JOURNAL OF ALGEBRA **197,** 231–245 (1997)
ARTICLE NO. JA977093

Stability for an Abelian Category

Alexei Rudakov*

Department of Mathematical Sciences, NTNU, 7034 Trondheim, Norway

Communicated by D. L. Buchsbaum

Received January 21, 1997

The main goal of the article is to give the definition of algebraic stability that would permit us to consider stability, not only for algebraic vector bundles or torsion-free coherent sheaves, but for the abelian category of coherent sheaves or for whatever abelian category. We present an axiomatic description of the algebraic stability on an abelian category and prove some general results. Then the stability for coherent sheaves on a projective variety is constructed which generalizes Gieseker stability. Stabilities for graded modules and for quiver representations are also discussed. The constructions could be used for other abelian categories as well. © 1997 Academic Press

Traditionally the stability is used as a technical tool while constructing moduli varieties. From its first appearance in the 1960s in D. Mumford's work in the geometric invariant theory the stable and semistable objects were defined in dozens of contexts (see, for example, [F; K; LT; M; R]). It seems practical to make a kind of general scheme for those definitions that would permit us to proceed with the stability considerations in an abelian category and this is done here in the article.

We would gain from this even in the classical case of algebraic coherent sheaves where stability was traditionally defined only for torsion-free coherent sheaves (see, for example, [OSS, Chap. 2]), and only recently the definition has been generalized to ‘‘coherent sheaves of pure dimension d’’ by Simpson and Maruyama [S; M]. In our approach we do not impose initially any condition on the sheaf in question. But ‘‘being of pure dimension’’ it becomes the property of stable sheaves a posteriori, one can derive this from the definition.

*E-mail: rudakov@math.ntnu.no.

0021-8693/97 $25.00

Section 1 is devoted to the definition and basic properties of a stability for an abelian category. In Section 2 the generalized Gieseker stability for algebraic coherent sheaves is constructed. Then in the next section we discuss another way to construct stability for a category and apply this to generalize the stability that was defined by King for quiver representations [K].

The author thanks the E. Schrödinger International Institute, where the first version of the text was written, and S. Kuleshov for helpful discussions. The research was also partly supported by INTAS grant.

1. GENERAL ALGEBRAIC STABILITY

Let $\mathcal{A}$ be an abelian category. To define stability in a category $\mathcal{A}$ we need first a preorder on the objects of $\mathcal{A}$. We will say that a preorder on $\mathcal{A}$ is given when we can compare nonzero objects of $\mathcal{A}$ that for $A, B \in \operatorname{Obj} \mathcal{A}$, $A \neq 0$, $B \neq 0$, either $A \prec B$, or $A \succ B$, or $A \asymp B$ is valid and it is possible to have $A \asymp B$ even when $A \neq B$.

DEFINITION 1.1. Let us say that the stability structure on $\mathcal{A}$ is given if there is a preorder on $\mathcal{A}$ such that for an exact sequence of nonzero objects $0 \to A \to B \to C \to 0$ we have

$$\begin{aligned} &\text{either} \quad A \prec B \Leftrightarrow A \prec C \Leftrightarrow B \prec C, \\ &\text{or} \quad A \succ B \Leftrightarrow A \succ C \Leftrightarrow B \succ C, \\ &\text{or} \quad A \asymp B \Leftrightarrow A \asymp C \Leftrightarrow B \asymp C. \end{aligned}$$

We call this property the seesaw property.

LEMMA 1.2. *Given an exact sequence of nonzero objects* $0 \to A \to B \to C \to 0$ *and a nonzero object* D *we have*

$$\begin{aligned} A \prec D \quad &\textit{and} \quad C \prec D \Rightarrow B \prec D, \\ A \succ D \quad &\textit{and} \quad C \succ D \Rightarrow B \succ D, \\ A \asymp D \quad &\textit{and} \quad C \asymp D \Rightarrow B \asymp D. \end{aligned}$$

Summarizing both properties one can say that the middle term B of the exact sequence is situated in between the side terms A and C in respect to the preorder.

To prove the lemma it is enough to notice that either $A \preccurlyeq B \preccurlyeq C$ or $A \succcurlyeq B \succcurlyeq C$ by the seesaw property and the result follows by the transitivity of the preorder.

