and Q are respectively arbitrary non-zero projective T-module and U-module, and $\alpha : P \longrightarrow P$ and $\beta : P \longrightarrow M \otimes_U Q$ are T-morphisms, satisfying the conditions $(i) - (v)$ in 2.1.

Remark. 1. A direct sum of a module in 1) and a module in 2) of Theorem 2.2 is of course again a SG-projective Λ-module: in fact it is in 3). In order to see this, just taking $\beta = 0$ in 3).

We stress that 3) really produces **new** SG-projective Λ-modules. See 2.3.

2. If M_U is flat, then by 2) of Theorem 2.2 each module in $\begin{pmatrix} M \otimes_U (U\text{-SGproj}) \\ U\text{-SGproj} \end{pmatrix}$ is a SG-projective Λ-module. However, this is **not** true in general. See Example 2.4.

Proof of Theorem 2.2. First, we justify the sufficiency. If $N \in \begin{pmatrix} (T\text{-SGproj}) \cap {}^{\perp}M \\ 0 \end{pmatrix}$, then $N = \mathrm{Ker}\,\alpha$, where $\mathcal{T}^\bullet : \cdots \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} \cdots$ is a strongly complete T-projective resolution. Note that $N \in {}^{\perp}M$ implies that

$$\mathrm{Hom}_\Lambda(\mathcal{T}^\bullet, \begin{pmatrix} M \otimes_U U \\ U \end{pmatrix}) \cong \mathrm{Hom}_T(\mathcal{T}^\bullet, M \otimes_U U) = \mathrm{Hom}_T(\mathcal{T}^\bullet, M)$$

is exact. Since $\mathrm{Hom}_\Lambda(\mathcal{T}^\bullet, \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix}) \cong \mathrm{Hom}_T(\mathcal{T}^\bullet, M \otimes_U Q)$ is a direct summand of $\mathrm{Hom}_\Lambda(\mathcal{T}^\bullet, M \otimes_U U^m)$ for some m, it follows that $\mathrm{Hom}_\Lambda(\mathcal{T}^\bullet, \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix})$ is exact for any projective U-module Q, i.e., $\mathcal{T}^\bullet$ is also a strongly complete Λ-projective resolution, and hence N is a SG-projective Λ-module.

Let $N = \begin{pmatrix} M \otimes_U L \\ L \end{pmatrix} \in \begin{pmatrix} M \otimes_U (U\text{-SGproj}) \\ U\text{-SGproj} \end{pmatrix}$, where $L = \mathrm{Ker}\, g \in U\text{-SGproj}$ with strongly complete U-projective resolution $\cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$ such that $\mathrm{Ker}(\mathrm{id}_M \otimes g) = M \otimes_U \mathrm{Ker}\, g$. Consider the sequence of projective Λ-modules

$$\mathcal{P}^\bullet : \cdots \xrightarrow{f} \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix} \xrightarrow{f} \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix} \xrightarrow{f} \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix} \xrightarrow{f} \cdots$$

with $f := \begin{pmatrix} \mathrm{id}_M \otimes g \\ g \end{pmatrix} : \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix} \longrightarrow \begin{pmatrix} M \otimes_U Q \\ Q \end{pmatrix}$. By assumption we have

$$\mathrm{Ker}\, f = \begin{pmatrix} \mathrm{Ker}(\mathrm{id}_M \otimes g) \\ \mathrm{Ker}\, g \end{pmatrix} = \begin{pmatrix} M \otimes_U \mathrm{Ker}\, g \\ \mathrm{Ker}\, g \end{pmatrix} = \begin{pmatrix} M \otimes_U \mathrm{Im}\, g \\ \mathrm{Im}\, g \end{pmatrix} = \mathrm{Im}\, f;$$

since $\operatorname{Hom}_\Lambda(\begin{pmatrix} M\otimes_U Q \\ Q \end{pmatrix}, \begin{pmatrix} P' \\ 0 \end{pmatrix}) = 0$ and

$$\operatorname{Hom}_\Lambda(\begin{pmatrix} M\otimes_U Q \\ Q \end{pmatrix}, \begin{pmatrix} M\otimes_U Q' \\ Q' \end{pmatrix}) \cong \operatorname{Hom}_U(Q, Q'),$$

it follows that $\mathcal{P}^\bullet$ is a strongly complete Λ-projective resolution. Thus $N = \operatorname{Ker} f$ is a SG-projective Λ-module.

