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On the Representation Dimension of Finite Dimensional Algebras

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1. INTRODUCTION

The importance of the representation finite Artin algebras for the whole representation theory of Artin algebras is well understood and there is much literature on the subject (see [1, 6, 12]). These are Artin algebras such that every indecomposable module is finitely generated and every module is a direct sum of indecomposable modules. Moreover there is a bijective correspondence between the class of representation finite Artin algebras and that of Artin algebras with global dimension of at most 2 and with dominant dimension of at least 2. This result was established by Auslander in [1]. Motivated by this correspondence, Auslander introduced the concept of representation dimension for Artin algebras to study the connection of arbitrary Artin algebras with representation finite Artin algebras. "It is hoped that this notion gives a reasonable way of measuring how far an Artin algebra is from being representation finite type" [1, p. 134].

Representation dimension is a Morita-invariant of Artin algebras and its definition involves homological dimensions and modules which are both generators and cogenerators. Unfortunately, little seems to be known about representation dimension. So there are a lot of essential questions on this invariant. Dealing with the computation and estimation of the representation dimension of an Artin algebra, there are a few cases known: Auslander proved that the representation dimension of an Artin algebra is two if and

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only if the algebra is representation finite, and that the global dimension being at most 1 implies that the representation dimension is at most 3. Later, Fossum et al. proved in [5] that for the 2×2 triangular matrix algebra over an algebra A the representation dimension is upper bounded by the representation dimension of A plus 2 (see Theorem 7.3 in [5]). The goal of this note is to enlarge the knowledge of representation dimensions. More precisely, in Section 3 we are going to study the representation dimension of the tensor product of two algebras and give a upper bound of the representation dimension for tensor products. From this we obtain that the representation dimension of an $n \times n$ triangular matrix algebra over an algebra A is still bounded by the representation dimension of A plus 2. In Section 4 we investigate the representation dimensions of algebras and their factor algebras by powers of the radicals. In Section 5 we consider the relationship of the representation dimension and the global dimension of an algebra. In the last section we discuss a special case of algebras which are one-point extensions. (Note that it is not yet known whether the representation dimension of an algebra is finite.)

Throughout this paper we work with Artin algebras which are finite dimensional k-algebras over a fixed field k with the identity 1. By a module we mean a finitely generated left module. The global dimension of an algebra A is denoted by $\mathrm{gl.dim}(A)$. By D we denote the duality $\mathrm{Hom}_k(-, k)$, and by A-mod the category of all A-modules. Given two homomorphisms $f: L \longrightarrow M$ and $g: M \longrightarrow N$, the composition of f and g is a homomorphism from L to N and is denoted in the paper by fg.

2. PRELIMINARIES AND DEFINITIONS

In this section we recall the definition of representation dimension from [1] and some relevant notion as well as some results which we need later.

Given a finite dimensional algebra A, we say that A has dominant dimension greater than or equal to n, denoted by $\mathrm{dom.dim}(A) \geq n$, if there is an exact sequence

$$0 \longrightarrow {}_A A \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

of A-modules such that X_i is projective and injective for $i = 1, \ldots, n$. We denote by $I_0(A)$ the module X_1.

For a representation finite Artin algebra, Auslander proved that the endomorphism algebra of the direct sum of all non-isomorphic indecomposable modules has global dimension of at most two and dominant dimension of at least two. More precisely, Auslander proved the following theorem, which motivated him to introduce the notion of representation dimension as a way of measuring how far a finite dimensional algebra is from being representation finite type.

THEOREM 2.1. *Suppose A is a finite dimensional algebra with* $\mathrm{gl.dim}(A) \leq 2$. *If P is a projective and injective A-module, then* $\mathrm{End}_A(P)$ *has representation finite type. Further, up to Morita equivalence, all finite dimensional algebras of representation finite type are obtained in this way.*

The representation dimension is defined as follows.

DEFINITION 2.2. Let A be a finite dimensional algebra over a field k. Consider the finite dimensional algebra Λ of dominant dimension of at least two such that $\mathrm{End}_\Lambda(I_0(\Lambda))$ is Morita equivalent to A. Then the representation dimension of A is defined to be the minimum of the global dimension of all possible Λ, and denoted by $\mathrm{rep.dim}(A)$.

In fact, Auslander also proved in [1] that the above definition is equivalent to the following definition:

$$\mathrm{rep.dim}(A) = \inf\{\mathrm{gl.dim}(\mathrm{End}_A(M)) \mid M \text{ is a generator-cogenerator}\}.$$

Note that an A-module M is called a generator-cogenerator if every indecomposable projective module and also every indecomposable injective module is isomorphic to a summand of M.

The following lemma collects some known results on the representation dimension which we shall need in the sequel.

LEMMA 2.3. *Let A be a non-semisimple k-algebra. Then*

(1) $\mathrm{rep.dim}(A) = 2$ *if and only if A is representation finite.*

(2) *If A is a selfinjective algebra, then* $\mathrm{rep.dim}(A) \leq \mathrm{LL}(A)$, *where* $\mathrm{LL}(A)$ *stands for the Loewy length of A.*

(3) *If* $\mathrm{gl.dim}(A) \leq 1$, *then* $\mathrm{rep.dim}(A) \leq 3$.