The result of Lemma 1.2 can be generalized as follows.

LEMMA 1.3. *Given nonzero objects B and D suppose that B has a filtration*

$$B = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

with the factors $G^i = F^i/F^{i+1}$. Then

$$\begin{aligned} G^i \prec D \quad & \text{for } i = 1, \ldots, m \quad \Rightarrow \quad B \prec D, \\ G^i \succ D \quad & \text{for } i = 1, \ldots, m \quad \Rightarrow \quad B \succ D, \\ G^i \asymp D \quad & \text{for } i = 1, \ldots, m \quad \Rightarrow \quad B \asymp D. \end{aligned}$$

We shall call this property the center of mass property.

LEMMA 1.4. *Suppose that B has a filtration*

$$B = F^0 \supset F^1 \supset \cdots \supset F^m \supset F^{m+1} = 0$$

with the nonzero factors $G^i = F^i/F^{i+1}$ and $G^0 \prec G^1 \prec \cdots \prec G^m$. Denote $G_p^k = F^k/F^{(k+p)}$ for $k \geq 0$, $p \geq 1$, $k + p \leq m + 1$. Then

$$G_p^k \prec G_q^n \Leftrightarrow (k, p) <_{\text{lex}} (n, q),$$

where "lex" stands for the lexicografic order.

Also we proceed by induction and leave the details to the reader.

DEFINITION 1.5. Let us call an object B stable when B is nonzero and for a nontrivial subobject $A \subset B$ we have $A \prec B$.

DEFINITION 1.6. Let us call an object B semistable when B is nonzero and for a nontrivial subobject $A \subset B$ we have $A \preccurlyeq B$.

Because of the seesaw property one can use factorobjects to define stable and semistable objects as well:

B is stable if and only if $B \prec C$ for a nontrivial factorobject C,

B is semi-stable means $B \preccurlyeq C$ for a nontrivial factorobject C.

In a sense stable objects are similar to irreducible ones and we have a general Schur lemma type result.

THEOREM 1. *Let A, B be semi-stable objects from A such that $A \succcurlyeq B$ and suppose there is a nonzero morphism φ: $A \to B$. Then*:

(a) $A \asymp B$,

(b) *if B is stable then φ is an epimorphism,*

(c) *if A is stable then φ is a monomorphism,*

(d) *if both A, B are stable then φ is an isomorphism.*

Remark. Suppose $\mathrm{Hom}(A, B)$ are finite dimensional vector spaces over a field $\Bbbk$ as it is for coherent sheaves on a projective variety over $\Bbbk$ and let $\Bbbk$ be algebraically closed. Then it follows from the theorem that for a stable object A we always have $\mathrm{Hom}(A, A) = \Bbbk$.

Proof of Theorem 1. Let us consider the usual ker-im and im-coker exact sequences for φ

$$0 \to K \to A \to I \to 0, \qquad 0 \to I \to B \to C \to 0.$$

As $\varphi \neq 0$ so $I \neq 0$. By the definition of semi-stability

$$I \preccurlyeq B, \qquad A \preccurlyeq I, \qquad \text{so } A \preccurlyeq B.$$

But $A \succcurlyeq B$, so $A \asymp I \asymp B$, thus (a) is proved.

For (b) we need to mention that $I \neq B$ implies $I \prec B$ (because B is stable) in contradiction with $I \asymp B$ that we have got above. We proceed similarly with (c) and (d). ∎

As usual one can expect also a kind of Harder−Narasimhan filtration to exist. For this we need to assume additional properties.

Let us use in the following the convenient shorthand notations like $A \subset ;\ \preccurlyeq B$, instead of writing $A \subset B$ and $A \preccurlyeq B$ (with obvious variations). Here $A \subset B$ does not exclude $A = B$.

As usual we call B noetherian if an ascending chain in B stabilizes and say $\mathcal{A}$ is noetherian when any object of $\mathcal{A}$ is noetherian.

DEFINITION 1.7. Let us call B quasi-noetherian (or q-noetherian) if a chain

$$A_1 \subset ;\ \preccurlyeq A_2 \subset ;\ \preccurlyeq \cdots$$

in B has to stabilize. We call $\mathcal{A}$ q-noetherian if any object in $\mathcal{A}$ is q-noetherian.

