A BRIEF INTRODUCTION TO GORENSTEIN PROJECTIVE MODULES

PU ZHANG

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, P. R. China

Since Eilenberg and Moore [EM], the relative homological algebra, especially the Gorenstein homological algebra ([EJ2]), has been developed to an advanced level. The analogues for the basic notion, such as projective, injective, flat, and free modules, are respectively the Gorenstein projective, the Gorenstein injective, the Gorenstein flat, and the strongly Gorenstein projective modules. One considers the Gorenstein projective dimensions of modules and complexes, the existence of proper Gorenstein projective resolutions, the Gorenstein derived functors, the Gorensteinness in triangulated categories, the relation with the Tate cohomology, and the Gorenstein derived categories, etc.

This concept of Gorenstein projective module even goes back to a work of Auslander and Bridger [AB], where the G-dimension of finitely generated module M over a two-sided Noetherian ring has been introduced: now it is clear by the work of Avramov, Martisinkovsky, and Rieten that M is Goreinstein projective if and only if the G-dimension of M is zero (the remark following Theorem (4.2.6) in [Ch]).

The aim of this lecture note is to choose some main concept, results, and typical proofs on Gorenstein projective modules. We omit the dual version, i.e., the ones for Gorenstein injective modules. The main references are [ABu], [AR], [EJ1], [EJ2], [H1], and [J1].

Throughout, R is an associative ring with identity element. All modules are left if not specified. Denote by R-Mod the category of R-modules, R-mof the

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E-mail: pzhang@sjtu.edu.cn.

category of all finitely generated R-modules, and R-mod the category of all finite-dimensional R-modules if R is an algebra over field k. Note that if R is a finite-dimensional k-algebra, then R-mof $=$ R-mod. Denote by R-Proj, or simply, Proj, the full subcategory of projective R-modules.

In the relative homological algebra, the following concept is fundamental. Let $\mathcal{X}$ be a full subcategory of R-Mod which is closed under isomorphism, and M an R-module. Recall from [AR] (also [EJ1]) that *a right $\mathcal{X}$-approximation* (or, *$\mathcal{F}$-precover*) of M is an R-homomorphism $f : X \longrightarrow M$ with $X \in \mathcal{X}$, such that the induced map $\operatorname{Hom}_R(X', X) \longrightarrow \operatorname{Hom}_R(X', M)$ is surjective for any $X' \in \mathcal{X}$. If every module M admits a right $\mathcal{X}$-approximation, then $\mathcal{X}$ is called *a contravariantly finite subcategory*, or $\mathcal{X}$ *is a precovering class*.

If every module M admits a surjective, right $\mathcal{X}$-approximation, then every module M has a (left) $\mathcal{X}$-resolution which is $\operatorname{Hom}(\mathcal{X}, -)$-exact. Such a resolution is called a *proper (left) $\mathcal{X}$-resolution of M*. Conversely, if every module admits a proper (left) $\mathcal{X}$-resolution, then every module admits a surjective, right $\mathcal{X}$-approximation.

Dually, one has the concept of *a left $\mathcal{X}$-approximation* (or, *a $\mathcal{X}$-preenvelope*) of M, *a covariantly finite subcategory* (or, *a preenveloping class*), and *a coproper (right) $\mathcal{X}$-resolution* of M.

By a $\operatorname{Hom}_R(\mathcal{X}, -)$-exact sequence $E^\bullet$, we mean that $E^\bullet$ itself is exact, and that $\operatorname{Hom}_R(X, E^\bullet)$ remains to be exact for any $X \in \mathcal{X}$. Dually, we use the terminology $\operatorname{Hom}(-, \mathcal{X})$-exact sequence.

1. **Gorenstein projective modules**

We recall some basic properties of Gorenstein projective modules.

1.1. *A complete projective resolution* is a $\mathrm{Hom_R}(-, \mathrm{Proj})$-exact sequence

$$(\mathcal{P}^\bullet, d) = \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$$

of projective R-modules. An R-module M is called *Gorenstein projective* if there is a complete projective resolution $(\mathcal{P}^\bullet, d)$ such that $M \cong \mathrm{Im}d^{-1}$ (Enochs-Jenda [EJ1], 1995).

Denote by R-GProj, or simply, GProj, the full subcategory of (left) Gorenstein projective modules.

Remark 1.1. *Auslander-Reiten [AR] have considered the following full subcategory for a full subcategory ω*

$$\mathcal{X}_\omega = \{M \in R\text{-Mod} \mid \exists \text{ an exact sequence}$$
$$0 \longrightarrow M \longrightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \cdots, \quad \text{with } T^i \in \omega, \quad \mathrm{Ker}d^i \in {}^\perp\omega, \ \forall\ i \geq 0\}.$$

*Note that if $\omega = R$-Proj, then $\mathcal{X}_\omega = R$-*GProj.

If R is an artin algebra, then the Gorenstein projective R-modules are also referred as the Cohen-Macaulay modules by some mathematician (see e.g. [Be]).

Facts 1.2. (i) *If A is a self-injective algebra, then* $\mathrm{GProj} = \mathrm{A\text{-}Mod}$.

