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FINITENESS OF REPRESENTATION DIMENSION

OSAMU IYAMA

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Abstract. We will show that any module over an artin algebra is a direct summand of some module whose endomorphism ring is quasi-hereditary. As a conclusion, any artin algebra has a finite representation dimension.

M. Auslander introduced a concept of representation dimension of artin algebras in [A], which was a trial to give a reasonable way of measuring homologically how far an artin algebra is from being of finite representation type ([X1], [FGR]). His methods given there have been effectively applied not only for the representation theory of artin algebras [ARS], but also for the theory of quasi-hereditary algebras of Cline-Parshall-Scott [CPS] by Dlab-Ringel in [DR2]. Unfortunately, much seems to be unknown about representation dimension itself. In particular, Reiten asked in 1998 whether any artin algebra has a finite representation dimension or not (cf. §2.3(2)). In this paper, we will give a positive answer to this question (§1.2) by showing that any module is a direct summand of some module whose endomorphism ring is quasi-hereditary (§1.1). Our method is to construct a certain chain of subcategories of $\mathrm{mod}\,\Lambda$ (§2.2), which was applied to solve Solomon's second conjecture on zeta functions of orders in [I3]. We will formulate it in terms of rejective subcategories (§2.1), which was effectively applied in [I1] to study the representation theory of orders and give a characterization of their finite Auslander-Reiten quivers in [I2].

Note. After the author submitted this paper, Professor Xi kindly informed him that Theorem 1.1 and Corollary 1.2 were stated in [X2] as conjectures, where the former was given by Ringel and Yamagata. He thanks Professor Xi and Professor Yamagata for valuable comments.

1.

In this paper, any module is assumed to be a left module. For an artin algebra Λ over R, let $\mathrm{mod}\,\Lambda$ be the category of finitely generated left Λ-modules, J_Λ the Jacobson radical of Λ, $\mathrm{dom.dim}\,\Lambda$ the dominant dimension of Λ [T], $\mathrm{I}_\Lambda(X)$ the injective hull of the Λ-module X and $(\ )^* := \mathrm{Hom}_R(\ ,\mathrm{I}_R(R/J_R)) : \mathrm{mod}\,\Lambda \leftrightarrow \mathrm{mod}\,\Lambda^{op}$ the duality. For $X \in \mathrm{mod}\,\Lambda$, we denote by $\mathrm{add}\,X$ the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of direct summands of a finite direct sum of X. The representation dimension of Λ is defined by $\mathrm{rep.dim}\,\Lambda := \inf\{\mathrm{gl.dim}\,\Gamma \mid \Gamma \in A(\Lambda)\}$, where $A(\Lambda)$ is the collection of all artin algebras Γ such that $\mathrm{dom.dim}\,\Gamma \geq 2$ and $\mathrm{End}_\Gamma(\mathrm{I}_\Gamma(\Gamma))$ is

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Morita-equivalent to Λ. Then $\text{rep.dim}\,\Lambda = \inf\{\text{gl.dim}\,\text{End}_\Lambda(M) \mid M \in \text{mod}\,\Lambda$ such that $\Lambda \oplus \Lambda^* \in \text{add}\, M\}$ holds by [A].

1.1. Theorem. *Let Λ be an artin algebra. Then any $M \in \text{mod}\,\Lambda$ is a direct summand of some $N \in \text{mod}\,\Lambda$ such that $\text{End}_\Lambda(N)$ is a quasi-hereditary algebra.*

1.2. Corollary. *Let Λ be an artin algebra. Then $\text{rep.dim}\,\Lambda$ has a finite value which is not greater than $2l-2$, where l is the length of a $(\Lambda, \text{End}_\Lambda(\Lambda\oplus\Lambda^*))$-module $\Lambda\oplus\Lambda^*$.*

2.

In the rest of this paper, any subcategory $\mathcal{C}'$ of an additive category $\mathcal{C}$ is assumed to be full and closed under direct sums. Let $\mathcal{J}_\mathcal{C}$ be the Jacobson radical of $\mathcal{C}$ and $[\mathcal{C}']$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object in $\mathcal{C}'$. Thus $\mathcal{J}_\mathcal{C}(X,X)$ forms the Jacobson radical of the ring $\mathcal{C}(X,X)$ for any $X \in \mathcal{C}$.