Of course the condition of being q-noetherian is weaker than being noetherian.

DEFINITION 1.8. Let us call B weakly artinian (or w-artinian) if a chain

$$A_1 \supset ;\ \preccurlyeq A_2 \supset ;\ \preccurlyeq \cdots$$

in B has to stabilize. The same way we call $\mathcal{A}$ w-artinian if any object in $\mathcal{A}$ is w-artinian.

Remark. B being w-artinian implies that a chain $A_1 \supset ;\ \prec A_2 \supset ;\ \prec \cdots$ in B has to be finite.

PROPOSITION 1.9. *Let B be q-noetherian and w-artinian then it exist a subobject $B^{\#}$ in B such that*:

(a) *if* $0 \neq A \subset B$ *is a subobject in* B *then* $A \preccurlyeq B^{\#}$,

(b) *if* $0 \neq A \subset B$ *and* $A \asymp B^{\#}$ *then* $A \subset B^{\#}$,

If in an object B there exists a subobject $B^{\#}$ with properties (a), (b) *then such $B^{\#}$ is uniquely defined.*

The object $B^{\#}$ itself is necessary semistable and clearly B is semistable if and only if $B = B^{\#}$.

Let B be under conditions of Proposition 1.9 further on.

LEMMA 1.10. *Let $0 \neq A \subset B$. Then either A is semistable or there is $0 \neq A' \subset B$ such that A' is semistable and $A' \succ A$.*

Proof of the lemma. Let $A_1 = A$. If A_1 is not semistable then there is A_2 such that

$$A_1 \supset ; \prec A_2 \neq 0.$$

The same is valid for A_2 and so on. We have to come to a semistable subobject after a finite number of steps because the infinite chain

$$A_1 \supset ; \prec A_2 \supset ; \prec \cdots$$

does not exist in the w-artinian B.

LEMMA 1.11. *Let $C \neq 0$ be a subobject in B. If there is a semistable subobject A in B satisfying $A \succ C$ then either $A \subset C$ or it exists $C' \subset B$ such that $C' \supset ; \succ C$.*

Proof of the lemma. We have two standard exact sequences

$$\begin{array}{lcccl} 0 \to & A \cap C & \to & A & \to U \to 0, \\ 0 \to & C & \to & A + C & \to U \to 0. \end{array}$$

If A is not in C then $A + C \neq C$ thus $U \neq 0$.

Now either $A \cap C = 0$ or $A \cap C \neq 0$. In the former case $A = U$, in the latter $A \cap C \preccurlyeq A$ because A is semistable, and $A \preccurlyeq U$ by the seesaw property applied to the first sequence. Hencefore $A \preccurlyeq U$ in both cases.

We know $C \prec A$ so $C \prec U$. Thus the second sequence implies that $C \prec (A + C)$ by the seesaw property.

We conclude that $C' = A + C$ satisfies the lemma.

Proof of Proposition 1.9. The uniqueness of $B^{\#}$ is clear, we are to prove the existence.

Let us call for the moment a subobject $C \neq 0$ in B greedy if for a semistable A in B the property $A \succ C$ implies $A \subset C$.

The B itself is greedy and for any $E \neq 0$ in B we can construct a greedy $C \succcurlyeq E$ as follows.

If the subobject $C_0 = E$ does not satisfy the condition then it exists a semistable A, $A \succ C_0$ and A is not a subobject of C_0. Then by Lemma 1.11 there is $C_1 \supset ; \succ C_0$. The same way if C_1 does not satisfy the condition we get to have $C_2 \supset ; \succ C_1$, and so on. But an infinite chain of the type

$$C_0 \subset ; \prec C_1 \subset ; \prec C_2 \cdots$$

is impossible as B is q-noetherian.

We would like first to prove the existence of B^* that satisfies the property (a).

If B does not satisfy (a) then we construct a greedy object $B_1 \subset ; \succ B$. Because of Lemma 1.10 and the fact that B_1 is greedy we can substitute B_1 for B in proving (a). Now the same way if B_1 does not satisfy (a) then we construct a greedy object $B_2 \subset ; \succ B_1$ and so on. Thus we are making the chain

$$B_1 \supset ; \prec B_2 \supset ; \prec B_3 \cdots$$

that must be finite because B is w-artinian.