(ii) *A projective module is Gorenstein projective.*

(iii) *If $(\mathcal{P}^\bullet, d)$ is a complete projective resolution, then all* $\mathrm{Im}d^i$ *are Gorenstein projective; and any truncations*

$$\cdots \longrightarrow P^i \longrightarrow \mathrm{Im}d^i \longrightarrow 0, \qquad 0 \longrightarrow \mathrm{Im}d^i \longrightarrow P^{i+1} \longrightarrow \cdots$$

and

$$0 \longrightarrow \mathrm{Im}d^i \longrightarrow P^{i+1} \longrightarrow \cdots \longrightarrow P^j \longrightarrow \mathrm{Im}d^j \longrightarrow 0, \quad i < j$$

are $\mathrm{Hom}(-, \mathrm{Proj})$*-exact.*

(iv) *If M is Gorenstein projective, then* $\mathrm{Ext}^i_R(M, L) = 0, \ \forall\ i > 0,$ *for all modules L of finite projective dimension.*

(v) *A module M is Gorenstein projective if and only if $M \in {}^\perp(\mathrm{Proj})$ and M has a right* Proj*-resolution which is* $\mathrm{Hom}(-, \mathrm{Proj})$*-exact; if and only if there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \cdots, \quad \text{with } T^i \in \mathrm{Proj}, \quad \mathrm{Ker}d^i \in {}^\perp(\mathrm{Proj}), \ \forall\ i \geq 0.$$

(*vi*) *For a Gorenstein projective module M, there is a complete projective resolution* $\cdots \longrightarrow F^{-1} \xrightarrow{d^{-1}} F^0 \xrightarrow{d^0} F^1 \longrightarrow \cdots$ *consisting of free modules, such that* $M \cong \mathrm{Im} d^{-1}$.

(*vii*) *The projective dimension of a Gorenstein projective module is either zero or infinite. So, Gorenstein projective modules make sense only to rings of infinite global dimension.*

Proof. (*vi*) There is a $\mathrm{Hom}(-, \mathrm{Proj})$-exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$ with each P^i projective. Choose projective modules $Q^0, Q^1, \cdots$, such that $F^0 = P^0 \oplus Q^0,\ F^n = P^n \oplus Q^{n-1} \oplus Q^n,\ n > 0$, are free. By adding $0 \longrightarrow Q^i \xrightarrow{=} Q^i \longrightarrow 0$ to the exact sequence in degrees i and $i+1$, we obtain a $\mathrm{Hom}(-, \mathrm{Proj})$-exact sequence of free modules. By connecting a deleted free resolution of M together with the deleted version of this sequence, we get the desired sequence.

(*vii*) Let $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow G \longrightarrow 0$ be a projective resolution of a Gorenstein projective R-module M, with n minimal. If $n \geq 1$, then by $\mathrm{Ext}^n_R(G, P_n) = 0$ we know $\mathrm{Hom}(P_{n-1}, P_n) \longrightarrow \mathrm{Hom}(P_n, P_n)$ is surjective, which implies $0 \longrightarrow P_n \longrightarrow P_{n-1}$ splits. This contradict the minimality of n. ■

1.2. A full subcategory $\mathcal{X}$ of R-Mod is resolving, if $\mathrm{Proj} \subseteq \mathcal{X}$, $\mathcal{X}$ is closed under extensions, the kernels of epimorphisms, and the direct summands.

Theorem 1.3. *For any ring R,* GProj *is resolving, and closed under arbitrary direct sums.*

Proof. In fact, this is a special case of [AR], Proposition 5.1. We include a direct proof given in [H1]. Easy to see GProj is closed under arbitrary direct sums; and it is closed under extension by using the corresponding Horseshoe Lemma. Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence with M, M_2 Gorenstein projective. Then $M_1 \in {}^{\perp}(\mathrm{Proj})$. Construct a $\mathrm{Hom}(-, \mathrm{Proj})$-exact, right Proj-resolution of M_1 as follows. Let

$$\mathbf{M} = 0 \longrightarrow M \longrightarrow P^0 \longrightarrow P_1 \longrightarrow \cdots$$

and

$$\mathbf{M_2} = 0 \longrightarrow M_2 \longrightarrow Q^0 \longrightarrow Q_1 \longrightarrow \cdots$$

be such resolutions of M and M_2, respectively. By $\mathrm{Hom}(-, \mathrm{Proj})$-exactness of $\mathbf{M}$, $M \longrightarrow M_2$ induces a chain map $\mathbf{M} \longrightarrow \mathbf{M_2}$, with mapping cone denoted by $\mathbf{C}$. Then $\mathbf{C}$ is exact, and $\mathrm{Hom}(-, \mathrm{Proj})$-exact by using the distinguished triangle and the induced long exact sequence. Consider a short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{M_1} & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P^0 & \longrightarrow & M_2 \oplus P^0 & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q^0 \oplus P^1 & = & Q^0 \oplus P^1 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Since $\mathbf{C}$, $\mathbf{D}$ are exact, so is $\mathbf{M_1}$. By a direct analysis on each row, we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}(\mathbf{D}, P) \longrightarrow \mathrm{Hom}(\mathbf{C}, P) \longrightarrow \mathrm{Hom}(\mathbf{M_1}, P) \longrightarrow 0$$

for every projective P. Since $\mathrm{Hom}(\mathbf{D}, P)$ and $\mathrm{Hom}(\mathbf{C}, P)$ are exact, so is $\mathrm{Hom}(\mathbf{M_1}, P)$. This proves that M_1 is Gorenstein projective. It remains to prove GProj is closed under arbitrary direct summands. Using Eilenberg's swindle. Let $X = Y \oplus Z \in$ GProj. Put $W = Y \oplus Z \oplus Y \oplus Z \oplus Y \oplus Z \oplus \cdots$. Then $W \in$ GProj, and $Y \oplus W \cong W \in$ GProj. Consider the split exact sequence $0 \longrightarrow Y \longrightarrow Y \oplus W \longrightarrow W \longrightarrow 0$. Then $Y \in$ GProj since we have proved that GProj is closed under the kernel of epimorphisms. ■

1.3. Finitely generated Gorenstein projective modules.

The full subcategory of finitely generated, Gorenstein projective modules is GProj $\cap$ R-mof. Denote by proj the full subcategory of finitely generated projective R-modules; and by Gproj the full category of modules M isomorphic to $\mathrm{Im} d^{-1}$, where

$$\cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$$

is a $\mathrm{Hom}(-, \mathrm{proj})$-exact sequence with each $P^i \in$ proj.

