Appendix 1. Subspace triples of vector spaces.

Subspace triples of vector spaces have been considered in these lectures at various stages, in particular see section 7, Example 1. As before, we will write $(V; U_1, U_2, U_3)$ in case U_1, U_2, U_3 are subspaces of the vector space V. Here, we want to outline in which way the classification of the indecomposable subspace triples can be proven directly, without the use of reflection functors. This turns out to be quite involved, but we obtain on the way valuable information concerning the lattice $\mathcal{L}(V; U_1, U_2, U_3)$. Of course, this subspace lattice may always be calculated without problems, once we have available the list of all indecomposable subspace triples.

Let us start with an arbitrary subspace triple $(V; U_1, U_2, U_3)$, we want to show that it is the direct sum of copies of the following subspace triples:

$$S(0) = (k; 0, 0, 0), \quad P(1) = (k; k, 0, 0), \quad P(2) = (k; 0, k, 0), \quad P(3) = (k; 0, k, 0)$$
$$R = (k^2; k0, 0k, \Delta), \quad N(1) = (k; 0, k, k), \quad N(2) = (k; k, 0, k), \quad N(3) = (k; k, k, 0)$$
$$I = (k; k, k, k)$$

with $\Delta = \{(x, x) \mid x \in k\} \subset k^2$. As we know, all these subspace triples are indecomposable, and we want to see that they are the only ones. But actually, we want to achieve more: we want an algorithms for decomposing the given triple $(V; U_1, U_2, U_3)$ effectively.

Before we consider decompositions in general, let us show that subspace triples which are direct sums of copies of R can be identified as follows:

Lemma: *Let $(V; U_1, U_2, U_3)$ be a subspace triple such that $V = U_i \oplus U_j$ for all pairs $i \neq j$. Let $U = U_1$. Then $(V; U_1, U_2, U_3)$ is isomorphic to $(U \oplus U; U \oplus 0, 0 \oplus U, \{(u, u) \mid u \in U\})$.*

Proof: By assumption, $V = U_1 \oplus U_2$ and we know that $U_3 = \Gamma(f)$ for some invertible linear map $f\colon U_1 \to U_2$. Define $F\colon U \oplus U \to U_1 \oplus U_2$ by $F(u, u') = (u, f(u'))$. Then we see that

$$F\colon (U \oplus U; U \oplus 0, 0 \oplus U, \{(u, u) \mid u \in U\}) \longrightarrow (V; U_1, U_2, U_3)$$

is an isomorphism.

Of course, taking a basis $b_1, \dots, b_m$ of U, we can write the triple $(U \oplus U; U \oplus 0, 0 \oplus U, \{(u, u) \mid u \in U\})$ as a direct sum of m copies of $R = (k^2; k0, 0k, \{(x, x) \mid x \in k\})$. This completes the proof of the lemma.

Now we return to consider an arbitrary subspace triple $(V; U_1, U_2, U_3)$.

(1) *We can assume that $\sum U_i = V$, otherwise we get direct summands of the form $S(0)$.*

See Exercise 13.

(2) *We split off copies of* $I = (k; k, k, k)$ *and then we can assume that* $\bigcap U_i = 0$.

Proof: Let $J = \bigcap U_i$ and $V = J \oplus C$. Then

$$(V; U_1, U_2, U_3) = (J; J, J, J) \oplus (C; U_1 \cap C, U_2 \cap C, U_3 \cap C).$$

Namely, by definition, the direct sum assertion is true for the total space V. But also $U_i = J \oplus (U_i \cap C)$ since $J \subseteq U_i$ (this was discussed as a consequence to the modular law).

(3) *We split of copies of* $N(3) = (k; k, k, 0)$ *and then we can assume that* $U_1 \cap U_2 = 0$.

Proof. Write $U_{12} = U_1 \cap U_2$, and let C be a complement for $U_{12} + U_3$ in V. Since $U_{12} \cap U_3 = 0$, the subspace $U_{12} + U_3$ is the direct sum $U_{12} \oplus U_3$, thus $V = (U_{12} \oplus U_3) \oplus C = U_{12} \oplus (U_3 + C)$. Write $V' = U_{12}, V'' = U_3 + C$. We claim that the decomposition $V = V' \oplus V''$ is compatible with all the subspaces U_i. This is clear for U_3, since U_3 is contained in the second summand V'', and for U_1, U_2 it follows from the modular law, since V' is contained in U_1 and also in U_2. Of course, $(V'; U_1 \cap V', U_2 \cap V', U_3 \cap V') = (U_{12}; U_{12}, U_{12}, 0)$ is isomorphic to a direct summand of copies of $N(3)$, whereas $(V''; U_1 \cap V'', U_2 \cap V'', U_3 \cap V'')$ satisfies $(U_1 \cap V'') \cap (U_2 \cap V'') = U_{12} \cap (U_3 + C) = 0$.