The case 3) follows directly from Lemma 2.1.

Secondly, we justify the necessity. If N is a SG-projective Λ-module, then $N = \operatorname{Ker} f$, where f is the Λ-map occurred in a strongly complete Λ-projective resolution $\mathcal{P}^\bullet$. By Lemma 2.1 f is of the form (2) where X and f are given in (1), satisfying the conditions $(i) - (v)$ in 2.1.

If $Q = 0$ in (1), then $\beta = 0,\ g = 0$, and hence by (ii) and (iv) in 2.1 we know that $\mathcal{T}^\bullet$ is a strongly complete T-projective resolution; and by taking $Q' = U$ in (v) we see that $\operatorname{Hom}_T(\mathcal{T}^\bullet, M)$ is exact, which implies $N = \operatorname{Ker} f \in {}^\perp M$. Thus N is of the form 1).

If $P = 0$ in (1), then $\alpha = 0,\ \beta = 0$. By (i) and (v) we see that $\cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$ is a strongly complete U-projective resolution; and by (iv) in 2.1 we have $\operatorname{Ker}(\mathrm{id}_M \otimes g) = M \otimes_U \operatorname{Ker} g$. It follows that

$$N = \operatorname{Ker} f = \begin{pmatrix} \operatorname{Ker}(\mathrm{id}_M\otimes g) \\ \operatorname{Ker} g \end{pmatrix} = \begin{pmatrix} M\otimes_U \operatorname{Ker} g \\ \operatorname{Ker} g \end{pmatrix} \in \begin{pmatrix} M\otimes_U (U\text{-SGProj}) \\ U\text{-SGProj} \end{pmatrix}.$$

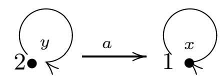
The remaining case is 3). This completes the proof. ■

2.3. Note that in the construction of upper triangular matrix artin algebra $\Lambda = \begin{pmatrix} T & M \\ 0 & U \end{pmatrix}$, T-mod is embedded into Λ-mod via the functor $F_T : T\text{-mod} \longrightarrow \Lambda\text{-mod}$ given by $F_T(X) = \begin{pmatrix} X \\ 0 \end{pmatrix}$; and U-mod is embedded into Λ-mod via the functor $F_U : U\text{-mod} \longrightarrow \Lambda\text{-mod}$ given by $F_U(Y) = \begin{pmatrix} M\otimes_U Y \\ Y \end{pmatrix}$, where the Λ-module structure of $\begin{pmatrix} M\otimes_U Y \\ Y \end{pmatrix}$ is given via $\mathrm{id}_{M\otimes_U Y}$. Also note that U-mod is embedded into Λ-mod via the functor $G_U : U\text{-mod} \longrightarrow \Lambda\text{-mod}$ given by $F_U(Y) = \begin{pmatrix} 0 \\ Y \end{pmatrix}$. A Λ-module $\begin{pmatrix} X \\ Y \end{pmatrix}$ is said to be **newly produced**, or simply **new**, provided that

$$\begin{pmatrix} X \\ Y \end{pmatrix} \notin F_T(T\text{-mod}) \oplus F_U(U\text{-mod}) \oplus G_U(U\text{-mod}).$$

In this sense a direct sum of module in 1) and a module in 2) of Theorem 2.2 is not **new**. What we want to stress is that 3) of Theorem 2.2 really produces **new** strongly Gorenstein projective Λ-modules, as the following example shows. This example also shows that a upper triangular matrix artin algebra really produces **new** Gorenstein projective Λ-modules.