(4) *Let $T_2(A)$ denote the 2×2 triangular matrix algebra over A. Then* $\mathrm{rep.dim}(T(A)) \leq \mathrm{rep.dim}(A) + 2$.

Statements (1)–(3) are proved in [1], and statement (4) was shown in [5, p. 115].

3. REPRESENTATION DIMENSION OF TENSOR PRODUCTS

Given two finite dimensional k-algebras A and B, we may form the tensor product $A \otimes_k B$ of A and B over k, which is still a finite dimensional k-algebra with the multiplication

$$(a \otimes b)(a_1 \otimes b_1) = aa_1 \otimes bb_1$$

for all $a, a_1 \in A$ and $b, b_1 \in B$. Here we write $\otimes$ for $\otimes_k$.

For any A-module M and B-module N, we have an $(A \otimes B)$-module $M \otimes N$ given by $(a \otimes b)(m \otimes n) = am \otimes bn$ for all $a \in A, b \in B$, $m \in M$, and $n \in N$. For $(A \otimes B)$-modules obtained in this way, we have the following properties.

LEMMA 3.1. *Suppose that M is an A-module and N is a B-module. Then*

(1) *If M is a projective A-module and N is a projective B-module, then $M \otimes N$ is a projective $(A \otimes B)$-module.*

(2) *If M is an injective A-module and N is an injective B-module, then $M \otimes N$ is an injective $(A \otimes B)$-module.*

Proof. (1) follows immediately from proposition 2.3 of Chapter IX in [3].

(2) Since $D(A \otimes B) = \mathrm{Hom}_k(A \otimes B, k) \cong (DA) \otimes (DB)$, we see that the tensor product of the injective A-module DA and the injective B-module DB is an injective $(A \otimes B)$-module. This implies statement (2).

One may consider the functor ${}_AM \otimes -\colon B\text{-mod} \longrightarrow (A \otimes B)\text{-mod}$. The following lemma shows that this functor has some nice properties.

LEMMA 3.2. *For A-modules X, Y and B-modules M, N, we have*

$$\mathrm{Hom}_A(X, Y) \otimes \mathrm{Hom}_B(M, N) \cong \mathrm{Hom}_{A\otimes B}(X \otimes M, Y \otimes N)$$

(*as an isomorphism of vector spaces*).

For the proof of this lemma, see [3, Chap. XI, Theorem 3.1, pp. 209–210]. From this lemma we get the following corollary.

COROLLARY 3.3. $\mathrm{End}_{A\otimes B}(X \otimes M) = \mathrm{End}_A(X) \otimes \mathrm{End}_B(M)$.

The following result on the global dimension is well known.

LEMMA 3.4. *Suppose k is a perfect field. Let A, B be two finite dimensional k-algebras. Then*

$$\mathrm{gl.dim}(A \otimes B) = \mathrm{gl.dim}(A) + \mathrm{gl.dim}(B).$$

Now let us prove the following result.

THEOREM 3.5. *Suppose k is a perfect field. Let A and B be two finite dimensional k-algebras. Then*

$$\mathrm{rep.dim}(A \otimes B) \leq \mathrm{rep.dim}(A) + \mathrm{rep.dim}(B).$$

Proof. Let M be an A-module which contains each indecomposable projective module and each indecomposable injective module as a direct summand such that $\text{rep.dim}(A) = \text{gl.dim}(\text{End}_A(M))$. Let N be a such B-module with the same property that $\text{rep.dim}(B) = \text{gl.dim}(\text{End}_B(N))$. Then there are natural numbers m and n such that A is a direct summand of M^m and such that B is a direct summand of N^n. Similarly, there are natural numbers m' and n' such that DA is a direct summand of $M^{m'}$ and such that DB is a direct summand of $N^{n'}$. Clearly, $M^{m+m'} \otimes N^{n+n'}$ contains $A \otimes B$ as a direct summand and also contains $D(A \otimes B) \cong (DA) \otimes (DB)$ as a direct summand. Let $X = M^{m+m'}$ and $Y = N^{n+n'}$. Then $X \otimes Y$ is a generator-cogenerator for the $(A \otimes B)$-mod. Note that $\text{gl.dim}(\text{End}_A(M^{m+m'})) = \text{gl.dim}(\text{End}_A(M))$ and $\text{gl.dim}(\text{End}_A(N^{n+n'})) = \text{gl.dim}(\text{End}_A(N))$. Thus

$$\begin{aligned}
\text{rep.dim}(A \otimes B) &\leq \text{gl.dim}(\text{End}_{A\otimes B}(X \otimes Y)) \\
&= \text{gl.dim}(\text{End}_A(X) \otimes \text{End}_B(Y)) \\
&= \text{gl.dim}(\text{End}_A(X)) + \text{gl.dim}(\text{End}_B(Y)) \\
&= \text{gl.dim}(\text{End}_A(M)) + \text{gl.dim}(\text{End}_B(N)) \\
&= \text{rep.dim}(A) + \text{rep.dim}(B).
\end{aligned}$$

As a consequence of the above result, we have the following corollary. Of course, statement (1) of the corollary is a generalization of Theorem 7.3 in [5].