Recall that a ring is left coherent, if each finitely generated left ideal of R is finitely presented; or equivalently, any finitely generated submodule of a finitely presented left module is finitely presented. A left noetherian ring is left coherent.

Proposition 1.4. *Let R any ring. Then* $\mathrm{Gproj} \subseteq \mathrm{GProj} \cap R\text{-mof}$.

If R is left cohenret, then $\mathrm{GProj} \cap R\text{-mof} = \mathrm{Gproj}$.

Proof. Let $M \in \mathrm{Gproj}$. Then there is a $\mathrm{Hom}(-, \mathrm{proj})$-exact sequence

$$\mathcal{P}^\bullet = \ \cdots \longrightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$$

with each $P^i \in \mathrm{proj}$, such that $M \cong \mathrm{Im} d^{-1}$. So M is finitely generated. Since each P^i is finitely generated, it is clear that $\mathcal{P}^\bullet$ is also $\mathrm{Hom}(-, \mathrm{Proj})$-exact, i.e., $M \in \mathrm{GProj} \cap R\text{-mof}$.

Let R be a coherent ring, and $M \in \mathrm{GProj} \cap R\text{-mof}$. By Facts 1.2($vi$) one can take an exact sequence $0 \longrightarrow M \xrightarrow{f} F \longrightarrow X \longrightarrow 0$ with F free and X Gorenstein projective. Since M is finitely generated, one can write $F = P^0 \oplus Q^0$ with $\mathrm{Im} f \subseteq P^0$. Then we have an exact sequence $0 \longrightarrow M \xrightarrow{f} P^0 \longrightarrow M' \longrightarrow 0$ with $X \cong M' \oplus Q^0$, and hence $M' \in \mathrm{GProj} \cap R\text{-mof}$ by Theorem 1.2. Repeating this procedure with M' replacing M, we get an exact sequence

$$0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with all images in $\mathrm{GProj} \cap R\text{-mof}$. Hence it is a $\mathrm{Hom}(-, \mathrm{proj})$-exact sequence. Since M is a finitely generated submodule of P^0 which is finitely presented, M is finitely presented. Repeating this we get a finitely generated projective resolution of M, which is $\mathrm{Hom}(-, \mathrm{proj})$-exact since $M \in {}^\perp(\mathrm{proj})$. So $M \in \mathrm{Gproj}$. ■

1.4. Strongly Gorenstein projective modules

A complete projective resolution of the form $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ is said to be *strong*, and $M \cong \mathrm{Ker} f$ is called *a strongly Gorenstein projective module* ([BM]). Denote by SGProj the full subcategory of strongly Gorenstein projective modules. Then it is known in [BM] that

$$\mathrm{Proj} \subsetneqq \mathrm{SGProj} \subsetneqq \mathrm{GProj};$$

and that *a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module*. So, a strongly Gorenstein projective module is an analogue of a free module.

Denote by SGproj the full subcategory of all the modules M isomorphic to Ker$f$, where $\cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$ is a complete projective resolution with P finitely generated. Then for any ring R, then category of all the finitely generated, strongly Groenstein projective modules is exactly SGproj, namely we have

$$(\mathrm{SGProj}) \ \cap \ R\text{-mof} = \mathrm{SGproj}.$$

Remark. Up to now there are no efficient ways of constructing concretely (especially finitely generated) Gorenstein projective modules. Strongly Gorenstein projective modules may provide an easier way to obtain the Gorenstein projective modules. In [GZ2] we determined all the finitely generated strongly Gorenstein projective modules over upper triangular matrix artin algebras.

2. Proper Gorenstein projective resolutions

A basic problem in Gorenstein homological algebra is, given a ring R, when R-GProj is contravariantly finite; or equivalently, when every module admits *a proper Gorenstein projective resolution*.

2.1. First, we recall a general result due to Auslander and Buchweitz [ABu].

Let $\mathcal{A}$ be an abelian category, $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ closed under extensions, direct summands, and isomorphisms. Let ω be a cogenerator of $\mathcal{X}$, which means ω is a full subcategory of $\mathcal{X}$ closed under finite direct sums and isomorphisms, and for any $X \in \mathcal{X}$, there is an exact sequence $0 \longrightarrow X \longrightarrow B \longrightarrow X' \longrightarrow 0$ in $\mathcal{X}$ with $B \in \omega$. Denote by $\widehat{\mathcal{X}}$ the full subcategory of $\mathcal{A}$ consisting of all objects X of finite $\mathcal{X}$-dimension n, i.e., there is an exact sequence $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow X \longrightarrow 0$ with $X_i \in \mathcal{X}$.