2.1. Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$.

(1) $\mathcal{C}'$ is called a *right rejective subcategory* of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \to \mathcal{C}$ has a right adjoint $\mathbb{F} : \mathcal{C} \to \mathcal{C}'$ with a counit ϵ [HS] such that ϵ_X is a monomorphism for any $X \in \mathcal{C}$ (cf. [I1], 5.1). This is equivalent to that, for any $X \in \mathcal{C}$, there exists a monomorphism $g \in \mathcal{C}(Y,X)$ with $Y \in \mathcal{C}'$ which induces an isomorphism $\mathcal{C}(\ ,Y) \xrightarrow{g} [\mathcal{C}'](\ ,X)$ on $\mathcal{C}$ (cf. [I1], 5.2).

(2) $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ is called a *right rejective chain* if $\mathcal{J}_{\mathcal{C}_n/[\mathcal{C}_{n+1}]} = 0$ holds and $\mathcal{C}_{n+1}$ is a right rejective subcategory of $\mathcal{C}_n$ for any n $(0 \le n < m)$. In this case, if $\Gamma := \mathcal{C}(M,M)$ is an artin algebra for an additive generator M of $\mathcal{C}$, then Γ is a quasi-hereditary algebra with a heredity chain $0 = [\mathcal{C}_m](M,M) \subseteq [\mathcal{C}_{m-1}](M,M) \subseteq \cdots \subseteq [\mathcal{C}_0](M,M) = \Gamma$.

Dually, we define a left rejective subcategory and a left rejective chain.

2.1.1. Let $\mathcal{C}'$ be a right rejective subcategory of $\mathcal{C}$ and $\mathcal{C}''$ a subcategory of $\mathcal{C}'$. Then $\mathcal{C}'/[\mathcal{C}'']$ is a right rejective subcategory of $\mathcal{C}/[\mathcal{C}'']$ since the isomorphism $\mathcal{C}(\ ,\mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}'](\ ,X)$ induces an isomorphism $[\mathcal{C}''](\ ,\mathbb{F}(X)) \xrightarrow{\epsilon_X} [\mathcal{C}''](\ ,X)$. Moreover, if $\mathcal{C}''$ is a right rejective subcategory of $\mathcal{C}'$, then it is a right rejective subcategory of $\mathcal{C}$.

2.1.2. *Proof of* 2.1(2). $\mathcal{C}_{m-1}$ is also a right rejective subcategory of $\mathcal{C}$ by 2.1.1. Let $\mathbb{F}$ be the right adjoint of the inclusion $\mathcal{C}_{m-1} \to \mathcal{C}$. Then $I := [\mathcal{C}_{m-1}](M,M)$ is isomorphic to a projective Γ-module $\mathcal{C}(M,\mathbb{F}(M))$, and $IJ_\Gamma I = 0$ holds by $\mathcal{J}_{\mathcal{C}_{m-1}} = 0$. Since $[\mathcal{C}_{m-1}]^2 = [\mathcal{C}_{m-1}]$ holds, I is a heredity ideal of Γ. Since $0 = \mathcal{C}_{m-1}/[\mathcal{C}_{m-1}] \subseteq \mathcal{C}_{m-2}/[\mathcal{C}_{m-1}] \subseteq \cdots \subseteq \mathcal{C}_0/[\mathcal{C}_{m-1}] = \mathcal{C}/[\mathcal{C}_{m-1}]$ is again a right rejective chain by 2.1.1, we obtain the assertion inductively. □

2.2. Our results 1.1 and 1.2 immediately follow from the following lemma (put $M := \Lambda \oplus \Lambda^*$ for 1.2).

Lemma. *Let Λ be an artin algebra and $M \in \text{mod}\,\Lambda$. Put $M_0 := M$, $M_{n+1} := M_n J_{\text{End}_\Lambda(M_n)} \subsetneq M_n$ and take a large m such that $M_m = 0$. Then $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ gives a right rejective chain for $\mathcal{C}_n := \text{add}\bigoplus_{l=n}^{m-1} M_l$. Thus $\Gamma := \text{End}_\Lambda(N)$ is a quasi-hereditary algebra for $N := \bigoplus_{l=0}^{m-1} M_l$ such that $\text{gl.dim}\,\Gamma \le 2m-2$.*

Proof. (i) Note that there exists a surjection $f_{n,l} \in \mathrm{Hom}_\Lambda(\bigoplus M_n, M_l)$ for any $n < l$.

