Borcherds Kac-Moody Lie algebras II

Seminar: Representation Theory

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Let A be a symmetrisable BKM matrix over a field k, $\mathfrak{g} := \mathfrak{g}(A)$.

Let (π, V) be a representation of $\mathfrak{g}$.
For $x \in \mathfrak{g}$, $v \in V$, we write just xv instead of $\pi(x)v$.

For $\lambda \in \mathfrak{h}^*$, set

$$V_\lambda := \{v \in V \mid hv = \lambda(h)v \ \forall h \in \mathfrak{h}\} \qquad =: \textit{weight space}$$

If $V_\lambda \neq 0$, we say that λ is a *weight* of the representation (π, V).

We set

$$\mathrm{mult}_V(\lambda) := \dim_k V_\lambda$$

and call this the *multiplicity of* λ *in* (π, V) (or *in* V).

We denote by $P(V) \subseteq \mathfrak{h}^*$ the set of weights of the representation (π, V).

A representation (π, V) is called $\mathfrak{h}$-*diagonalisable* (and V an $\mathfrak{h}$-*diagonalisable module*), if

1. $\dim V_\lambda < \infty$ for all $\lambda \in \mathfrak{h}^*$, and
2. $V = \bigoplus_{\lambda \in P(V)} V_\lambda$.

If V is $\mathfrak{h}$-diagonalisable, we set

$$\operatorname{ch} V := \sum_{\lambda \in P(V)} (\dim_k V_\lambda) \cdot e^\lambda$$

and call this the *character of* V.

$$\operatorname{ch} V := \sum_{\lambda \in P(V)} (\dim_k V_\lambda) \cdot e^{\lambda}$$

The value of $\operatorname{ch} V$ at $h \in \mathfrak{h}$ is defined as

$$(\operatorname{ch} V)(h) := \sum_{\lambda \in P(V)} (\dim_k V_\lambda) \cdot e^{\lambda(h)},$$

if the sum converges. (In this case, this can be rewritten as:

$$(\operatorname{ch} V)(h) = \sum_{\lambda \in P(V)} \operatorname{tr}_V(e^{\pi(h)}),$$

which is the trace of the linear operator $e^{\pi(h)}$.)

Let V be an $\mathfrak{h}$-diagonalisable $\mathfrak{g}$-module. If $\pi(e_i)$ and $\pi(f_i)$ act on V locally nilpotently for all $i \in I^{\mathrm{re}}$, then V is called *integrable*.

If V is integrable, then also e_α and f_α act locally nilpotently for all $\alpha \in \Delta^{\mathrm{re}}$. If we set

$$r_\alpha^V := e^{\pi(f_\alpha)} e^{\pi(-e_\alpha)} e^{\pi(f_\alpha)},$$

then

$$v \in V_\lambda \;\Rightarrow\; r_\alpha^V v \in V_{r_\alpha \lambda}$$

It follows:

If V is an $\mathfrak{h}$-diagonalisable integrable $\mathfrak{g}$-module, then $\mathrm{ch}V$ is invariant under the action of the Weyl group.

Let V be an $\mathfrak{h}$-diagonalisable integrable $\mathfrak{g}$-module. If there exists a $\Lambda \in P(V)$ and an element $0 \neq v_\Lambda \in V_\Lambda$ with the following properties

1. $xv_\Lambda = 0$ for all $x \in \mathfrak{n}_+$, and
2. $\mathfrak{U}(\mathfrak{g}) \cdot v_\Lambda = V$,

then V is called *highest weight representation* (or *highest weight* $\mathfrak{g}$*-module*) *with highest weight* Λ and v_Λ a *highest weight vector*.

Note that:

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{n}_-) \cdot \mathfrak{U}(\mathfrak{h}) \cdot \mathfrak{U}(\mathfrak{n}_+)$$

(by the PBW-Theorem) and that this implies that already

$$\mathfrak{U}(\mathfrak{n}_-) \cdot v_\Lambda = V$$

If V is a highest weight $\mathfrak{g}$-module with highest weight Λ, then we have:

- Any $\lambda \in P(V)$ can be written as

$$\lambda = \Lambda - \sum_{i=1}^{n} m_i \alpha_i$$

  with $m_i \in \mathbb{Z}_{\geq 0}$.
- $V_\Lambda = \mathbb{C} v_\Lambda$ $(\dim_k V_\Lambda = 1)$

Since

$$\mathfrak{U}(\mathfrak{n}_-) \cdot v_\Lambda = V,$$

the linear map $f : \mathfrak{U}(\mathfrak{n}_-) \to V,\ x \mapsto xv_\Lambda$ is clearly surjective.