Theorem 2.1. *([ABu], Theorems 1.1, 2.3, 2.5) (i) Every object $C \in \widehat{\mathcal{X}}$ has a surjective right $\mathcal{X}$-approximation. More precisely, for any $C \in \widehat{\mathcal{X}}$ there is a* $\mathrm{Hom}(\mathcal{X}, -)$*-exact sequence*

$$0 \longrightarrow Y_C \longrightarrow X_C \longrightarrow C \longrightarrow 0$$

with $X_C \in \mathcal{X},\ Y_C \in \widehat{\omega}$; and $Y_C \in \ \mathcal{X}^{\perp}$.

(ii) Every object $C \in \widehat{\mathcal{X}}$ has a injective left $\widehat{\omega}$-approximation. More precisely, for any $C \in \widehat{\mathcal{X}}$ there is a $\mathrm{Hom}(-, \widehat{\omega})$*-exact sequence*

$$0 \longrightarrow C \longrightarrow Y^C \longrightarrow X^C \longrightarrow 0$$

with $X^C \in \mathcal{X},\ Y^C \in \widehat{\omega}$; and $X^C \in \ {}^{\perp}\widehat{\omega}$.

2.2. *A proper Gorenstein projective resolution* of an R-module M is a $\mathrm{Hom}(\mathrm{GProj}, -)$-exact sequence $\mathcal{G}^\bullet : \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$ with each G_i Gorenstein projective.

Note that $\mathrm{Hom}(\mathrm{GProj}, -)$-exactness guarantee the uniqueness of such a resolution in the homotopy category.

2.3. *The Gorenstein projective dimension*, $\mathrm{Gpd}M$, of R-module M is defined as the smallest integer $n \geq 0$ such that M has a GProj-resolution of length n.

Theorem 2.2. *Let M be an R-module of finite Gorenstein projective dimension n. Then M admits a surjective right* GProj*-approximation $\phi : G \longrightarrow M$, with pd* $\mathrm{Ker}\phi = n-1$ *(if $n = 0$, then $K = 0$). In particular, a module of finite*

Gorenstein projective dimension n admits a proper Gorenstein projective resolution of length at most n.

Proof. This follows from Theorem 2.1(i) by letting $\mathcal{X} = \mathrm{GProj}, \ \omega = \mathrm{Proj}$. However, we include a direct proof given in [H1]. Recall the Auslander-Bridger Lemma ([AB], Lemma 3.12), which in particular showing that any two "minimal" resolutions are of the same length.

Auslander-Bridger Lemma Let $\mathcal{X}$ be a resolving subcategory of an abelian category $\mathcal{A}$ having enough projective objects. If

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \longrightarrow A \longrightarrow 0$$

are exact sequences with $X_i, \ Y_i \in \mathcal{X}, \ 0 \leq i \leq n-1$, then $X_n \in \mathcal{X}$ if and only if $Y_n \in \mathcal{X}$.

Coming back to the proof. Take an exact sequence $0 \longrightarrow K' \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ with P_i projective. By Auslander-Bridger Lemma, K' is Gorenstein projective. Hence there is a $\mathrm{Hom}(-, \mathrm{Proj})$-exact sequence $0 \longrightarrow K' \longrightarrow Q^0 \longrightarrow Q^1 \longrightarrow \cdots Q^{n-1} \longrightarrow G \longrightarrow 0$, where Q^i are projective, G is Gorenstein projective. Thus there exist homomorphisms $Q^i \longrightarrow P_{n-1-i}$ for $i = 0, ..., n-1$, and $G \longrightarrow M$, such that the following diagram is commutative

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & K' & \longrightarrow & Q^0 & \longrightarrow & Q^1 & \longrightarrow \cdots \longrightarrow & Q^{n-1} & \longrightarrow & G' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

Let $C_1^\bullet$ and $C_2^\bullet$ denote the upper and the lower row, respectively. Then we have a distinguished triangle in the homotopy category

$$C_1^\bullet \xrightarrow{f^\bullet} C_2^\bullet \longrightarrow \mathrm{Con}(f^\bullet) \longrightarrow C_1^\bullet[1].$$

Since H^0 is a cohomology functor, it follows that $\mathrm{Con}(f^\bullet)$ is also exact, i.e., we have exact sequence

$$0 \longrightarrow K' \xrightarrow{\alpha} Q^0 \oplus K' \longrightarrow Q^1 \oplus P_{n-1} \longrightarrow \cdots \longrightarrow Q^{n-1} \oplus P_1 \longrightarrow G' \oplus P_0 \longrightarrow M \longrightarrow 0$$

with α splitting mono. It follows that we have exact sequence

$$0 \longrightarrow Q^0 \longrightarrow Q^1 \oplus Q_{n-1} \longrightarrow \cdots \longrightarrow Q^{n-1} \oplus Q_1 \longrightarrow G' \oplus P_0 \xrightarrow{\phi} M \longrightarrow 0$$

with $G' \oplus P_0$ Gorenstein projective, $pd\ \operatorname{Ker}\phi \le n-1$ (then necessarily $pd\ \operatorname{Ker}\phi = n-1$). Since $\operatorname{Ext}^i(H, \operatorname{Proj}) = 0$ for $i \ge 1$ and Gorenstein projective module H, so in particular $\operatorname{Ext}^1(H, \operatorname{Ker}\phi) = 0$, and hence ϕ is a left GProj-approximation. ■

Corollary 2.3. *If* $0 \longrightarrow G' \longrightarrow G \longrightarrow M \longrightarrow 0$ *is a short exact sequence with* G'*, G Gorenstein projective, and* $\operatorname{Ext}^1(M, \operatorname{Proj}) = 0$*, then* M *is Gorenstein projective.*