Example 2.3. *Let Λ be the k-algebra given by the quiver*



with relations

$$x^2,\ y^2,\ ay - xa$$

(We write the conjunction of paths from right to left). Let $T = k[x]/\langle x^2\rangle,\ U = k[y]/\langle y^2\rangle$, and ${}_TM_U = ka \oplus kxa = ka \oplus kay$ with the natural T-U-actions by the conjunction of paths. Then $\Lambda = \begin{pmatrix} T & M \\ 0 & U \end{pmatrix}$, with ${}_TM \cong {}_TT$ and $M_U \cong U_U$. Write P for $P_1 = \Lambda e_1 = ke_1 \oplus kx$: it is the unique indecomposable projective T-module; and write Q for $U = ke_2 \oplus ky$: it is the unique indecomposable projective U-module. Let $\alpha : P \longrightarrow P$ be the T-morphism given by multiplication by x, $g : Q \longrightarrow Q$ the U-morphism given by multiplication by $-y$, and $\beta : P \longrightarrow M$ the T-morphism given by the right multiplication by a:

$$\beta(e_1) = a, \quad \beta(x) = xa.$$

Note that $\mathrm{id}_M \otimes g : M \otimes_U Q \longrightarrow M \otimes_U Q$ is given by the right multiplication by $-y$, if $M \otimes_U Q$ is identified with M. Then $\alpha,\ \beta,\ g$ satisfy all the conditions $(i)-(v)$ in 2.1, and by Lemma 2.1 we get a strongly complete Λ-projective resolution

$$\mathcal{P}^\bullet : \ \cdots \xrightarrow{f} \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix} \xrightarrow{f} \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix} \xrightarrow{f} \begin{pmatrix} P \oplus (M \otimes_U Q) \\ Q \end{pmatrix} \xrightarrow{f} \cdots$$

with

$$f := \begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & \mathrm{id}_M \otimes g \end{pmatrix} \\ g \end{pmatrix}.$$

By Theorem 2.2 3) we obtain a strongly Gorenstein projective Λ-module

$$\begin{aligned} \mathrm{Ker} f &= \begin{pmatrix} \{(p,m) \in P \oplus (M \otimes_U Q) \mid \alpha(p) = 0,\ \beta(p) + (\mathrm{id}_M \otimes g)(m) = 0\} \\ \mathrm{Ker} g \end{pmatrix} \\ &= \begin{pmatrix} \{(cx, ca + day) \in P \oplus M \mid c,\ d \in k\} \\ ky \end{pmatrix}, \end{aligned}$$

whose Λ-module structure is given via the T-morphism

$$M \otimes_U ky \longrightarrow \{(cx, ca + day) \in P \oplus M \mid c,\ d \in k\} : \quad a \otimes y \mapsto (0, ay), \quad ay \otimes y \mapsto (0,0).$$

Observe that there is a T-isomorphism

$$T \cong \{(cx, ca + day) \in P \oplus M \mid c,\ d \in k\} : \quad e_1 \mapsto (x, a), \quad x \mapsto (0, ay).$$

From these we see that $\mathrm{Ker} f$ *is a* **new***, strongly, indecomposable Gorenstein projective Λ-module.*

Let $P_2 = \Lambda e_2 = ke_2 \oplus ky \oplus ka \oplus kay = ke_2 \oplus ky \oplus ka \oplus kxa$. Alternatively, $\mathcal{P}^\bullet$ can be written as

$$\cdots \xrightarrow{\begin{pmatrix} \alpha & 0 \\ \beta & g \end{pmatrix}} P_1 \oplus P_2 \xrightarrow{f=\begin{pmatrix} \alpha & 0 \\ \beta & g \end{pmatrix}} P_1 \oplus P_2 \xrightarrow{\begin{pmatrix} \alpha & 0 \\ \beta & g \end{pmatrix}} P_1 \oplus P_2 \xrightarrow{\begin{pmatrix} \alpha & 0 \\ \beta & g \end{pmatrix}} \cdots .$$

In this expression we have $\mathrm{Ker} f = \{(cx, ca + day + by) \in P_1 \oplus P_2 \mid b, c, d \in k\}$. ■

The following example shows that a module in $\begin{pmatrix} M \otimes_U (U\text{-SGproj}) \\ U\text{-SGproj} \end{pmatrix}$ may **not** be a SG-projective Λ-module.