If f is also injective, then we call V a *Verma module*. We denote the Verma module with highest weight Λ by $M(\Lambda)$.

It can be shown that, given a $\Lambda \in \mathfrak{h}^*$, a Verma module with highest weight Λ exists (explicit construction) and that it is unique (using universal properties).

We get a basis of the Verma module $M(\Lambda)$ if we have a basis of $\mathfrak{U}(\mathfrak{n}_-)$, because $f : \mathfrak{U}(\mathfrak{n}_-) \to V,\ x \mapsto xv_\Lambda$ is a linear isomophism.

We take a basis of $\mathfrak{n}_-$, from which we can construct by the PBW-Theorem a basis of $\mathfrak{U}(\mathfrak{n}_-)$.

From this basis (and an application of the geometric series) we obtain:

$$\mathrm{ch} M(\Lambda) = e^\Lambda \prod_{\alpha\in\Delta_+} \frac{1}{(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$$

We set $R := \prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}$. Then: $\mathrm{ch} M(\Lambda) = \frac{e^\Lambda}{R}$.

The second condition for highest weight modules does *not* imply that they are irreducible. To study what can happen, we will now take a look at "maximal weights" for invariant subspaces.

Let V be a highest weight $\mathfrak{g}$-module and $\lambda \in P(V)$ with $\lambda \neq \Lambda$, the highest weight of V.
We call a vector $0 \neq v \in V_\lambda$

- a *singular vector of weight* λ if $e_i v = 0$ for all $i \in I$
- a *primitive vector of weight* λ if there is an invariant subspace $U \subset V$ s.t.
  1. $v \notin U$, and
  2. $e_i v \in U$ for all $i \in I$.

If there is a singular (primitive) vector of weight λ, we call λ a *singular (primitive) weight*.

Casimir element

$$\Omega := 2\varrho + \sum_{i=1}^{\dim \mathfrak{h}} u^i u_i + 2 \sum_{\alpha \in \Delta_+} e_{-\alpha}^{(i)} e_{\alpha}^{(i)}$$

Forget about the exact definition. The following properties are more important.

Casimir element Ω

If V is a highest weight $\mathfrak{g}$-module with highest weight Λ, then Ω is a scalar operator with eigenvalue $(\Lambda|\Lambda + 2\varrho)$, where ϱ is the half sum of the positive roots.
Idea of proof: Use the definition and show the identity for v_Λ. Now use that $\mathfrak{U}(\mathfrak{n}_-) \cdot v_\Lambda = V$ and that the action of the Casimir element commutes with the action of $\mathfrak{g}$.

Further, if λ is a primitive weight, then

$$(\lambda|\lambda + 2\varrho) = (\Lambda|\Lambda + 2\varrho).$$

Idea of proof: Show the identity for singular vectors and use that primitive vectors are singular vectors in V/U!

Let V be a highest weight $\mathfrak{g}$-module with highest weight Λ. If V is integrable, then

$$\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I^{\mathrm{re}}.$$

The converse is in general *not* true!

One can show:
Every highest weight $\mathfrak{g}$-module V with highest weight Λ is isomorphic to a quotient of the Verma module $M(\Lambda)$ by an appropriate invariant subspace U:

$$V \cong M(\Lambda)/U$$

(Construct U explicitly!)

Let us denote by v_Λ a highest weight vector for V, and by $\tilde{v}_\Lambda$ a highest weight vector for $M(\Lambda)$.