Proof. Since Gpd $M \le 1$, by the theorem above there is an exact sequence $0 \longrightarrow Q \longrightarrow E \longrightarrow M \longrightarrow 0$ with E Gorenstein projective and Q projective. By assumption $\operatorname{Ext}^1(M, Q) = 0$, hence M is Gorenstein projective by Theorem 1.2. ■

Remark 2.4. *If* R *is left noetherian and* M *is a finitely generated left module with* $\operatorname{Gpd}M = n < \infty$*, then* M *has a surjective left* Gproj*-approximation* $G \longrightarrow M$ *with kernel of projective dimension* $n-1$*. Hence* M *has a proper finitely generated Gorenstein projective resolution of length at most* n*. The proof is same as Proposition 1.4.*

2.4. We list some facts on the Gorenstein projective dimensions of modules.

Proposition 2.5. *([H1]) 1. We have* $\operatorname{Gpd}(\bigoplus M_i) = \sup\{\operatorname{Gpd}M_i \mid i \in I\}$.

2. Let n *be an integer. Then the following are equivalent.* (i) $\operatorname{Gpd}M \le n$. (ii) $\operatorname{Ext}^i(M, L) = 0$ *for all* $i > n$ *and modules* L *with finite* $\operatorname{pd}L$. (iii) $\operatorname{Ext}^i(M, Q) = 0$ *for all* $i > n$ *and projective modules* Q. (iv) *For every exact sequence* $0 \longrightarrow K^{-n} \longrightarrow G^{-n+1} \longrightarrow \cdots \longrightarrow G^{-1} \longrightarrow G^0 \longrightarrow M \longrightarrow 0$ *with all* G^i *Gorenstein projective, then also* K^{-n} *is Gorenstein projective.*

3. Let $0 \longrightarrow M' \longrightarrow M \to M'' \longrightarrow 0$ *be a short exact sequence. If any two of the modules have finite Gorenstein projective dimension, then so has the third.*

4. If M *is of finite projective dimension, then* $\operatorname{Gpd}M = \operatorname{pd}M$.

2.5. If M has a proper Gorenstein projective resolution $G^\bullet \longrightarrow M \longrightarrow 0$, then for any module N, *the Gorenstein right derived functor* $\operatorname{Ext}^n_{\mathrm{GProj}}(-, N)$ of $\operatorname{Hom}_R(-, N)$ is defined as

$$\operatorname{Ext}^n_{\mathrm{GProj}}(M, N) := H^n \mathrm{Hom_R(G^\bullet, N)}.$$

Note that it is only well-defined on the modules having proper Gorenstein projective resolutions. Dually, fix a module M, one has *the Gorenstein right derived functor* $\operatorname{Ext}^n_{\mathrm{GInj}}(M, -)$ of $\operatorname{Hom}_R(M, -)$, which is defined on the modules having coproper Gorenstein injective resolutions.

Theorem 2.6. *([AM], [H3]) For all modules M and N with* $\mathrm{Gpd}_R M < \infty$ *and* $\mathrm{Gid}_R N < \infty$*, one has isomorphisms*

$$\mathrm{Ext}^n_{\mathrm{GProj}}(M, N) \cong \mathrm{Ext}^n_{\mathrm{GInj}}(M, N),$$

which are functorial in M and N; and if either pd $M < \infty$ or id $N < \infty$, then the group above coincides with $\mathrm{Ext}^n(M, N)$*.*

Remark 2.7. (i) *If* $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ *is a short* $\mathrm{Hom}(\mathrm{GProj}, -)$*-exact sequence, where all M_i have proper Gorenstein projective resolutions, then for any N,* $\mathrm{Ext}^n_{\mathrm{GProj}}(-, N)$ *induce a desired long exact sequence ([AM], [V]).*

(ii) *If* $0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$ *is a short* $\mathrm{Hom}(\mathrm{GProj}, -)$*-exact sequence, and M has a proper Gorenstein projective resolution, then* $\mathrm{Ext}^n_{\mathrm{GProj}}(M, -)$ *induce a desired long exact sequence ([EJ2], [AM], [V]).*

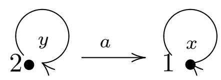
(iii) *Over some special rings (e.g. the Gorenstein rings), the Gorenstein* Ext *groups, the usual* Ext *groups, and the Tate cohomology groups can be related by a long exact sequence ([AM], [J2]).*

3. Gorenstein rings

By a Gorenstein ring (an n-Gorenstein ring) R, we mean that R is left and right noetherian, and that ${}_RR$ and R_R have finite injective dimension (at most n).

Finite-dimensional k-algebras which are self-injective, or are of finite global dimension, are examples of Gorenstein rings. But there are many other examples of Gorenstein rings.

For an example, let Λ be the k-algebra given by the quiver



with relations $x^2,\ y^2,\ ay - xa$ (we write the conjunction of paths from right to left). Then Λ is a Gorenstein algebra which is of infinite global dimension, and not self-injective.

3.1. One has the following basic property of an n-Gorenstein ring.

Theorem 3.1. *(Iwanaga, 1980) Let R be an n-Gorenstein ring, and M a left R-module. Then the following are equivalent.* (i) $id\ M < \infty$. (ii) $id\ M \leq n$. (iii) $pd\ M < \infty$. (iv) $pd\ M \leq n$. (v) $fd\ M < \infty$. (vi) $fd\ M \leq n$.

Proof. Before proving the theorem, we first recall some useful facts.