Now we are to prove the existence of $B^{\#}$ among the subobjects that already have the property (a).

If (a) is valid for B_0 but (b) is wrong then it exists A, $A \asymp B_0$, A is not a subobject in B_0 and we can suppose that A is semistable by Lemma 1.10. Let $B_1 = B_0 + A$. By the reasoning similar to this of the proof of Lemma 1.11 it is easy to show that $B_1 \succcurlyeq B_0$, clearly B_1 is strictly larger than B_0, and, of course, (a) is valid for B_1 as well.

We would repeat this getting

$$B_0 \subset ; \preccurlyeq B_1 \subset ; \preccurlyeq B_2 \cdots$$

with the strict inclusion on every step until we come to a subobject satisfying both (a) and (b) because the infinite chain of this type is impossible as B is q-noetherian. ∎

Definition 1.12. Let us call B weakly-noetherian (or w-noetherian) if B is q-noetherian and a chain

$$A_1 \subset ; \succcurlyeq A_2 \subset ; \succcurlyeq \cdots$$

in B has to stabilize also. We call A w-noetherian if any object of A is w-noetherian.

Of course "noetherian" implies "w-noetherian" and "q-noetherian."

THEOREM 2. *Suppose A is w-artnian and w-noetherian then for an object B of A there exists a filtration*

$$B = F_H^0 B \supset F_H^1 B \supset \cdots \supset F_H^m B \supset F_H^{m+1} B = 0$$

such that:

(i) *factors $G_H^i B = F_H^i B / F_H^{i+1} B$ are semistable,*

(ii) $G_H^0 B \prec G_H^1 B \prec \cdots \prec G_H^m B$.

If the filtration with the properties (i), (ii) *exists in an object B then it is unique*.

We need to prove some propositions first.

PROPOSITION 1.13. *Let B have a filtration with the properties* (i), (ii) *from Theorem* 2. *Then $B^{\#} = F_H^m B$.*

Proof of the proposition. We can proceed by induction on m. For $m = 0$ the statement is obvious. So let us consider the general case.

Let A be a subobject in B. Let us write F^i instead of $F_H^i B$ and the same for G^i. By induction $F^{m-1}/F^m = (B/F^m)^{\#}$, thus

$$A/(F^m \cap A) \preccurlyeq F^{m-1}/F^m = G^{m-1}.$$

But $G^{m-1} \prec G^m$ so $A/(F^m \cap A) \prec F^m$ as $G^m = F^m$.

Notice that $(F^m \cap A) \preccurlyeq F^m$ because F^m is semistable. Then by the center of mass property we have

$$A \preccurlyeq F^m,$$

so F^m satisfies the condition (a) from Proposition 1.9.

To prove that F^m satisfies (b) consider $A \asymp F^m$. Now we have $(F^m \cap A) \preccurlyeq F^m \asymp A$. By the seesaw property this implies

$$A/(F^m \cap A) \succcurlyeq A,$$

provided that $A/(F^m \cap A) \neq 0$. But $A \asymp F^m = G^m \succ G^{m-1}$, hence

$$A/(F^m \cap A) \succ G^{m-1},$$

which is impossible by induction. Whence $A/(F^m \cap A) = 0$ and $F^m \cap A = A$. Thus we conclude that $A \subset F^m$ so F^m satisfies (b), and the uniqueness statement from Proposition 1.9 gives us exactly what is needed. ∎

Proof of Theorem 2. To prove the uniqueness let us notice first that the last term of a filtration is uniquely defined by Propositions 1.13 and 1.9. From this it is easy to get the result by induction.