COROLLARY 3.6. *Suppose k is a perfect field. Let A be a finite dimensional k-algebra.*

(1) *Let $T_n(A)$ be the $n \times n$ triangular matrix algebra with entries in A:*

$$T_n(A) = \begin{pmatrix} A & A & \dots & A \\ 0 & A & \dots & A \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}.$$

Then $\text{rep.dim}\, T_n(A) \leq \text{rep.dim}(A) + 2$.

(2) *If B is a selfinjective k-algebra with* $\text{LL}(B) = m$, *then* $\text{rep.dim}(A \otimes B) \leq \text{rep.dim}(A) + m$.

Proof. (2) follows from 2.3(2) and 3.5. As to (1), just note that $T_n(A) = T_n(k) \otimes A$ and that $\text{rep.dim}T_n(k) = 2$ for $n \geq 2$ and $\text{rep.dim}\, T_1(k) = 0$.

Clearly, we have that $\text{rep.dim}(A \oplus B) = \max\{\text{rep.dim}(A), \text{rep.dim}(B)\}$. So if B is a direct sum of matrix algebras over an arbitrary field k, then, by definition, rep.dim$(B) = 0$. In this case, we have rep.dim$(A \otimes B) =$ rep.dim$(A) +$ rep.dim(B). If B is not semisimple, the upper bound may not be attained. This can be seen from the following examples which show also that in general one cannot hope to have the equality in 3.5:

(1) If $A = B = T_2(k)$, then rep.dim$(A \otimes B) = 2 =$ rep.dim(A) by 2.3.

(2) If $A = k[T]/(T^2)$, then since the Loewy length of $A \otimes A$ is 3, we have rep.dim$(A \otimes A) = 3$, but rep.dim$(A) = 2$ again by 2.3.

PROPOSITION 3.7. *Let A and B be two selfinjective algebras over an arbitrary field k such that $A/\mathrm{rad}(A)$ or $B/\mathrm{rad}(B)$ is separable. Then*

$$\mathrm{rep.dim}(A \otimes B) \leq \mathrm{LL}(A) + \mathrm{LL}(B) - 1,$$

where $\mathrm{LL}(A)$ *denotes the Loewy length of A.*

Proof. Suppose $A/\mathrm{rad}(A)$ is separable, that is, for any extension field L of k, the L-algebra $L \otimes (A/\mathrm{rad}(A))$ is semisimple. Then the radical of $A \otimes B$ is $\mathrm{rad}(A) \otimes B + A \otimes \mathrm{rad}(B)$ (see [4, p. 116]). One can verify that the Loewy length of $A \otimes B$ is $\mathrm{LL}(A) + \mathrm{LL}(B) - 1$. Thus the proposition follows from 2.3 since the tensor product of two selfinjective algebras is again selfinjective by a result in [10].

One special case of tensor products is that one of the algebras A and B is a field extension of k. The following lemma is true by [8].

LEMMA 3.8. *Let L be a finite dimensional separable extension of a field k. We denote by* $\mathrm{rep.dim}_k(A)$ *the representation dimension of the k-algebra A. Then* $\mathrm{rep.dim}_k(A) = 2$ *if and only if* $\mathrm{rep.dim}_L(L \otimes A) = 2$.

Usually, the representation dimension of A changes when we change the base field. This can be seen by the following example.

Let k be a non-perfect field of characteristic p. Then there is an extension field of k which is not separable. So there is a purely inseparable extension L of k of exponent one (i.e., $L = k(\alpha)$ with $\alpha \notin k$ and $\alpha^p \in k$). Then, by [7, p. 491], the tensor product $L \otimes L$ has a nilpotent element, so the L-algebra $L \otimes L$ is not semisimple. Note that L is a finite dimensional semisimple k-algebra. Thus $\mathrm{rep.dim}_L(L \otimes L) \geq 2$ and $\mathrm{rep.dim}_k(L) = 0$. For other examples one may see [8].

Remark. It is unknown whether over a perfect field k the tensor product of two algebras A and B has a representation dimension of at least $\max\{\mathrm{rep.dim}(A), \mathrm{rep.dim}(B)\}$.

4. REPRESENTATION DIMENSION AND THE RADICAL OF AN ALGEBRA

Auslander proved in [1] that if $\text{rep.dim}(A/\text{rad}^{n-1}(A)) \leq 2$, then $\text{rep.dim}(A) \leq 3$. And then he asked a question whether $\text{rep.dim}(A) \leq \text{rep.dim}(A/\text{rad}^{n-1}(A)) + 1$ holds, where n is the nilpotent index of the radical $\text{rad}(A)$. Motivated by this question we shall provide a related bound in this section. The main result in this section is the following theorem which can be used to estimate the upper bound for representation dimensions of certain "complicated" algebras.