Example 2.4. *Let Λ be the k-algebra given by the quiver in Example 2.3 with relations x^2, y^2, ay, and ${}_TM_U = ka \oplus kxa$ with the natural T-U-actions given by the conjunction of paths, where T and U are same as in Example 2.3. Then $\Lambda = \begin{pmatrix} T & M \\ 0 & U \end{pmatrix}$ with ${}_TM \cong {}_TT$; $M_U \cong S_2 \oplus S_2$, where S_2 is the unique simple U-module (and also the simple Λ-module corresponding vertex 2). Note that $ky \cong S_2$ is a SG-projective U-module, and $M \otimes_U ky = 0$. We claim that $\begin{pmatrix} M \otimes_U ky \\ ky \end{pmatrix} = \begin{pmatrix} 0 \\ ky \end{pmatrix} = S_2$ is* **not** *a strongly Gorenstein Λ-module.*

By Theorem 2.2 it suffices to prove that S_2 is neither in 2) nor in 3) of Theorem 2.2. If S_2 is in 2) of Theorem 2.2, then we have a strongly complete U-projective resolution $\cdots \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} Q \xrightarrow{g} \cdots$ such that $ky = \mathrm{Ker}\, g$ and $\mathrm{Ker}(\mathrm{id}_M \otimes g) = M \otimes_U \mathrm{Ker}\, g$. Then $\dim_k Q = \dim_k \mathrm{Ker} g + \dim_k \mathrm{Im} g = 2$, and $Q = U$ and g is given by multiplication by y. Thus we have the desired contradiction

$$\mathrm{Ker}(\mathrm{id}_M \otimes g) = \mathrm{Ker}\, 0 = M \otimes_U U = M \neq 0 = M \otimes_U ky = M \otimes_U \mathrm{Ker}\, g.$$

If S_2 is in 3) of Theorem 2.2, then in Theorem 2.2 3) we have $ky = \mathrm{Ker}\, g$, and by the same argument above we have $Q = U$ and that g is given by multiplication by y. Using the same notation in Theorem 2.2 3) we have

$$\begin{aligned} S_2 = \mathrm{Ker} f &= \begin{pmatrix} \{(p, m) \in P \oplus (M \otimes_U Q) \mid \alpha(p) = 0,\ \beta(p) + (\mathrm{id}_M \otimes g)(m) = 0\} \\ \mathrm{Ker} g \end{pmatrix} \\ &= \begin{pmatrix} \{(p, m) \in P \oplus M \mid \alpha(p) = 0,\ \beta(p) = 0\} \\ ky \end{pmatrix}. \end{aligned}$$

Write $\mathrm{Ker} f = \begin{pmatrix} X \\ ky \end{pmatrix}$. *Then $X \neq 0$ since $0 \neq m \in M$, again a contradiction.* ■

3. *SG*-PROJECTIVE MODULES OVER ONE-POINT EXTENSIONS

This section is to specialize the result in the last section when U is a field. Thus, throughout this section $\Lambda = \begin{pmatrix} T & M \\ 0 & k \end{pmatrix}$, where T is a finite-dimensional k-algebra, k is a field which is assumed to be algebraically closed, and M is a finite-dimensional T-module.