Suppose that A satisfies the conditions of Theorem 2. To construct the filtration let us define

$$F^0 = 0, \qquad F^{-1} = B^{\#}, \qquad F^{-(i+1)} = \text{preimage}\left(B/F^{-i}\right)^{\#}.$$

Clearly a factor $G^{-(i+1)} = (B/F^{-i})^{\#}$ is semistable and $G^{-(i+2)} \prec G^{-(i+1)}$ by the seesaw property applied to the sequence

$$0 \to G^{-(i+1)} \to F^{-(i+2)}/F^{-i} \to G^{-(i+2)} \to 0.$$

From Proposition 1.4 one concludes that

$$F^{-1} \subset ; \succ F^{-2} \subset ; \succ \cdots .$$

Since B is q-noetherian so $F^{-(m+1)} = B$ for some m and we have only to shift the indices to get the filtration as it is needed for the theorem. ∎

Remark. The author can prove a kind of a dual statement to Proposition 1.9 but with a bit stronger condition on B. Let B be noetherian and w-artinian then it exist a factorobject $B_{\#}$ for B such that:

(a′) if $B \to A \neq 0$ is a factor object for B then $A \succcurlyeq B_{\#}$,

(b′) if $B \to A \neq 0$ is a factor object and $A \asymp B_{\#}$ then the morphism $B \to A$ factors through $B \to B_{\#} \to A$, so A is a factor object for $B_{\#}$.

The factor object $B_{\#}$ is uniquely defined by these properties. In this case obviously $B_{\#}$ coincides with the first factor $G_H^0 B$ of the Harder–Narasimhan filtration in B.

One can also construct a Jordan–Hölder filtration in a semistable object.

THEOREM 3. *Suppose* A *is w-artinian and q-noetherian and* B *is a semistable object of* A. *Then* B *has a filtration*

$$B = F_J^0 B \supset F_J^1 B \supset \cdots \supset F_J^m B \supset F_J^{m+1} B = 0$$

such that:

(i) *factors* $G_J^i B = F_J^i B / F_J^{i+1} B$ *are stable*,

(ii) $G_J^0 B \asymp G_J^1 B \asymp \cdots \asymp G_J^m B$,

and the set $\{G_J^i B\}$ *of factors is uniquely defined by the properties* (i), (ii).

Proof of the theorem. Clearly the subobjects X in B such that $X \asymp B$ satisfy the ascending and descending chain conditions. So the result becomes standard. ∎

2. POLYNOMIAL STABILITY

It is well known that the category of algebraic coherent sheaves on a projective variety is noetherian. The same is the category of finitely generated graded R-modules where the algebra R is commutative and finitely generated over a field $\Bbbk$. We would like to construct a stability structure for these categories.

In both cases an object of a category has "a characteristic function." For a sheaf A on a variety X it is:

$$P_{[A]}(n) = \dim_{\Bbbk} H^0(X, A(n)).$$

For a graded module $A = \oplus_{q \in \mathbb{Z}} A_q$ let it be the Hilbert−Samuel function:

$$P_{[A]}(n) = \dim_{\Bbbk} \oplus_{q > -\infty}^{q \leq n} A_q.$$

This justifies the following definition.

DEFINITION 2.1. Let us say that a characteristic function is defined for a category $\mathcal{A}$ if it is defined for any object A a function $P_{[A]}: \mathbb{Z} \to \mathbb{Z}$ is defined with the properties:

(i) given an exact sequence $0 \to A \to B \to C \to 0$ we have

$$P_{[B]}(n) = P_{[A]}(n) + P_{[C]}(n) \quad \text{for } n \gg 0;$$

(ii) $P_{[A]} = 0$ if and only if $A = 0$;

(iii) for $n \gg 0$ the function $P_{[A]}$ becomes a polynomial which has a positive highest coefficient when $A \neq 0$.

Remark. It is well known for the functions we have discussed above for coherent sheaves and R-modules that they have these properties.

It follows from the definition that if $A \subset B$ then

$$P_{[A]}(n) \leq P_{[B]}(n) \quad \text{for } n \gg 0.$$

Without loss of generality we can suppose from now on that $P_{[A]}$ denotes the polynomial one obtains via condition (iii) of the definition.