THEOREM 4.1. *Let A be a finite dimensional k-algebra with Jacobson radical N of nilpotence index n, that is, $N^n = 0 \neq N^{n-1}$. Suppose that for each indecomposable injective A-module I with $N^{n-1}I \neq 0$ the indecomposable direct summands of $I/N^{n-1}I$ are either injective A/N^{n-1}-modules or projective A/N^{n-1}-modules, then* $\text{rep.dim}(A) \leq \text{rep.dim}(A/N^{n-1}) + 3$.

Before we give the proof of this theorem, let us first recall some definitions and facts needed.

Let $\mathscr{X}$ be a full subcategory of A-mod which is closed under direct sums and isomorphisms, and $X, M \in A$-mod. A homomorphism $f: X \longrightarrow M$ is called minimal if an endomorphism $g: X \longrightarrow X$ is an automorphism whenever $f = gf$. The morphism f is called a right $\mathscr{X}$-approximation of M if $X \in \mathscr{X}$ and for each homomorphism $g: Y \longrightarrow M$ with $Y \in \mathscr{X}$ there is a homomorphism $h: Y \longrightarrow X$ such that $g = hf$. If in addition f is minimal then we call f a minimal right $\mathscr{X}$-approximation of M.

The following lemma is true (see, for example, [2]).

LEMMA 4.2. *Let $\mathscr{X}$ be an additive and full subcategory of A-mod closed under direct sums and isomorphisms. If there are only finitely many indecomposable A-modules in $\mathscr{X}$ (up to isomorphisms), then for every A-module M in A-mod there is a minimal right $\mathscr{X}$-approximation of M. If ${}_AA \in \mathscr{X}$ then each minimal right $\mathscr{X}$-approximation is surjective.*

Let M be an A-module. We denote by $\text{add}(M)$ the full subcategory of A-mod whose objects are isomorphic to direct summands of direct sums of finite copies of M.

Let $\mathscr{C}$ be a skeletally small category. We denote by $\mathscr{C}^{\text{op}}$ the opposite category of $\mathscr{C}$ and by $\text{Funct}(\mathscr{C}^{\text{op}}, \text{Ab})$ the abelian category of all functors from $\mathscr{C}^{\text{op}}$ to the category Ab of abelian groups. Let $\widehat{\mathscr{C}}$ be the full subcategory of $\text{Funct}(\mathscr{C}^{\text{op}}, \text{Ab})$ consisting of all coherent functors G, that is, those functors G for which there is a morphism $C_1 \longrightarrow C_2$ in $\mathscr{C}$ such that the sequence

$$(\,, C_1) \longrightarrow (\,, C_2) \longrightarrow G \longrightarrow 0$$

is exact. Here and in the sequel we denote by $(\,,C)$ the Hom functor $\mathrm{Hom}_{\mathscr{C}}(\,,C)\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$ for $C \in \mathscr{C}$.

The following lemma is proved in [1].

LEMMA 4.3. *Let M be in A-mod. Then the category $\widehat{\mathrm{add}(M)}$ and $\mathrm{End}(M)$-mod are equivalent. In particular,*

$$\mathrm{gl.dim}(\mathrm{End}_A(M)) = \mathrm{gl.dim}(\widehat{\mathrm{add}(M)}).$$

Note that for each module $X \in A$-mod the functor $(\,,X)$ belongs to $\widehat{\mathrm{add}(M)}$ by 4.2.

Now we prove Theorem 4.1.

Let $B = A/N^{n-1}$ and V_0 be a B-module such that $\mathrm{gl.dim}(\mathrm{End}_B(V_0)) = \mathrm{rep.dim}(B)$. We may assume that $\mathrm{rep.dim}(B) = m < \infty$. (Otherwise there is nothing to prove.) Let $P_1, \ldots, P_t$ be a complete set of non-isomorphic indecomposable projective A-modules and $I_1, \ldots, I_s$ a complete set of non-isomorphic indecomposable injective A-modules. Put $V := V_0 \oplus \bigoplus_{i=1}^t P_i \oplus \bigoplus_{j=1}^s I_j$ and define $\mathscr{X} = \mathrm{add}(V_0)$. We want to show that $\mathrm{gl.dim}(\mathrm{End}_A(V)) \leq \mathrm{rep.dim}(B) + 3$.

Suppose M is an indecomposable A-module. We show that there is an exact sequence $0 \longrightarrow X_{m+3} \longrightarrow \cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$ with all $X_i \in \mathrm{add}(V)$ such that the induced sequence

$$0 \longrightarrow (X, X_{m+3}) \longrightarrow \cdots \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact for all $X \in \mathrm{add}(V)$.

(1) If $M \in \mathrm{add}(V)$ then we simply define $X_2 = M$ and $X_3 = \cdots = X_{m+3} = 0$ and the morphism $X_2 \longrightarrow M$ the identity.

(2) Suppose that M is not in $\mathrm{add}(V)$. There are two cases to be considered.