(A) If $id\ {}_RR \leq n$, then $id\ {}_RP \leq n$, for any left projective module P.

In fact, note that P may be a direct summand of an infinite direct sum $R^{(I)}$. Then it suffices to see $id\ R^{(I)} \leq n$, or equivalently, $\operatorname{Ext}_R^{n+i}(N, R^{(I)}) = 0$ for any $i \geq 1$ and for any module N. Since N is a direct limit of all finitely generated submodules N_i of N, and since

$$\operatorname{Ext}_R^{n+i}(\underrightarrow{lim} N_i, R^{(I)}) = \underleftarrow{lim}\, \operatorname{Ext}_R^{n+i}(N_i, R^{(I)}),$$

it suffices to prove $\operatorname{Ext}_R^{n+i}(N, R^{(I)}) = 0$ for any $i \geq 1$ and for any finitely generated module N. This is true since for finitely generated module N one has

$$\operatorname{Ext}_R^{n+i}(N, R^{(I)}) = \operatorname{Ext}_R^{n+i}(N, R)^{(I)} = 0.$$

(B) A module is flat if and only if it is a direct limit of projective modules.

The sufficiency follows from

$$\operatorname{Tor}_i(X,\ \underrightarrow{lim} N_j) = \underrightarrow{lim}\, \operatorname{Tor}_i(X,\ N_j), \quad \forall\ i \geq 1,\ \forall\ X_R.$$

The proof of the necessity is not a short one. The first proof was given by V.E. Govorov in "On flat modules" (Russian), Sib. Math. J. VI (1965), 300-304; and then it was proved by D. Lazard in "Autour de la Platitude", Bull. Soc. Math. France 97 (1969), 81-128.

(C) If R is left noetherian, and $id\ _RR \le n$, then $id\ _RF \le n$, for any left flat module F.

In fact, by (B) we have $id\ F = id\ \varinjlim P_i$. Since R is left noetherian, we have

$$\mathrm{Ext}^i_R(M, \varinjlim N_i) = \varinjlim \mathrm{Ext}^i_R(M, N_i)$$

for any finitely generated module M. It follows that $id\ \varinjlim N_i \le \sup\{id\ N_i\}$. Now the assertion follows from (A).

$(iii) \Longrightarrow (ii)$: Since $pd\ M < \infty$, and $id\ P \le n$ for any projective module P by (A), it follows that $id\ M \le n$.

$(v) \Longrightarrow (iv)$: Let $m = fd\ M < \infty$. Take a projective resolution of M

$$\cdots \longrightarrow P_t \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If $m > n$, then by dimension shift we see $F = \mathrm{Im}(P_m \longrightarrow P_{m-1})$ is flat, and hence $id\ F \le n$ by (C). Then $\mathrm{Ext}^m(M, F) = 0$, which implies $\mathrm{Hom}(P_{m-1}, F) \longrightarrow \mathrm{Hom}(F, F)$ is surjective, and hence $F \hookrightarrow P_{n-1}$ splits, say $P_{n-1} = F \oplus G$. Hence we have exact sequence

$$0 \longrightarrow G \longrightarrow P_{m-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with G projective. If $m-1 = n$ then we are done. If $m-1 > n$ then we repeat the procedure with G replacing F. So we see $pd\ M \le n$. If $m \le n$, then we also have $pd\ M \le n$. Otherwise $d = pd\ M > n$, then one can choose $d' < \infty,\ n < d' \le d$. Note that $F' = \mathrm{Im}(P_{d'} \longrightarrow P_{d'-1})$ is again flat by dimension shift, and hence $id\ F' \le n$ by (C). Then $\mathrm{Ext}^{d'}(M, F') = 0$, which implies $\mathrm{Hom}(P_{d'-1}, F') \longrightarrow \mathrm{Hom}(F', F')$ is surjective, and hence $F' \hookrightarrow P_{d'-1}$ splits, say $P_{d'-1} = F' \oplus G'$. Hence we have exact sequence

$$0 \longrightarrow G' \longrightarrow P_{d'-2} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with G' projective. This contradicts $d = pd\ M$.

$(i) \Longrightarrow (vi)$: It suffices to prove $fd\ I \le n$ for any injective left R-module. Note that $I^+ = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat right R-module. Then by (C) on the right one has

$id\ I^+ \leq n$, and hence $fd\ I^{++} \leq n$. However, I is a pure submodule of I^{++}, it follows that (for details see Appendix)

$$fd\ I \leq fd\ I^{++} \leq n.$$

Now we have $(i) \Longrightarrow (vi) \Longrightarrow (v) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$. ■

3.2. As we will see, for an n-Gorenstein ring, the full subcategory of Gorenstein projective modules is exactly the left perpendicular of projective modules.

Lemma 3.2. *([EJ2], Lemma 10.2.13) Let R be an n-Gorenstein ring. Then every module M has an injective left $\mathcal{L}$-approximation $f : M \longrightarrow L$, where $\mathcal{L}$ is the full subcategory of R-modules of finite injective dimension.*

Theorem 3.3. *([EJ2]) Let R be an n-Gorenstein ring. Then* $\mathrm{GProj} = {}^{\perp}(\mathrm{Proj})$.