DEFINITION 2.2. Let A, B be nonzero objects of $\mathcal{A}$ and

$$P_{[A]}(n) = \sum_{i=0}^{m} a_i n^i, \qquad P_{[B]}(n) = \sum_{i=0}^{m} b_i n^i$$

be the corresponding polynomials (m being unspecified large number). Denote

$$\lambda_{i,j} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$$

and let

$$\Lambda_{(A,B)} = (\lambda_{m,m-1}, \lambda_{m,m-2}, \ldots, \lambda_{m,0}, \lambda_{m-1,m-2}, \ldots, \lambda_{2,1})$$

be the line of 2×2-minors of the matrix

$$\begin{bmatrix} a_m, a_{m-1}, \ldots, a_0 \\ b_m, b_{m-1}, \ldots, b_0 \end{bmatrix}.$$

The polynomial stability structure is define by conditions:

$$A \asymp B \Leftrightarrow \Lambda_{(A,B)} = 0$$
$$A \prec B \Leftrightarrow \text{the first nonzero term in } \Lambda_{(A,B)} \text{ is positive.}$$

Remark. Of course it follows that

$$A \succ B \Leftrightarrow \text{ the first nonzero term in } \Lambda_{(A,B)} \text{ is negative.}$$

We have to check the transitivity of the preorder and the seesaw property.

LEMMA 2.3. *If* $\deg P_{[A]} > \deg P_{[B]}$ *then* $A \prec B$.

Clearly the first nonzero minor in $\Lambda_{(A,B)}$ will be equal to the product of the highest coefficients of $P_{[A]}$ and $P_{[B]}$ which are positive.

LEMMA 2.4. *If* $\deg P_{[A]} = \deg P_{[B]} = d$ *then* $A \prec B$ *if and only if*

$$\left(\frac{a_{d-1}}{a_d}, \frac{a_{d-2}}{a_d}, \ldots, \frac{a_0}{a_d}\right) <_{\text{lex}} \left(\frac{b_{d-1}}{b_d}, \frac{b_{d-2}}{b_d}, \ldots, \frac{b_0}{b_d}\right)$$

(where "$<_{\text{lex}}$" is used for "lexicographically less").

This amounts to the straight check according to the definition. It follows from Lemmas 2.3 and 2.4 that the preorder is transitive.

Remark. In a sense one can consider Definition 2.2 as a way to define an order on a projective space. This order coincides with the lexicographic order on the affine chart $\{(1 : a_{m-1} : \cdots : a_0)\}$ and the "infinite" points $\{(0 : a_{m-1} : \cdots : a_0)\}$ are all bigger than "finite" points. And the structure repeats itself then—points in $\{(0:1:a_{m-2}:\cdots:a_0)\}$ are in lexicographic order between themselves, are smaller than points of the type $\{(0:0:a_{m-2}:\cdots:a_0)\}$, and so on.

LEMMA 2.5. *The polynomial preorder defines a stability structure.*

Proof of the proposition. We are to check the seesaw property. Let $0 \to A \to B \to C \to 0$ be an exact sequence. Then

$$P_{[B]}(n) = P_{[A]}(n) + P_{[C]}(n).$$

Hence

$$\begin{vmatrix} a_j & a_i \\ b_j & b_i \end{vmatrix} = \begin{vmatrix} a_j & a_i \\ a_j + c_j & a_i + c_i \end{vmatrix} = \begin{vmatrix} a_j & a_i \\ c_j & c_i \end{vmatrix}$$

$$= \begin{vmatrix} a_j + c_j & a_i + c_i \\ c_j & c_i \end{vmatrix} = \begin{vmatrix} b_j & b_i \\ c_j & c_i \end{vmatrix}$$

and this implies the seesaw property. ∎

PROPOSITION 2.6. *If the characteristic function with the properties* (i)–(ii) *is defined for an abelian category* A, *then* A *is w-artinian.*

Proof of the proposition. By the contrary let us have an infinite chain

$$A_1 \supset ; \preccurlyeq A_2 \supset ; \preccurlyeq \cdots,$$

with strict inclusions and let

$$P_r = \sum a_i^{[r]} x^i$$

be the corresponding polynomials. As $A_r \supset A_{r+1}$ strictly so

$$P_r(n) > P_{r+1}(n) \qquad \text{for } n \gg 0.$$

Hence $\deg P_r \geq \deg P_{r+1}$ and, therefore, $\deg P_r = \deg P_{r+1} = \cdots = d$ for large enough r and $a_d^{[r]} \geq a_d^{[r+1]} \geq \cdots$.