The first case. M is a B-module. Then we consider the functor $(\,,M)\colon \mathscr{X}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$. Since $\mathrm{gl.dim}(\mathrm{End}_B(V_0)) = \mathrm{gl.dim}(\widehat{\mathrm{add}(V_0)}) = m$ by 4.3, we have a minimal projective resolution for $(\,,M)$,

$$0 \longrightarrow (\,,X_{m+2}) \longrightarrow \cdots \longrightarrow (\,,X_3) \longrightarrow (\,,X_2) \longrightarrow (\,,M) \longrightarrow 0$$

with all $X_i \in \mathscr{X}$. Since $B \in \mathscr{X}$, we have an exact sequence of B-modules

$$0 \longrightarrow X_{m+2} \longrightarrow \cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0.$$

We will show that this is a desired sequence . If X is a projective A-module, then the sequence

$$(*)\quad 0 \longrightarrow (X, X_{m+2}) \longrightarrow \cdots \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact. Now assume that X is an indecomposable injective A-module. If X is also a B-module, then X is a direct summand of V_0, thus in $\mathscr{X}$. So there is nothing to prove. We assume that X is not a B-module. Then $0 \neq N^{n-1}X$ is the socle of X and $X/\mathrm{Soc}(X)$ is either an injective or a projective B-module by assumption. Since V_0 contains all indecomposable injective B-modules and $(X, M_1) = (X/\mathrm{Soc}(X), M_1)$ for any indecomposable B-module M_1, we see that $(*)$ is exact if X is an indecomposable injective A-module with $N^{n-1}X \neq 0$. Hence for all $X \in \mathrm{add}(V)$ the sequence $(*)$ is exact.

The second case. M is not a B-module. We take $M' = \{m \in M \mid N^{n-1}m = 0\}$. Then M' is a B-module. Suppose that $l: X_0 \longrightarrow M'$ is a right minimal $\mathscr{X}$-approximation of M', $g: P \longrightarrow M/M'$ is a projective cover of A-module, and $h: P \longrightarrow M$ is a lifting such that $g = h\pi$, where π is the canonical homomorphism $M \longrightarrow M/M'$. Define $f: X_0 \oplus P \longrightarrow M$ by $(x, p) \mapsto l(x) + h(p)$. If $(x, p) \in \mathrm{Ker}(f)$ with $x \in X_0$ and $p \in P$ then $p \in \mathrm{Ker}(g)$. Since P is a projective cover, we have $\mathrm{Ker}(g) \subset NP$. Hence $\mathrm{Ker}(f)$ is a B-module because $N^{n-1}(x, p) \subset (N^{n-1}x, N^{n-1}p) = 0$. Now we show that for any $X \in \mathrm{add}(V)$ the induced map $(X, X_0 \oplus P) \longrightarrow (X, M)$ is surjective. If $X \in \mathscr{X}$ then the image of any homomorphism from X to M is a B-module, and thus lies in M'. This implies that the induced map is surjective. If X is projective then there is nothing to show. So we assume that X is an indecomposable injective A-module not in $\mathscr{X}$. Since $\mathscr{X}$ contains all indecomposable injective B-modules, we know that X is not a B-module and hence $N^{n-1}X \neq 0$. Let g' be a homomorphism from X to M. Then g' is not injective. Otherwise we would have $M \cong X \in \mathrm{add}(V)$. This yields that $N^{n-1}X = \mathrm{Soc}(X) \subset \mathrm{Ker}(g')$. Hence $(X, M) = (X/\mathrm{Soc}(X), M) = (X/\mathrm{Soc}(X), M')$. Since every homomorphism from $X/\mathrm{Soc}(X)$ to M' factors through $l: X_0 \longrightarrow M'$, it follows that the map $(X, X_0 \oplus P) \longrightarrow (X, M)$ is surjective. Hence we have proved that for all $X \in \mathrm{add}(V)$ the induced map is surjective.

Now by the previous result we have an exact sequence

$$0 \longrightarrow X_{m+3} \longrightarrow X_{m+2} \longrightarrow \cdots \longrightarrow X_3 \longrightarrow \mathrm{Ker}(f) \longrightarrow 0$$

with all $X_i \in \mathscr{X} \subset \mathrm{add}(V)$ such that for all $X \in \mathrm{add}(V)$ the sequence

$$\begin{aligned} 0 \longrightarrow (X, X_{m+3}) \longrightarrow (X, X_{m+2}) \longrightarrow \cdots \longrightarrow (X, X_3) \\ \longrightarrow (X, \mathrm{Ker}(f)) \longrightarrow 0 \end{aligned}$$

is exact. Define $X_2 := X_0 \oplus P$, then we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow (X, X_{m+3}) \longrightarrow (X, X_{m+2}) \longrightarrow \cdots \longrightarrow (X, X_3) \\ \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0 \end{aligned}$$

for all $X \in \mathrm{add}(V)$, where all X_i are in $\mathrm{add}(V)$.