3.1. Denote by

$$X := \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix}, \qquad f := \begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix} \\ A \end{pmatrix} : \quad X \longrightarrow X \tag{3}$$

with P a projective T-module, where the Λ-module structure of X is given by $\begin{pmatrix} 0 \\ \mathrm{id}_{M^{2n}} \end{pmatrix}$: $M \otimes_k k^{2n} \longrightarrow P \oplus M^{2n}$ (here we identify $M \otimes_k k^{2n}$ with M^{2n}); $A : k^{2n} \to k^{2n}$, and $\widetilde{A} : M^{2n} \to M^{2n}$ are given by the block matrix

$$\begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}. \tag{4}$$

Note that $X = \begin{pmatrix} P \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M^{2n} \\ k^{2n} \end{pmatrix}$ is a projective Λ-module. It is clear that $f : X \longrightarrow X$ is a Λ-morphism.

Consider the following conditions $(i) - (iii)$:

(i) $\cdots \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} P \xrightarrow{\alpha} \cdots$ is a strongly complete T-projective resolution;

(ii) $\beta_{2i} = -\beta_{2i-1}\alpha, \ \ \beta_{2i}\alpha = 0, \ \ 1 \le i \le n$, where $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{2n} \end{pmatrix} : P \to M^{2n}$;

(iii) For any $s : P \longrightarrow M$ with $s\alpha = \sum\limits_{1 \le i \le n} t_{2i}\beta_{2i-1}\alpha$ for some $t_{2i} \in k, \ 1 \le i \le n$, there exists $s' : P \longrightarrow M$, such that $s = s'\alpha + \sum_i t_{2i}\beta_{2i-1}$.

Lemma 3.1. *With the notations above, if the conditions $(i) - (iii)$ are satisfied, then*

$$\mathcal{P}^\bullet : \ \cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots$$

is a strongly complete Λ-projective resolution; conversely, any strongly complete Λ-projective resolution is of this form, where X and f are given in (3) *and* (4)*, satisfying the conditions $(i) - (iii)$.*

Proof. First, we prove the sufficiency. It is clear that $\operatorname{Ker} A = \operatorname{Im} A$. By (ii) we have $\beta\alpha + \widetilde{A}\beta = 0$, and hence $\operatorname{Im} \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix} \subseteq \ker \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix}$. For $(p, \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{2n} \end{pmatrix}) \in \operatorname{Ker} \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix}$. Then $\alpha(p) = 0, \ \beta_{2i}(p) = 0, \ m_{2i} = -\beta_{2i-1}(p), \ 1 \le i \le n$. By $\operatorname{Ker}\alpha =$

Imα we have $p' \in P$ such that $p = \alpha(p')$. By (ii) we verify that

$$(p, \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{2n} \end{pmatrix}) = \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix} (p', \begin{pmatrix} 0 \\ m_1 - \beta_1(p') \\ 0 \\ m_3 - \beta_3(p') \\ 0 \\ \vdots \\ 0 \\ m_{2n-1} - \beta_{2n-1}(p') \end{pmatrix}).$$

It follows that $\mathrm{Ker} \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix} = \mathrm{Im} \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix}$, i.e., $\mathcal{P}^\bullet$ is exact.

Since $\mathrm{Hom}_\Lambda(\begin{pmatrix} M^{2n} \\ k^{2n} \end{pmatrix}, \begin{pmatrix} P' \\ 0 \end{pmatrix}) = 0$, it follows from (i) that $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} P' \\ 0 \end{pmatrix})$ is exact.

For the exactness of $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} M \otimes_k k^m \\ k^m \end{pmatrix})$, we only need the exactness in the case of $m = 1$. The reason is $\begin{pmatrix} M \otimes_k k^m \\ k^m \end{pmatrix} = \begin{pmatrix} M \\ k \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} M \\ k \end{pmatrix}$. Since $\mathrm{Hom}_\Lambda(\begin{pmatrix} P \\ 0 \end{pmatrix}, \begin{pmatrix} M \\ k \end{pmatrix}) \cong \mathrm{Hom}_T(P, M)$ and $\mathrm{Hom}_\Lambda(\begin{pmatrix} M^{2n} \\ k^{2n} \end{pmatrix}, \begin{pmatrix} M \\ k \end{pmatrix}) \cong \mathrm{Hom}_k(k^{2n}, k)$, it follows that the exactness of $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} M \otimes_k k^m \\ k^m \end{pmatrix})$ is same as the exactness of