Since the polynomials have positive integer values for $n \gg 0$ so their highest coefficients $a_d^{[r]}$ belong to $(1/d!)\mathbb{N}$, hence it is impossible for them to decrease infinitely, and we get

$$a_d^{[s]} = a_d^{[s+1]} = \cdots = q$$

for some large enough s.

Then the property $P_r(n) > P_{r+1}(n)$ for $n \gg 0$, and $r \geq s$ is equivalent to

$$\left(q, a_{d-1}^{[r]}, a_{d-2}^{[r]}, \ldots, a_0^{[r]}\right) >_{\text{lex}} \left(q, a_{d-1}^{[r+1]}, a_{d-2}^{[r+1]}, \ldots, a_0^{[r+1]}\right)$$

and this is the same as

$$\left(\frac{a_{d-1}^{[r]}}{q}, \frac{a_{d-2}^{[r]}}{q}, \ldots, \frac{a_0^{[r]}}{q}\right) >_{\text{lex}} \left(\frac{a_{d-1}^{[r+1]}}{q}, \frac{a_{d-2}^{[r+1]}}{q}, \ldots, \frac{a_0^{[r+1]}}{q}\right).$$

Now because of Lemma 2.4 this means $A_r \succ A_{r+1}$ which contradicts to the presupposition that $A_r \preccurlyeq A_{r+1}$. ∎

DEFINITION 2.7. Let us say that the stability structure defined on a category of algebraic coherent sheaves over a projective variety by the polynomial preorder described above is the generalized Gieseker stability.

COROLLARY 2.8. *The statements of Theorems* 1, 2, *and* 3 *are valid for algebraic coherent sheaves on a projective variety in respect to the generalized Gieseker stability.*

One only have to remember that coherent sheaves are noetherian and they are w-artinian as well, by the above proposition.

Remark. Let us remind [M] that a coherent sheaf F on X is said to be of pure dimension d if $\dim \operatorname{Supp}(F) = d$ and for every nonzero coherent subsheaf F' of F, we have $\dim \operatorname{Supp}(F) = d$.

Clearly it follows from our definitions and Lemma 2.3 that a stable (for the generalized Gieseker stability) sheaf is "of pure dimension" and that the Simpson−Maruyama-stable sheaves are the same as the generalized-Gieseker-stable sheaves in the end result.

3. RATIO OF ADDITIVE FUNCTIONS STABILITY

Another, perhaps more usual way to define the stability [F; K; LT; OSS] is via a ratio of two additive functions and this is what we are going to discuss in this section.

DEFINITION 3.1. Let c and r be two additive functions on $\mathcal{A}$ and let $r(A) > 0$ for any nonzero object A of $\mathcal{A}$. We call the ratio

$$\mu(A) = c(A)/r(A)$$

the $(c:r)$-slope of A and define the slope order by the conditions:

$$A \prec B \Leftrightarrow \mu(A) < \mu(B),$$
$$A \asymp B \Leftrightarrow \mu(A) = \mu(B).$$

This is the way stability for algebraic vector bundles is usually defined [OSS; M; LT].

LEMMA 3.2. *The* $(c:r)$*-slope preorder defines a stability structure*.

Proof of the lemma. Let us notice that

$$\frac{c(A)}{r(A)} - \frac{c(B)}{r(B)} = \frac{1}{r(A)r(B)} \begin{vmatrix} r(B) & c(B) \\ r(A) & c(A) \end{vmatrix}.$$

So the ordering between A and B is determined by the positivity, negativity, or nullity of the determinant

$$\begin{vmatrix} r(B) & c(B) \\ r(A) & c(A) \end{vmatrix}.$$

Now it is easy to see that the same transformations of determinants that were used in the proof of Lemma 2.5 also work here. We leave details to the reader. ∎

Remark. The function c is not obliged to take values in $\mathbb{Z}$. For example, $\mathbb{Q}$, $\mathbb{C}$, or an ordered $\mathbb{Z}$-module could be the target set as well. The latter one was the case for the stability used in [R].

King [K] has used the notion of stability to construct moduli spaces of the representations of a quiver. In his case stability is discussed only for representations with a fixed K_0-image α and it depends on a choice of an additive function θ such that $\theta(\alpha) = 0$. This approach makes it possible to construct a moduli space, but at the same moment it does not allow us to compare stable representations with different α as their stabilities often have to be defined with respect to different functions of θ.