If $\Lambda \in \mathfrak{h}^*$ with

$$\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$$

for all $i \in I^{\text{re}}$, define $U_0 \subset M(\Lambda)$ as follows:

$$U_0 := \sum_{i \in I^{\text{re}}} \mathfrak{U}(\mathfrak{n}_-) f_i^{\langle \Lambda, \alpha_i^\vee \rangle + 1} \tilde{v}_\Lambda.$$

(Here, $\langle \Lambda, \alpha_i^\vee \rangle =: m_i$ is the minimal possible integer with $f^{m_i+1} v_\Lambda = 0$.)
Then:

- U_0 is an invariant subspace of $M(\Lambda)$, and
- if U is an invariant subspace of $M(\Lambda)$, then

$$V := M(\Lambda)/U \text{ is integrable } \Leftrightarrow U_0 \subset U.$$

Integrable representations

Kac-Moody vs. Borchers case: Integrability does *not* imply irreducibility

$\mathfrak{g}(A)$ is a Kac-Moody Lie algebra $\Leftrightarrow I^{\mathrm{re}} = I$
So, the elements of the set $\{f_i \mid i \in I^{\mathrm{re}}\}$ contain all generators of $\mathfrak{n}_-$.

One can show that in the Kac-Moody case, the subspace U_0 as defined above is a maximal proper invariant subspace. So every U with $U_0 \subset U$ is already U_0.
From this, one can derive that in the Kac-Moody case, every integrable highest weight $\mathfrak{g}$-module is irreducible.

In the non Kac-Moody case, a U for which $U_0 \subset U$ is *not* unique. There exist integrable highest weight $\mathfrak{g}$-modules which are not irreducible if $\mathfrak{g}$ is not a Kac-Moody Lie algebra. In particular, if $I^{\mathrm{re}} = \emptyset$, then *all* highest weight modules are integrable.

If V is a highest weight module with highest weight $\Lambda \in \mathfrak{h}^*$, then there is a maximal proper invariant submodule of V.

For $\Lambda \in \mathfrak{h}^*$, we denote the maximal proper invariant subspace of the Verma module $M(\Lambda)$ by J_Λ.

For $\Lambda \in \mathfrak{h}^*$, there is a unique irreducible highest weight module $L(\Lambda)$ with highest weight Λ, namely $L(\Lambda) = M(\Lambda)/J_\Lambda$.

Category $\mathcal{O}$?

For $\lambda \in \mathfrak{h}^*$, set

$$D(\lambda) := \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\} := \{\lambda - \alpha \mid \alpha \in Q_+\}.$$

($Q_+ =$ positive root lattice)

The *objects* of the category $\mathcal{O}$ are the $\mathfrak{h}$-diagonalisable $\mathfrak{g}$-modules V such that there exist $\lambda_1, \ldots, \lambda_k \in \mathfrak{h}^*$ with

$$P(V) \subset \bigoplus_{i=1}^{k} D(\lambda_i).$$

The *morphisms* are homomorphisms of $\mathfrak{g}$-modules.

- The category $\mathcal{O}$ is closed under taking subobjects and quotients.
- For an object in the category $\mathcal{O}$, there are only finitely many maximal weights.
- $\lambda \in P(V)$ is called *primitive weight*, if there exists $0 \neq v \in V_\lambda$ and an invariant subspace $U \subset V$ with
  1. $v \notin U$, and
  2. $e_i v \in U$ for all $1 \leq i \leq n$.
- If $U = 0$, then we call λ a *singular weight*.

For an object V in the category $\mathcal{O}$, primitive vectors of sub- and factor objects give rise to primitive vectors for V itself.

If V is an object in the category $\mathcal{O}$, then there is a proper invariant submodule U of V such that V/U is an irreducible $\mathfrak{g}$-module.

In particular, if one takes one of the maximal weights Λ of V, then there exists a proper invariant submodule U of V such that $V/U \cong L(\Lambda)$.

Let V be an object in the category $\mathcal{O}$.
Then there exists a series

$$\cdots \subset W_{k+1} \subset W_k \subset \cdots \subset W_1 \subset W_0 = V$$

such that

1. W_k/W_{k+1} are irreducible $\mathfrak{g}$-modules, $k = 0, 1, 2, \ldots$, and
2. $\bigcap_{k=0}^{\infty} W_k = 0$.

If we have such a series and we denote the highest weights ($\neq 0$) of W_k/W_{k+1} by μ_k, $k = 0, 1, 2, \ldots$, then

$$\mathrm{ch} V = \sum_k \mathrm{ch} L(\mu_k).$$

Rewriting the formula for Verma modules, we get:

$$\mathrm{ch}M(\Lambda) = \sum_{\substack{\lambda \leq \Lambda, \\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} b(\Lambda, \lambda)\mathrm{ch}L(\lambda),$$

where $b(\Lambda, \lambda) \in \mathbb{Z}_{\geq 0}$ are uniquely determined by $M(\Lambda)$ and $b(\Lambda, \Lambda) = 1$.