Now we establish that $\mathrm{gl.dim}(\widehat{\mathrm{add}(V)}) \le m+3$. Take a functor G in $\widehat{\mathrm{add}(V)}$. Then there is a morphism $f\colon X_1 \longrightarrow X_0$ in $\mathrm{add}(V)$ such that $(\,,X_1) \longrightarrow (\,,X_0) \longrightarrow G \longrightarrow 0$ is exact. Letting $M = \mathrm{Ker}(f)$, we know that there is an exact sequence

$$0 \longrightarrow X_{m+3} \longrightarrow X_{m+2} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

with all $X_i \in \mathrm{add}(V)$ such that the induced sequence

$$0 \longrightarrow (X, X_{m+3}) \longrightarrow (X, X_{m+2}) \longrightarrow \cdots \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact for all $X \in \mathrm{add}(V)$. Thus the sequence

$$0 \longrightarrow (\,, X_{m+3}) \longrightarrow \cdots \longrightarrow (\,, X_2) \longrightarrow (\,, X_1) \longrightarrow (\,, X_0) \longrightarrow G \longrightarrow 0$$

is exact in $\mathrm{add}(V)$. This shows that $\mathrm{proj.dim}(G) \le m+3$. By 4.3, we get $\mathrm{gl.dim}(\mathrm{End}_A(V)) \le m+3 = \mathrm{rep.dim}(B)+3$. Thus the proof is completed.

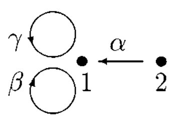
Remark. From the proof we may formulate the following fact: Let M be an A/N^{n-1}-module such that $\mathrm{rep.dim}(A/N^{n-1}) = \mathrm{gl.dim}(\mathrm{End}_{A/N^{n-1}}(M))$. If each injective indecomposable A-module I with $N^{n-1}I \neq 0$ has the factor module $I/N^{n-1}I$ in $\mathrm{add}(M)$, then $\mathrm{rep.dim}(A) \le \mathrm{rep.dim}(A/N^{n-1}) + 3$. In particular, if $\mathrm{rep.dim}(A/N^{n-1}) \le 2$, then $\mathrm{rep.dim}(A) \le 5$. (Compare this with Auslander's result.)

As a consequence of the theorem, we obtain the following result of Auslander [1].

COROLLARY 4.4. *Let A be a finite dimensional k-algebra. If* $\mathrm{rad}^2(A) = 0$, *then* $\mathrm{rep.dim}(A) \le 3$.

Finally, let us give an example of a finite dimensional algebra satisfying the condition in Theorem 4.1.

EXAMPLE. Consider the k-algebra A with the quiver



with relations $\alpha\beta = \alpha\gamma = \gamma\beta = \beta\gamma = 0$ and $\beta^2 = \gamma^2$. Thus we have the following decomposition of the regular module ${}_AA$:

$${}_AA = \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array}.$$

Then the injective module $D(A_A)$ appears as follows:

$$D(A_A) = \begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{array} \oplus \; 2 \; .$$

We can see that the condition in 4.1 is satisfied. Hence $\text{rep.dim}(A) \leq \text{rep.dim}(A/N^2) + 3$. It follows further from 2.3 that $\text{rep.dim}(A) \leq 6$.

5. REPRESENTATION DIMENSION AND GLOBAL DIMENSION

In this section we will discuss the relationship between the global dimension and the representation dimension of a finite dimensional algebra under certain conditions. Our result generalizes a result of Auslander which says that if $\text{gl.dim}(A) \leq 1$ then $\text{rep.dim}(A) \leq 3$.

Let M be an A-module. We denote by $\text{Fac}(M)$ the full subcategory of A-mod consisting of all modules which are generated by M. Recall that a full subcategory of A-mod closed under direct summands is said to be of finite type if it contains only finitely many indecomposable A-modules (up to isomorphisms).

THEOREM 5.1. *Let A be a finite dimensional algebra. If* $\text{Fac}(D(A))$ *is of finite type and if* $\text{Hom}_A(X, M) = 0$ *for all* $X \in \text{Fac}(D(A))$ *and all indecomposable modules* $M \notin \text{Fac}(D(A))$, *then* $\text{rep.dim}(A) \leq \text{gl.dim}(A) + 2$.

Proof. Let $P_1, \ldots, P_t$ be a complete set of all non-isomorphic indecomposable projective A-modules. Since $\text{Fac}(D(A))$ is of finite type, we may assume that $\{N_1, \ldots, N_s\}$ is a complete set of non-isomorphic indecomposable modules in $\text{Fac}(D(A))$. Define $V = P_1 \oplus \cdots \oplus P_t \oplus N_1 \oplus \cdots \oplus N_s$. Let M be an A-module. We decompose M into $M_1 \oplus M_2$ where $M_2 \in \text{add}(V)$, and M_1 has no indecomposable direct summands in $\text{add}(V)$. We take a projective cover $P_0(M_1) \longrightarrow M_1$ and define $X_2 = P_0(M_1) \oplus M_2$ as well as the canonical surjective homomorphism $f_2\colon X_2 \longrightarrow M$. We demonstrate that for any indecomposable $X \in \text{add}(V)$ the induced map $(X, X_2) \longrightarrow (X, M)$ is surjective. If X is projective there is nothing to prove. Now let X be some module N_i. Since M_1 does not belong to $\text{add}(V)$, we know that $\text{Hom}(N_i, M_1) = 0$ by the hypothesis. So each homomorphism from N_i to M is, in fact, a homomorphism from N_i to M_2; this implies that for $X = N_i$ the induced map is surjective. Hence for all $X \in \text{add}(V)$ the induced map is surjective. Note that the module $P_0(M_1)$ is a direct summand of the projective cover P_0 of M.