Proof. Note that $\mathrm{GProj} \subseteq {}^{\perp}(\mathrm{Proj})$. Assume $M \in {}^{\perp}(\mathrm{Proj})$. By Lemma 3.2 M has an injective left $\mathcal{L}$-approximation $f : M \longrightarrow L$. Take an exact sequence $0 \longrightarrow K \longrightarrow P^0 \stackrel{\theta}{\longrightarrow} L \longrightarrow 0$ with P^0 projective. By Theorem 3.1 K has finite projective dimension. It follows from this and the assumption $M \in {}^{\perp}(\mathrm{Proj})$ that $\mathrm{Ext}^i(M, K) = 0$ for $i \geq 1$. In particular $\mathrm{Ext}^1(M, K) = 0$. Thus, θ induces a surjective map $\mathrm{Hom}_R(M, P^0) \longrightarrow \mathrm{Hom}_R(M, L)$. Hence we get $g : M \longrightarrow P^0$ such that $f = \theta g$. Since f is an injective left $\mathcal{L}$-approximation and $P^0 \in \mathcal{L}$, we deduce that g is also an injective left $\mathcal{L}$-approximation, and hence $\mathrm{Ext}^i(P^0/M, \mathrm{Proj}) = 0$ for $i \geq 1$. Applying the same argument to P^0/M and continuing this process, we obtain a long exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$, which is $\mathrm{Hom}(-, \mathrm{Proj})$-exact. Putting a (deleted) projective resolution of M together with (the deleted version of) this exact sequence, we see M is Gorenstein projective. ■

Corollary 3.4. *Let R be an n-Gorenstein ring, and*

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence with P_i projective. Then K is Gorenstein projective.

Proof. Note that $\mathrm{Ext}^i_R(K, \mathrm{Proj}) = \mathrm{Ext}^{n+i}_R(M, \mathrm{Proj}) = 0$ for all $i \geq 1$, by Theorem 3.1. Then K is Gorenstein projective by Theorem 3.3. ■

Theorem 3.3 together with Corollary 3.4 are few existed way of producing Gorenstein projective modules.

3.3. The following shows, in particular, that for an n-Gorenstein ring, the subcategory of Gorenstein projective modules is covariantly finite; and moreover, that any module has a proper Gorenstein projective resolution of bounded length n.

Theorem 3.5. *([EJ2]) Let R be an n-Gorenstein ring. Then every R-module M has a surjective left* GProj*-approximation $\phi : G \longrightarrow M \longrightarrow 0$, with pd $\operatorname{Ker}\phi \leq n-1$ (if $n = 0$ then $\operatorname{Ker}\phi = 0$). Thus, every R-module has Gorenstein projective dimension at most n.*

Proof. By the corollary above there is an exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with every P_i projective and K Gorenstein projective. Since K is Gorenstein projective, there is a $\operatorname{Hom}(-, \operatorname{Proj})$-exact sequence

$$0 \longrightarrow K \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots \longrightarrow P^{n-1} \longrightarrow G' \longrightarrow 0 \qquad (*)$$

with every P^i projective and G' Gorenstein projective. Since $(*)$ is $\operatorname{Hom}(-, \operatorname{Proj})$-exact, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & K & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots & \longrightarrow & P^{n-1} & \longrightarrow & G' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

With the same argument as in the proof of Theorem 2.4 we get a surjective left GProj-approximation $\phi : G \longrightarrow M \longrightarrow 0$, with *pd* $\operatorname{Ker}\phi \leq n-1$. ■

Together with Theorem 2.2 we have

Corollary 3.6. *Let R be an n-Gorenstein ring. Then every module has a proper Gorenstein projective resoluiton, of length at most n.*

4. Some recent results

We state more recent results.

Theorem 4.1. *([H2]) Let R be a ring. If M is an R-module with* $\mathrm{pd}_R M < \infty$, *then* $\mathrm{Gid}_R M = \mathrm{id}_R M$. *In particular, if* $\mathrm{Gid}_R R < \infty$, *then also* $\mathrm{id}_R R < \infty$.

Let A, B be k-algebra. *A dualizing dualizing complex* ${}_B D^\bullet_A$ is a bounded complex of B-A-bimodules, such that the cohomology modules of $D^\bullet$ are all finitely generated both over B and A^{op}, $D^\bullet \cong I^\bullet \in D^b(B \otimes_k A^{op})$ with each component of $I^\bullet$ being injective both over B and A^{op}, and the canonical maps

$$A \longrightarrow \mathrm{RHom}_B(D^\bullet, D^\bullet), \quad B \longrightarrow \mathrm{RHom}_{A^{op}}(D^\bullet, D^\bullet)$$

are isomorphisms in $D(A^e)$ and in $D(B^e)$, respectively.

Theorem 4.2. *([J1, Thm. 1.10, Thm. 2.11]) If a ring A satisfies one of the following two conditions, then A-*GProj *is contravariantly finite.*

(i) A is a noetherian commutative ring with a dualizing complex.

(ii) A is a left coherent and right noetherian k-algebra over the field k for which there exists a left noetherian k-algebra B and a dualizing complex ${}_B D_A$.

Note that if A is a finite-dimensional k-algebra, then A has a dualizing complex $A^* = \mathrm{Hom}_k(A, k)$. So A-GProj is contravariantly finite.

Recall from [Be] that a ring R is called *Cohen-Macaulay finite* (CM-finite for short) if there are only finitely many isomorphism classes of finitely generated indecomposable Gorenstein projective R-modules.

Theorem 4.3. *([C]) Let A be a Gorenstein artin algebra. Then A is CM-finite if and only if every Gorenstein projective module is a direct sum of finitely generated Gorenstein projective modules.*

A class $\mathcal{T}$ of modules is called *a tilting class* if there is a (generalized) tilting module T such that $\mathcal{T} = T^\perp$.