$$\cdots \longrightarrow \mathrm{Hom}_T(P, M) \oplus \mathrm{Hom}_k(k^{2n}, k) \xrightarrow{\phi} \mathrm{Hom}_T(P, M) \oplus \mathrm{Hom}_k(k^{2n}, k) \xrightarrow{\phi} \cdots,$$

where ϕ is defined by

$$\phi(s, t) := (s\alpha + \widetilde{t}\beta,\ tA)$$

for $(s, t) \in \mathrm{Hom}_T(P, M) \oplus \mathrm{Hom}_k(k^{2n}, k)$, $t = (t_1, t_2, ..., t_{2n}) : k^{2n} \longrightarrow k$, here $\widetilde{t} = (t_1, t_2, ..., t_{2n}) : M^{2n} \longrightarrow M$ is given by $\begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{2n} \end{pmatrix} \longmapsto \sum_i t_i m_i$. By (ii) we have Im$\phi \subseteq$ Kerϕ; and (iii) implies that Ker$\phi \subseteq$ Imϕ. This completes the sufficiency.

Secondly, we prove the necessity. Let $\mathcal{P}^\bullet$ be a strongly complete Λ-projective resolution. Then $\mathcal{P}^\bullet$ is of the form

$$\cdots \xrightarrow{f} X' \oplus Y \xrightarrow{f} X' \oplus Y \xrightarrow{f} X' \oplus Y \xrightarrow{f} \cdots \quad (*)$$

with $X' = \begin{pmatrix} P \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} M \otimes_k k^r \\ k^r \end{pmatrix}$. Note that P and r could be 0. We identity $M \otimes_k k^r$ with M^r. Since $\mathrm{Hom}_\Lambda(\begin{pmatrix} M^r \\ k^r \end{pmatrix}, \begin{pmatrix} P \\ 0 \end{pmatrix}) = 0$, it follows that f is of the form (3), with $\alpha \in \mathrm{Hom}_T(P,P)$, $\beta \in \mathrm{Hom}_T(P,M^r)$, $A \in \mathrm{Hom}_k(k^r,k^r)$ and $\widetilde{A} \in \mathrm{Hom}_T(M^r,M^r)$. Since $\cdots \xrightarrow{A} k^r \xrightarrow{A} k^r \xrightarrow{A} k^r \xrightarrow{A} \cdots$ is exact, it follows from the Jordan canonical form of A that r is even, say, $r=2n$, and that A is of form (4). By $\mathrm{Ker}\begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix} = \mathrm{Im}\begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix}$ we get (ii) and $\mathrm{Ker}\alpha = \mathrm{Im}\alpha$ (we omit the details). By the exactness of $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} P' \\ 0 \end{pmatrix})$ we get (i).

By the exactness of $\mathrm{Hom}_\Lambda(\mathcal{P}^\bullet, \begin{pmatrix} M\otimes_k k \\ k \end{pmatrix})$ we can get (iii). We omit the details. This completes the proof. ■

3.2. In order to simplify the strongly complete Λ-projective resolution obtained in Lemma 3.1, we note that if a complex is isomorphic to a strongly complete projective resolution, then this complex itself is also a strongly complete projective resolution with the same SG-projective module, up to an isomorphism.