Similarly, we can split off copies of $N(1)$ and $N(2)$ and can assume that we also have $U_2 \cap U_3 = 0$ and $U_1 \cap U_3 = 0$.

Actually, it is worthwhile to look at $E = U_{12} + U_{13} + U_{23}$, where we write $U_{ij} = U_i \cap U_j$ for $i \neq j$.

(4) *If* $U_1 \cap U_2 \cap U_3 = 0$, *then* $E = U_{12} \oplus U_{13} \oplus U_{23}$.

Proof: Let $u_{ij} \in U_{ij}$ and assume $u_{12} + u_{13} + u_{23} = 0$. Then $u_{12} = -u_{13} - u_{23}$ and the right hand side element lies in U_3, the left hand side element in $U_1 \cap U_2$, thus this element lies in $U_1 \cap U_2 \cap U_3 = 0$. Similarly, also $u_{13} = 0 = u_{23}$.

(5) Also, *let* $D = (U_1 + U_2) \cap (U_1 + U_3) \cap (U_2 + U_3)$. *Then* $E \subseteq D$.

Proof: E is the sum of U_{12}, U_{13}, U_{23}. Let us show that say $U_{12} \subseteq D$. But U_{12} is a subset of U_1, thus of $(U_1 + U_2) \cap (U_1 + U_3)$, and it is a subset of U_2, thus also of $U_2 + U_3$.

Now let us assume that $E = 0$ (this has been achieved after splitting off copies of I and of $N(1), N(2), N(3)$).

(6) *Let* C_i *be a complement for* $U_i \cap D$ *in* U_i, *thus* $(U_i \cap D) \oplus C_i = U_i$. *Then* $V = D \oplus C_1 \oplus C_2 \oplus C_3$.

Proof. Since $U_i \subseteq D + C_i$, for all i, we see that $V = \sum U_i \subseteq C + \sum U_i$, thus it remains to show that this sum is a direct sum. Let $d \in D$ and $c_i \in C_i$, for $1 \leq i \leq 3.$, and assume that $d + c_1 + c_2 + c_3 = 0$. Then $c_1 = -d - c_2 - c_3$ is an element which belongs to C_1

(according to the left side) as well as to $U_2 + U_3$ (according to the right hand side), thus also to $U_1 \cap (U_2 + U_3) \subseteq D$, thus to $(U_1 \cap D) \cap C_1 = 0$. Similarly, $c_2 = 0$ and $c_3 = 0$, therefore also $d = 0$.

(7) *The decomposition* $V = D \oplus C_1 \oplus C_2 \oplus C_3$ *is compatible with the subspaces* U_1, U_2, U_3 *and yields a decomposition of* V *into direct summands of the form* $R, P(1), P(2), P(3)$.

Proof: The decomposition is compatible with U_1, since

$$(U_1 \cap D) \oplus (U_1 \cap C_1) \oplus (U_1 \cap C_2) \oplus (U_1 \cap C_3) = (U_1 \cap D) \oplus C_1 \oplus 0 \oplus 0 = U_1.$$

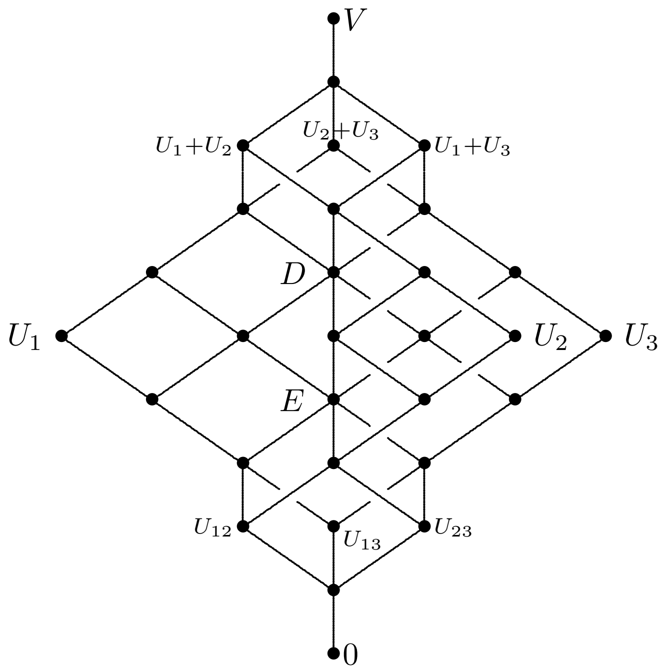
Similarly, one sees that the decomposition is compatible with U_2 and with U_3.