We can assume that $\text{gl.dim}(A) = m < \infty$. To prove the theorem, we show that for each module M we can find an exact sequence

$$0 \longrightarrow X_{m+2} \longrightarrow X_{m+1} \longrightarrow \cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

with all $X_i \in \operatorname{add}(V)$ such that for each $X \in \operatorname{add}(V)$ the induced sequence

$$0 \longrightarrow (X, X_{m+2}) \longrightarrow (X, X_{m+1}) \longrightarrow \cdots \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact.

Let M be an A-module, and let $0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a minimal projective resolution for M. We have defined the module X_2 and the homomorphism $f_2: X_2 \longrightarrow M$. Consider the kernel of f_2. This kernel is in fact the first syzygy of M_1. Replacing M by the kernel of f_2 and repeating the above procedure, we define a module X_3 and the corresponding homomorphism $f_3: X_3 \longrightarrow \operatorname{Ker}(f_2)$. All these datums have the desired properties. Note again that $P_0(\operatorname{Ker}(f_2)_1)$ is a direct summand of P_1. Since the global dimension of A is m, the above procedure must stop at most after m steps, so we have an exact sequence

$$0 \longrightarrow X_{m+2} \longrightarrow X_{m+1} \longrightarrow \cdots \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

with $X_i \in \operatorname{add}(V)$ and the desired property.

As in the proof of 4.1, we get that $\operatorname{rep.dim}(A) \le m + 2 = \operatorname{gl.dim}(A) + 2$. The proof is finished.

Let us mention that hereditary algebras and tame concealed algebras satisfy the conditions in Theorem 5.1.

COROLLARY 5.2. (1) *If A is a hereditary algebra, then* $\operatorname{rep.dim}(A) \le 3$.

(2) *If A is a tame concealed algebra, then* $3 \le \operatorname{rep.dim}(A) \le 4$.

For the definition of tame concealed algebras and also the definition of tubular algebras below, we refer to [11].

Finally, we mention another bound of the representation dimension related to the global dimension.

PROPOSITION 5.3. *Let A be a finite dimensional k-algebra. If* $\operatorname{Hom}_A(D(A), A) = 0$, *then* $\operatorname{rep.dim}(A) \le 1 + 2\operatorname{gl.dim}(A)$. *In particular, the representation dimension of a tubular algebra is at most* 5.

Proof. One just needs to take the module $A \oplus D(A)$ as generator-cogenerator and to use a result in [9, p. 246] to compute the global dimension of its endomorphism algebra.

6. REPRESENTATION DIMENSION OF ONE-POINT EXTENSIONS

In this section we shall consider the representation dimension of algebras which are one-point extensions. These are algebras of the form

$$A = \Lambda[M] = \begin{pmatrix} \Lambda & M \\ 0 & k \end{pmatrix},$$

where Λ is a finite dimensional k-algebra, and M is a Λ-module. The multiplication and the addition are defined as the usual ones of matrices. This algebra A is called the one-point extension of Λ by the Λ-module M (see [11, p. 90]).

While we cannot say much about the bounds of the representation dimension for arbitrary one-point extensions, we get some results when we impose certain conditions on M. Our result in this direction is

PROPOSITION 6.1. *Let Λ be a finite dimensional algebra over an arbitrary field k, and let M be a Λ-module. Suppose A is the one-point extension of Λ by M. If M is a simple injective Λ-module, then*

$$\text{rep.dim}(\Lambda) \leq \text{rep.dim}(A) \leq \text{rep.dim}(\Lambda) + 2.$$

Proof. Let X be a Λ-module such that rep.dim(Λ) = gl.dim(End$_\Lambda(X)$). Let $P(\omega)$ be the projective A-module with rad$(P(\omega)) = M$. Then $P(\omega)/M$ is an injective A-module; we denote this simple injective module by $I(\omega)$. Now set $Y = X \oplus P(\omega) \oplus I(\omega)$ and consider the endomorphism algebras of ${}_AY$. An easy computation shows that

$$\text{End}_A(Y) = \begin{pmatrix} \text{End}_\Lambda(X) & \text{Hom}_\Lambda(X, P(\omega)) & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

Let B be the algebra $T_2(k)$, and V be the End$_\Lambda(X)$-B bimodule $(\text{Hom}_\Lambda(X, P(\omega)), 0)$. Then we can rewrite End$_A(Y)$ as

$$\text{End}_A(Y) \cong \begin{pmatrix} \text{End}_\Lambda(X) & V \\ 0 & B \end{pmatrix}.$$

By Proposition 5.1 of [9, p. 246], we have

$$\begin{aligned} \text{gl.dim}\,\text{End}_A(Y) &\leq \max\{\text{gl.dim}\,\text{End}_\Lambda(X) + \text{proj.dim} V_B + 1, \text{gl.dim}(B)\} \\ &\leq \max\{\text{rep.dim}(\Lambda) + 1 + 1, 1\} = \text{rep.dim}(\Lambda) + 2. \end{aligned}$$

Note that the last inequality follows from the fact that gl.dim$(B) = 1$. Note that the condition on M implies that each indecomposable injective A-module is either an injective Λ-module or isomorphic to $P(\omega)$, or $I(\omega)$. Since Y contains all non-isomorphic indecomposable projective A-modules and all indecomposable injective A-modules as direct summands, we have $\text{rep.dim}(A) \leq \text{gl.dim}(\Lambda) + 2$ by definition.