In order to relate the King's definition with ours let us first remember the definition from King's paper.

DEFINITION 3.3 [K, p. 516]. Let $\mathcal{A}$ be an abelian category and $\theta\colon K_0(\mathcal{A}) \to \mathbb{R}$ an additive function on the Grothendieck group. An object $M \in \mathcal{A}$ is called θ-semistable if $\theta(M) = 0$ and every subobject $M' \subset M$ satisfies $\theta(M') \geq 0$. Such an M is called θ-stable if the only subobjects M' with $\theta(M') = 0$ are M and 0.

PROPOSITION 3.4. *Given a stability for an abelian category* A *that is defined via the* $(c:r)$*-slope preorder and* $M \in \mathsf{A}$ *let us consider an additive function* θ *such that*

$$\theta = -c + \frac{c(M)}{r(M)} r.$$

Then $\theta(M) = 0$ *and* M *is stable by the* $(c:r)$*-stability if and only if it is* θ*-stable in the sense of Definition* 3.3.

Proof. Let us notice that

$$\theta(M') \geq 0 \Leftrightarrow -c(M') + \frac{c(M)}{r(M)} r(M') \geq 0 \Leftrightarrow \frac{c(M')}{r(M')} \leq \frac{c(M)}{r(M)}. \qquad \blacksquare$$

So King's results about moduli spaces θ-stable objects are relevant to our stability. The existence theorems from [K] for moduli spaces of θ-stable representations of a finite-dimensional algebra imply the existence theorems for moduli spaces of $(c:r)$-stable representations.

Remark. The Harder−Narasimhan filtration of Theorem 2 in general depends on the stability in question. This is easy to see with the following example.

Let (1) → (2) → (3) be a quiver of type A_3 and

$$V = \{V_1 \to V_2 \to V_3\}$$

the representation of the quiver (for the definitions consult, for example, [K]). If we put

$$r(V) = \sum \dim V_i, \qquad c(V) = \sum a_i \dim V_i,$$

then different choices for the coefficients $\{a_i\}$ define different stabilities.

Now let V be the representation where $\dim V_i = 1$ and the maps are isomorphisms. The only subobjects of V are the following two:

$$V^{[1]} = \left\{V_1^{[1]} = 0,\ V_2^{[1]} = 0,\ V_3^{[1]} = V_3\right\};$$
$$V^{[2]} = \left\{V_1^{[2]} = 0,\ V_2^{[2]} = V_2,\ V_3^{[2]} = V_3\right\},$$

and one can investigate how the Harder−Narasimhan filtrations are made out of them.

It is not difficult to check that if $a_1 = 3$, $a_2 = 2$, $a_3 = 1$ then V is stable and the Harder−Narasimhan filtration is trivial. But if $a_i = i$ then V is not stable and

$$V \supset V^{[2]} \supset V^{[1]} \supset 0$$

is the Harder–Narasimhan filtration in this case.

REFERENCES

[F] G. Faltings, Mumford-Stabilität in der algebraischen Geometrie, *in* "Proc. of the Intern. Congress of Math., Zürich, 1994," pp. 648–655, Birkhäuser, Basel, 1995.

[K] A. D. King, Moduli of representations of finite dimensional algebras, *Quart. J Math. Oxford* (2) **45** (1994), 515–530.

[LT] M. Lübke and A. Teleman, "The Kobayashi–Hitchin Correspondence," World Scientific, Singapore/London, 1995.

[M] M. Maruyama, Construction of moduli spaces of stable sheaves via Simpson's idea, *in* "Moduli of Vector Bundles" (M. Maruyama, Ed.) pp. 147–187, Marcel Dekker, New York, 1996.

[OSS] C. Okonek, M. Schneider, and H. Spindler, "Vector Bundles on Complex Projective Spaces," Birkhäuser, Boston/Basel/Stuttgart, 1980.

[R] A. Rudakov, A description of Chern classes of semistable sheaves on a quadric surface, *J. Reine Angew. Math.* **453** (1994), 113–135.

[S] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety, to appear.