Applying this to a general highest weight module $V \cong M(\Lambda)/U$ of highest weight $\Lambda \in \mathfrak{h}^*$ we obtain:

$$\mathrm{ch}V = \sum_{\substack{\lambda \leq \Lambda, \\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} c(\Lambda, \lambda)\mathrm{ch}L(\lambda),$$

where $c(\Lambda, \lambda) \in \mathbb{Z}$ and $c(\Lambda, \Lambda) = 1$.

Let A be a BKM matrix. We set:

- $P_+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \forall i \in I^{\mathrm{re}}\}$
- $P_{++} := \{\lambda \in P_+ \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0} \ \forall i \in I^{\mathrm{re}}\}$
- $P_{(+)} := \{\lambda \in P_+ \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ \forall i \in I\}$
  $=:$ *dominant integral weights*

Aim:
Compute $\mathrm{ch}_\Lambda := \mathrm{ch} L(\Lambda)$, where $L(\Lambda)$ is the irreducible highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in P_{(+)}$.

$$\mathrm{ch}_\Lambda = \sum_{\substack{\lambda \le \Lambda, \\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} c(\Lambda,\lambda)\,\mathrm{ch}\,M(\lambda)$$

$$= \frac{1}{R} \sum_{\substack{\lambda \le \Lambda, \\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} c(\Lambda,\lambda) e^{\lambda}$$

$$= \frac{1}{e^{\varrho} R} \sum_{\substack{\lambda \le \Lambda, \\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} c(\Lambda,\lambda) e^{\lambda+\varrho}$$

Since ch_Λ is W-invariant and $w(e^\varrho) = \varepsilon(w)e^\varrho$ for $w \in W$, we get:

- If $\lambda, \mu \in P(\Lambda)$ with $w(\mu + \varrho) = \lambda + \varrho$ and $|\mu + \varrho|^2 = |\lambda + \varrho|^2$, then
$$c(\Lambda, \lambda) = \varepsilon(w)c(\Lambda, \mu).$$
- If $c(\Lambda, \mu) \neq 0$, then $w(\mu + \varrho) \leq \Lambda + \varrho$ for all $w \in W$.

From this we obtain:
If $\Lambda \in P_{(+)}$ and $\mu \in \mathfrak{h}^*$ with $w(\mu + \varrho) \leq \Lambda + \varrho$ for all $w \in W$, then there exists $w_0 \in W$ with

$$\langle w_0(\mu + \varrho), \alpha_i^\vee \rangle \geq 0 \ \forall i \in I^{\mathrm{re}}.$$

Idea of proof: Use the heights of the corresponding vectors.

It follows from a direct calculation, that

$$r_i(\mu + \varrho) = \mu + \varrho,$$

if $\langle w_0(\mu + \varrho), \alpha_i^\vee \rangle = 0$, and therefore (using the skew symmetry w.r.t. W above) that

$$c(\Lambda, \mu) = 0.$$

Taking into account that now $\langle w_0(\mu + \varrho), \alpha_i^\vee \rangle \geq 1$ has to be fulfilled, one can show that one has to summerise in the character formula only over the $\mu \in P_+$, and that all the μ are such that

$$\mu + \varrho = w(\lambda + \varrho) \text{ for some } w \in W.$$

So, the character formula simplifies to

$$e^\varrho R \cdot \mathrm{ch}_\Lambda = \sum_{w \in W} \varepsilon(w) \sum_{\substack{\lambda \in P_+ \\ \lambda \leq \Lambda,\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} c(\Lambda, \lambda) e^{w(\lambda+\varrho)}.$$

For simplification, we set:

$$S_\Lambda := \sum_{\substack{\lambda \in P_+ \\ \lambda \leq \Lambda,\ |\lambda+\varrho|^2 = |\Lambda+\varrho|^2}} c(\Lambda, \lambda) e^{w(\lambda+\varrho)}.$$

Now, the question is: What are the $\lambda \in P_+$ satisfying the two conditions

$$\lambda \leq \Lambda, \text{ and}$$
$$|\lambda + \varrho|^2 = |\Lambda + \varrho|^2 \quad ?$$