Consider the following complex of projective Λ-modules

$$\cdots \xrightarrow{h} \begin{pmatrix} P\oplus M^{2n} \\ k^{2n} \end{pmatrix} \xrightarrow{h} \begin{pmatrix} P\oplus M^{2n} \\ k^{2n} \end{pmatrix} \xrightarrow{h} \begin{pmatrix} P\oplus M^{2n} \\ k^{2n} \end{pmatrix} \xrightarrow{h} \cdots \tag{5}$$

with

$$h := \begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \widetilde{A} \end{pmatrix} \\ A \end{pmatrix}, \tag{6}$$

where $A: k^{2n} \to k^{2n}$, and $\widetilde{A}: M^{2n} \to M^{2n}$ are given by the block matrix (4); $\alpha: P \longrightarrow P$ satisfies the condition (i) and the following condition

(iv) For any $s: P \longrightarrow M$ with $s\alpha = 0$, there exists $s': P \longrightarrow M$, such that $s = s'\alpha$.

Observe that (iv) is equivalent to (iii) in 3.1.

Theorem 3.2. *A sequence $\mathcal{P}^\bullet$ of Λ-modules is a strongly complete Λ-projective resolution if and only if it is isomorphic as a complex to a sequence of form* (5) *with α satisfying* (i) *and* (iv).

Proof. By Lemma 3.1 any strongly complete Λ-projective resolution is of the form

$$\mathcal{P}^\bullet: \quad \cdots \xrightarrow{f} \begin{pmatrix} P\oplus M^{2n} \\ k^{2n} \end{pmatrix} \xrightarrow{f} \begin{pmatrix} P\oplus M^{2n} \\ k^{2n} \end{pmatrix} \xrightarrow{f} \begin{pmatrix} P\oplus M^{2n} \\ k^{2n} \end{pmatrix} \xrightarrow{f} \cdots$$

where f is given in (3) with α and β satisfying the conditions $(i) - (iii)$ in 3.1.

Put $\widetilde{\beta} := \begin{pmatrix} 0 \\ \beta_1 \\ 0 \\ \beta_3 \\ \vdots \\ 0 \\ \beta_{2n-1} \end{pmatrix} : P \to M^{2n}$. Then by condition (ii) in 3.1 we have

$$\widetilde{\beta}\alpha + \beta = \widetilde{A}\widetilde{\beta},$$

and hence we have the commutative diagram of Λ-morphisms

$$\begin{array}{ccc} \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix} & \xrightarrow{f = \left(\begin{smallmatrix} \begin{pmatrix} \alpha & 0 \\ \beta & \widetilde{A} \end{pmatrix} \\ A \end{smallmatrix} \right)} & \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix} \\ {\scriptstyle \left(\begin{smallmatrix} \begin{pmatrix} \mathrm{id}_P & 0 \\ \widetilde{\beta} & \mathrm{id}_{M^{2n}} \end{pmatrix} \\ \mathrm{id}_{k^{2n}} \end{smallmatrix} \right)} \downarrow & & \downarrow {\scriptstyle \left(\begin{smallmatrix} \begin{pmatrix} \mathrm{id}_P & 0 \\ \widetilde{\beta} & \mathrm{id}_{M^{2n}} \end{pmatrix} \\ \mathrm{id}_{k^{2n}} \end{smallmatrix} \right)} \\ \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix} & \xrightarrow{h = \left(\begin{smallmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \widetilde{A} \end{pmatrix} \\ A \end{smallmatrix} \right)} & \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix} \end{array}$$

Note that $\left(\begin{smallmatrix} \begin{pmatrix} \mathrm{id}_P & 0 \\ \widetilde{\beta} & \mathrm{id}_{M^{2n}} \end{pmatrix} \\ \mathrm{id}_{k^{2n}} \end{smallmatrix} \right)$ is an isomorphism.

Conversely, the assertion follows from Lemma 3.1. This completes the proof. ■

3.3. Keep the notations as in 2.2. The following result shows in particular that one-point extensions **do not produce** new strongly Gorenstein projective modules.