To prove the first inequality of the theorem, we note that if X_0 is an indecomposable A-module with composition factors $I(\omega)$ then the top $X_0/\text{rad}(X_0)$ of X_0 lies in $\text{add}\, I(\omega)$ and the radical of X_0 has no composition factor isomorphic to $I(\omega)$. Hence any generator-cogenerator for the A-mod is of the form $Y = X \oplus P(\omega) \oplus I(\omega) \oplus X_0$, where X is a generator-cogenerator for Λ-mod and where X_0 is a direct sum of indecomposable A-modules such that X_0 has no direct summand isomorphic to $P(\omega)$ or $I(\omega)$ and such that each direct summand of X_0 has its top in $\text{add}\, I(\omega)$. Let us consider the endomorphism algebra of Y. Since $\text{Hom}_A(X_0, X) = 0 = \text{Hom}_A(I(\omega), X_0)$ and $\text{Hom}_A(X_0, P(\omega)) = 0$, the endomorphism algebra of Y is isomorphic to the triangular matrix algebra:

$$\begin{pmatrix} \text{End}_\Lambda(X) & \text{Hom}_\Lambda(X, P(\omega)) & 0 & \text{Hom}_\Lambda(X, X_0) \\ 0 & k & k & \text{Hom}_A(P(\omega), X_0) \\ 0 & 0 & k & 0 \\ 0 & 0 & \text{Hom}_A(X_0, I(\omega)) & \text{End}_A(X_0) \end{pmatrix}.$$

Again by [9, p. 246], it holds that $\text{gl.dim}\,\text{End}_A(Y) \geq \text{gl.dim}\,\text{End}_\Lambda(X)$. This implies that $\text{gl.dim}\,\text{End}_A(Y) \geq \text{rep.dim}(\Lambda)$ and that $\text{rep.dim}(A) \geq \text{rep.dim}(\Lambda)$ because Y is an arbitrary generator-cogenerator. The proof is finished.

Dually, we have the notion of one-point coextension $[M]\Lambda$ of Λ by a Λ-module M defined by

$$A := [M]\Lambda = \begin{pmatrix} k & DM \\ 0 & \Lambda \end{pmatrix} \cong (\Lambda^{\text{op}}[DM])^{\text{op}},$$

where Λ^{op} denotes the opposite algebra of Λ. In this case, we have the same statement as in the above proposition. This follows from the fact that $\text{rep.dim}(A) = \text{rep.dim}(A^{\text{op}})$ for all finite dimensional algebras A.

PROPOSITION 6.2. *If M is a simple projective Λ-module, then*

$$\text{rep.dim}(\Lambda) \leq \text{rep.dim}[M]\Lambda \leq \text{rep.dim}(\Lambda) + 2.$$

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REFERENCES

1. M. Auslander, "Representation Dimension of Artin Algebras," Queen Mary College Mathematics Notes, Queen Mary College, London, 1971.
2. M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* **85**, (1990), 111–152.
3. H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, NJ, 1956.
4. Yu A. Drozd and V. V. Kirichenko, "Finite Dimensional Algebras," Springer-Verlag, Berlin/New York, 1995.
5. R. M. Fossum, P. A. Griffith, and I. Reiten, "Trivial Extension of Abelian Categories," Lecture Notes in Mathematics 456, Springer-Verlag, Berlin/New York, 1975.
6. P. Gabriel, Auslander–Reiten sequences and representation-finite algebras. In: Lecture Notes in Mathematics **831**, (1980), 1–71.
7. N. Jacobson, "Basic Algebra II," Freeman, New York, 1980.
8. C. U. Jensen & H. Lenzing, Homological dimension and representation type of algebras under base field extension, *Manuscripta Math.* **39**, (1982), 1–13.
9. J. C. McConnell & J. C. Robson, "Noncommutative Noetherian Rings," Wiley, New York, 1987.
10. K. Morita, Duality for modules and its applications to the theory of rings with minimum condition, *Tokyo Kyoiku Daigaku, Ser. A* **6**, (1958), 83–142.
11. C. M. Ringel, "Tame Algebras and Integral Quadratic Forms," Lecture Notes in Mathematics 1099, Springer-Verlag, Berlin/New York, 1984.
12. H. Tachikawa, "Quasi-Frobenius Rings and Generalizations," Lecture Notes in Mathematics 351, Springer-Verlag, Berlin/New York, 1973.