(ii) Define a functor $\mathbb{F}_n : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ by

$$\mathbb{F}_n(X) := \sum_{Y \in \mathcal{C}_n,\ f \in \mathcal{J}_{\mathrm{mod}\,\Lambda}(Y,X)} f(Y).$$

Then a natural transformation $\epsilon : \mathbb{F}_n \to 1$ is defined by the inclusion $\epsilon_X : \mathbb{F}_n(X) \to X$. By (i), $\mathbb{F}_n(M_n) = M_n J_{\mathrm{End}_\Lambda(M_n)} = M_{n+1} \in \mathcal{C}_{n+1}$ holds. Thus $\mathcal{J}_{\mathcal{C}_n}(\ ,X) = [\mathcal{C}_{n+1}](\ ,X) = \mathcal{C}_n(\ ,\mathbb{F}_n(X))\epsilon_X$ holds on $\mathcal{C}_n$ for any indecomposable $X \in \mathcal{C}_n - \mathcal{C}_{n+1}$.

(iii) Fix indecomposable $X \in \mathcal{C}_n$. Put $Y := \mathbb{F}_n(X)$ and $g := \epsilon_X$ if $X \notin \mathcal{C}_{n+1}$, and $Y := X$ and $g := 1_X$ if $X \in \mathcal{C}_{n+1}$. By (ii), $Y \in \mathcal{C}_{n+1}$ and $\mathcal{C}_n(\ ,Y) \xrightarrow{g} [\mathcal{C}_{n+1}](\ ,X)$ is an isomorphism on $\mathcal{C}_n$. Thus $\mathcal{C}_{n+1}$ is a right rejective subcategory of $\mathcal{C}_n$. Since $\mathcal{J}_{\mathcal{C}_n/[\mathcal{C}_{n+1}]} = 0$ holds by (ii), our chain is right rejective. Now $\mathrm{gl.dim}\,\Gamma \leq 2m-2$ follows from [DR1]. □

2.3. *Remark.* (1) The dual version of 2.2 is the following lemma, which gives a variation of the theorem of Auslander and Dlab-Ringel in [A] and [DR2] by putting $M := \Lambda$.

Lemma. *Let Λ be an artin algebra and $M \in \mathrm{mod}\,\Lambda$. Put $M_0 := M$, $M_{n+1} := M_n/\{x \in M_n \mid xJ_{\mathrm{End}_\Lambda(M_n)} = 0\}$ and take a large m such that $M_m = 0$. Then $0 = \mathcal{C}_m \subseteq \mathcal{C}_{m-1} \subseteq \cdots \subseteq \mathcal{C}_0 = \mathcal{C}$ gives a left rejective chain for $\mathcal{C}_n := \mathrm{add} \bigoplus_{l=n}^{m-1} M_l$. Thus $\Gamma := \mathrm{End}_\Lambda(N)$ is a quasi-hereditary algebra for $N := \bigoplus_{l=0}^{m-1} M_l$ such that* $\mathrm{gl.dim}\,\Gamma \leq 2m-2$.

(2) By a result of Igusa-Todorov ([IT], 0.8), $\mathrm{rep.dim}\,\Lambda \leq 3$ implies $\mathrm{fin.dim}\,\Lambda < \infty$. Thus, from the viewpoint of the finitistic global dimension conjecture, it is an interesting question whether any artin algebra Λ satisfies $\mathrm{rep.dim}\,\Lambda \leq 3$ or not [A].

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Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan

E-mail address: iyama@kusm.kyoto-u.ac.jp

Current address: Department of Mathematics, Himeji Institute of Technology, Himeji, 671-2201, Japan

E-mail address: iyama@sci.himeji-tech.ac.jp