Theorem 4.4. *([HHT]) Let R be a noetherian ring. Denote by* GInj-R *the category of right Gorenstein injective R-modules. The following statements are equivalent:*

(i) R is Gorenstein.

*(ii) Both R-*GInj *and* GInj-R *form tilting classes.*

Theorem 4.5. *([BMO]) Let R be a ring. Then the following are equivalent.*

(i) Any Gorenstein projective module is Gorenstein injective.

(ii) Any Gorenstein injective module is Gorenstein projective.

(iii) R is quasi-Frobenius.

Theorem 4.6. *([WSW]) Let R be a commutative ring. Given an exact sequence of Gorenstein projective R-modules*

$$G^\bullet = \cdots \to G^{-2} \xrightarrow{d^{-2}} G^{-1} \xrightarrow{d^{-1}} G^0 \xrightarrow{d^0} G^1 \xrightarrow{d^1} \cdots$$

such that the complexes $\mathrm{Hom}_R(G^\bullet, H)$ and $\mathrm{Hom}_R(H, G^\bullet)$ are exact for each Gorenstein projective R-module H. Then all the images are Gorenstein projective.

5. Appendix: Character modules

5.1. An injective left R-module E is *an injective cogenerator* of left R-modules, if $\operatorname{Hom}(M, E) \neq 0$ for any nonzero module M, or equivalently, for any nonzero module M and any $0 \neq x \in M$, there is $f \in \operatorname{Hom}(M, E)$ such that $f(x) \neq 0$.

Example-definition. $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathbb{Z}$-modules.

For any right R-module $M \neq 0$, the nonzero left R-module $M^+ = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called *the character module* of M.

By the adjoint pair one can see that R^+ is an injective left R-modules. So R^+ is an injective cogenerator of left R-modules since $\operatorname{Hom}(M, R^+) \cong M^+$.

5.2. A submodule N of M is *pure* if $0 \longrightarrow X \otimes N \longrightarrow X \otimes M$ is exact for any right R-module X. For any module M, M is a submodule of M^{++}: In fact, the R-homomorphism $M \longrightarrow M^{++} = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ given by

$$m \mapsto f: \text{``}g \mapsto g(m)\text{''}$$

is injective. Moreover, we have

Lemma 5.1. *For any module M, M is a pure submodule of M^{++}.*

Proof. Applying $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z})$ to the canonical injection $M \hookrightarrow M^{++}$, we get the surjection $\pi : M^{+++} \twoheadrightarrow M^+$. Put $\sigma : M^+ \hookrightarrow M^{+++}$. Then by direct verification one see $\pi\sigma = \operatorname{Id}_{M^+}$, which implies M^+ is a direct summand of M^{+++}. Thus for any right module X we have exact sequence

$$\operatorname{Hom}(X, M^{+++}) \longrightarrow \operatorname{Hom}(X, M^+) \longrightarrow 0,$$

i.e., we have exact sequence (by the adjoint pair)

$$(X \otimes M^{++})^+ \longrightarrow (X \otimes M)^+ \longrightarrow 0;$$

and then it is easily verified that

$$0 \longrightarrow X \otimes M \longrightarrow X \otimes M^{++}$$

is exact. ■

5.3.

Lemma 5.2. *Let N be a pure submodule of M. Then $fd\ N \le fd\ M$.*

In particular, $fd\ M \le fd\ M^{++}$ for any module M.

Proof. We may assume $fd\ M = n < \infty$. For any right module X, take a partial projective resolution

$$0 \longrightarrow K \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

Consider the commutative diagram

$$\begin{array}{ccccc} & & K \otimes N & \longrightarrow & P_n \otimes N \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K \otimes M & \longrightarrow & P_n \otimes M. \end{array}$$

Since $\mathrm{Tor}_{n+1}(X, M) = 0$, the bottom row is exact. Two vertical maps are injective since N is a pure submodule of M. It follows that $K \otimes N \longrightarrow P_n \otimes N$ is injective, and hence $\mathrm{Tor}_{n+1}(X, N) = 0$. So $fd\ N \le n$. ■

5.4. Flat modules can be related with injective modules as follows.

Proposition 5.3. *Let M be an R-S-bimodule, E an injective cogenerator of right S-modules. Then*

(*i*) $fd\ {}_RM = id\ \mathrm{Hom}_S(M, E)_R$.

In particular, ${}_RM$ is flat $\iff$ $(M^+)_R$ is injective.

(*ii*) *If furthermore R is left noetherian, then $id\ {}_RM = fd\ \mathrm{Hom}_S(M, E)_R$.*

In particular, ${}_RM$ is injective $\iff$ $(M^+)_R$ is flat.

Proof. The assertion (i) follows from the identity

$$\mathrm{Hom}_S(\mathrm{Tor}_i^R(X, M), E) \cong \mathrm{Ext}_R^i(X, \mathrm{Hom}_S(M, E)), \ \forall\ X_R.$$

For (ii), note that if R is left noetherian, then for any finitely presented module ${}_RX$ there holds

$$\mathrm{Tor}_i^R(\mathrm{Hom}_S(M, E), X) \cong \mathrm{Hom}_S(\mathrm{Ext}_R^i(X, M), E);$$

also, by using direct limit one has $id\ M \le n$ if and only if $\mathrm{Ext}_R^i(X, M) = 0, \ \forall\ i \ge n+1$, and for all finitely generated modules X.

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