Let A be a symmetrisable BKM matrix and $\Lambda \in P_{(+)}$ and $\lambda \in \mathfrak{h}^*$ with $\lambda \leq \Lambda$ and $(\lambda + \varrho|\alpha_i) \geq 0$ for all $i \in I^{\mathrm{re}}$. Then:

- If $i \in \mathrm{supp}(\Lambda - \lambda) \cap I^{\mathrm{re}}$, then $(\Lambda + \lambda + 2\varrho|\alpha_i) > 0$.
- If $i \in \mathrm{supp}(\Lambda - \lambda) \cap I^{\mathrm{im}}$, then $(\Lambda + \lambda + 2\varrho|\alpha_i) \geq 0$.
- $|\Lambda + \varrho|^2 \geq |\lambda + \varrho|^2$
- Write $\lambda = \Lambda - \sum_{i \in J} m_i \alpha_i$ with $J \subset I$ and $m_i \in \mathbb{Z}_{\geq 0}$. Then: $|\Lambda + \varrho|^2 = |\lambda + \varrho|^2$ if and only if
  - $J \subset I^{\mathrm{im}}$, and
  - if $i \in J$, then $(\Lambda|\alpha_i) = 0$, and
  - if $i, j \in J$ with $i \neq j$, then $(\alpha_i|\alpha_j) = 0$, and
  - if $i \in J$ [with $a_{ii} < 0$], then $m_i = 1$.

Claim:

If $c(\Lambda, \lambda) \neq 0$, then

$$c(\Lambda, \lambda) = (-1)^{|J|}$$

for J defined as above.
Since $(\Lambda|\alpha_i) = 0$ for $i \in J$, it follows that

$$f_i v_\Lambda \in M(\Lambda)$$

is a singular vector. Thus it cannot appear in the irreducible representation $L(\Lambda)$. It follows that $\Lambda - \alpha_i$ is *not* a weight of $L(\Lambda)$. So the coefficient $c(\Lambda, \Lambda - \alpha_i)$ is zero.

Now, compare the two hand sides in the simplified character formula!

Let now $i_1, \ldots, i_k \in J$ be distinct roots. Then $\Lambda - \alpha_{i_1} - \ldots - \alpha_{i_k}$ cannot be a weight of $L(\Lambda)$, this follows from the fact that the α_{i_j} are pairwise orthogonal (and some calculations).

But the expression $e^{-\alpha_{i_1} - \cdots - \alpha_{i_k}}$ appears in the expansion of

$$R = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}.$$

It can only be obtained from

$$(1 - e^{-\alpha_{i_1}}) \cdots (1 - e^{-\alpha_{i_k}}),$$

since one can show that $\alpha_{i_j} + \alpha_{i_\ell}$ cannot be a root.

So, the coefficient is $(-1)^k$, as stated above.

Theorem

If $\mathfrak{g}(A)$ is a symmetrisable BKM Lie algebra, then, for $\Lambda \in P_{(+)}$,

$$\mathrm{ch}_\Lambda = \frac{1}{e^{\varrho} R} \sum_{w \in W} \varepsilon(w) w(S_\Lambda)$$

with $S_\Lambda := \sum_{k \in \mathbb{Z}_{\geq 0}; i_1, \ldots, i_k \in I^{\mathrm{im}}; \text{(i)},\text{(ii)}} (-1)^k e^{\Lambda + \varrho - (\alpha_{i_1} + \cdots + \alpha_{i_k})}$, where

(i) $(\Lambda | \alpha_{i_j}) = 0$ for $j = 1, \ldots, k$

(ii) $i_1, \ldots, i_k$ distinct with $\alpha_{i_1}, \ldots, \alpha_{i_k}$ pairwise orthogonal.

From the previous considerations, one can derive that only element λ fulfilling all the conditions above (and over which we have to summerise in the expression S_Λ) is Λ itself.

So we get:

Theorem

If $\mathfrak{g}(A)$ is a symmetrisable Kac-Moody Lie algebra, then, for $\Lambda \in P_+ (= P_{(+)})$,

$$\mathrm{ch}_\Lambda = \frac{1}{e^\varrho R} \sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\varrho)}.$$