Corollary 3.3. *Let* $\Lambda = \begin{pmatrix} T & M \\ 0 & k \end{pmatrix}$ *be a finite-dimensional k-algebra with k an algebraically closed field, and N a Λ-module. Then N is a SG-projective Λ-module if and only if N is one of the following form*

1) $N \in \begin{pmatrix} (T\text{-SGproj}) \cap {}^{\perp}M \\ 0 \end{pmatrix}$;

2) $N = \begin{pmatrix} M^n \\ k^n \end{pmatrix}$, *i.e., N is the n copy of the last projective Λ-module* $\begin{pmatrix} M \\ k \end{pmatrix}$, *for any positive integer n;*

3) A direct sum of a module in 1) and a module in 2).

Proof. By Theorem 3.2 we need to compute Ker$h$, where

$$h := \begin{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \widetilde{A} \end{pmatrix} \\ A \end{pmatrix} : \ \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix} \longrightarrow \begin{pmatrix} P \oplus M^{2n} \\ k^{2n} \end{pmatrix},$$

P is an arbitrary projective T-module; $A: \ k^{2n} \to k^{2n}$, and $\widetilde{A}: \ M^{2n} \to M^{2n}$ are given by the block matrix in (4) (here n could be zero); $\alpha : P \longrightarrow P$ satisfy the condition (i) in 3.1 and (iv) in 3.2. Thus

$$\mathrm{Ker}h = \begin{pmatrix} \{(p,x) \in P \oplus M^{2n} \mid \alpha(p) = 0, \ \widetilde{A}(x) = 0\} \\ \mathrm{Ker}A \end{pmatrix}.$$

Write $x = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{2n} \end{pmatrix} \in M^{2n}$. Note that Ker$A = k^n \subseteq k^{2n}$: more precisely, if we write $k^{2n} = k_1 \oplus k_2 \oplus \cdots \oplus k_{2n-1} \oplus k_{2n}$ with $k_1 = k_2 = \cdots = k_{2n-1} = k_{2n} = k$, then

$$\mathrm{Ker}A = k^n = k_1 \oplus k_3 \oplus \cdots \oplus k_{2n-1}$$

(it is understood to be zero if $n = 0$). Then we rewrite Ker$h$ as the following form

$$\begin{aligned} \mathrm{Ker}h &= \begin{pmatrix} \{(p,x) \in P \oplus M^{2n} \mid \alpha(p) = 0, \ \widetilde{A}(x) = 0\} \\ \\ \mathrm{Ker}A \end{pmatrix} \\ &= \begin{pmatrix} \{(p, \begin{pmatrix} m_1 \\ 0 \\ m_3 \\ 0 \\ \vdots \\ m_{2n-1} \\ 0 \end{pmatrix} \in P \oplus M^{2n} \mid \alpha(p) = 0\} \\ \\ k_1 \oplus k_3 \oplus \cdots \oplus k_{2n-1} \end{pmatrix} \subseteq \begin{pmatrix} \mathrm{Ker}\alpha \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M^{2n} \\ k^{2n} \end{pmatrix}. \end{aligned}$$

With this expression we immediately see that

$$\mathrm{Ker}h \cong \begin{pmatrix} \mathrm{Ker}\alpha \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M^n \\ k^n \end{pmatrix}$$

via the isomorphism

$$\mathrm{Ker}h \ni (p, \begin{pmatrix} m_1 \\ 0 \\ m_3 \\ 0 \\ \vdots \\ m_{2n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{2n-1} \end{pmatrix}) \mapsto (p, \begin{pmatrix} m_1 \\ m_3 \\ \vdots \\ m_{2n-1} \end{pmatrix}, \begin{pmatrix} c_1 \\ c_3 \\ \vdots \\ c_{2n-1} \end{pmatrix}) \in \begin{pmatrix} \mathrm{Ker}\alpha \\ 0 \end{pmatrix} \oplus \begin{pmatrix} M^{2n} \\ k^{2n} \end{pmatrix}.$$

Now, by (i) we know Ker$\alpha \in T$-SGproj, and by (iv) we know Ker$\alpha \in {}^{\perp}M$. This completes the proof. ■

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