Now we claim that $(D; U_1 \cap D, U_2 \cap D, U_3 \cap D)$ is a subspace triple with $0 = (U_i \cap D) \cap (U_j \cap D)$ and $D = (U_i \cap D) + (U_j \cap D)$ for all $i \neq j$. The first property is trial, since we even have $U_i \cap U_j = 0$ for $i \neq j$. In order to see the second property, let us consider the case $i = 1, j = 2$. We start with the direct decomposition $V = D \oplus C_1 \oplus C_2 \oplus C_3$ and look at the corresponding decomposition of the subspaces $U_1 = (U_1 \cap D) \oplus C_1 \oplus 0 \oplus 0$ and $U_2 = (U_2 \cap D) \oplus 0 \oplus C_2 \oplus 0$, thus $U_1 + U_2 = (U_1 \cap D) + (U_2 \cap D) \oplus C_1 \oplus C_2 \oplus 0$. But by the definition of D, we know that $D \subseteq U_1 + U_2$, thus $D \oplus 0 \oplus 0 \oplus 0 \subseteq (U_1 \cap D) + (U_2 \cap D) \oplus C_1 \oplus C_2 \oplus 0$ and therefore $D \subseteq (U_1 \cap D) + (U_2 \cap D)$, thus $D = (U_1 \cap D) + (U_2 \cap D)$. We now use the lemma at the beginning of the section in order to conclude that $(D; U_1 \cap D, U_2 \cap D, U_3 \cap D)$ is isomorphic to a direct sum of copies of R.

Of course, the remaining summands $(C_i; U_1 \cap C_i, U_2 \cap C_i, U_3 \cap C_i)$ clearly are isomorphic to direct sums of copies of $P(i)$, for $1 \leq i \leq j$. This completes the proof of (7).

Altogether we have seen how we can decompose any subspace triple $(V; U_1, U_2, U_3)$ into a direct sum of copies of $S(0), P(1), P(2), P(3), R, N(1), N(2), N(3), I$.

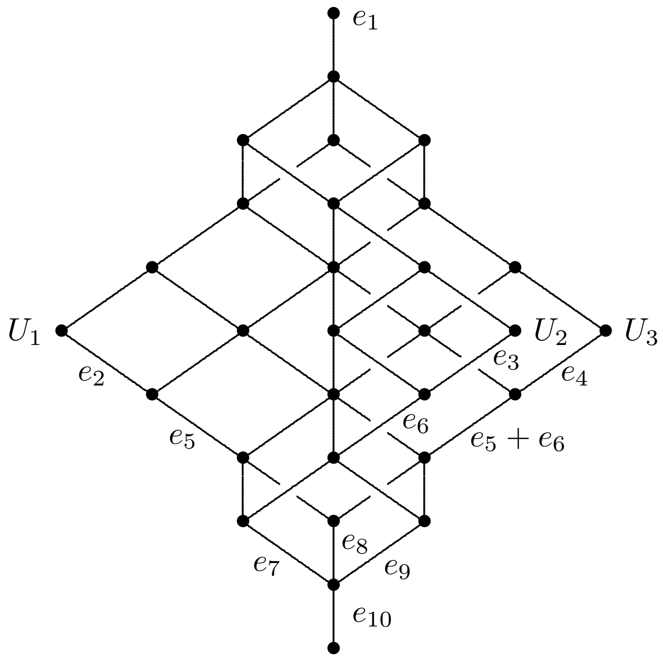
Let us look at the subspace lattices $\mathcal{L}(V; U_1, U_2, U_3)$ where $(V; U_1, U_2, U_3)$ is a subspace triple. Recall that $\mathcal{L}(V; U_1, U_2, U_3)$ is the smallest set of subspaces of V which contains the three given subspaces U_1, U_2, U_3 and is closed under intersections and sums. We claim that the most complicated lattice which we can obtain in this way is the following lattice $\mathcal{L}_3$, it occurs in case $(V; U_1, U_2, U_3)$ has all possible indecomposable subspace triples as direct summands:



Indeed, first consider the case of $(V; U_1, U_2, U_3)$ being the direct sum of one copy of each of $S(0), P(1), P(2), P(3), R, N(1), N(2), N(3), I$, thus V is a 10-dimensional vector space and there is a basis $e_1, \ldots, e_{10}$ of V such that

$$U_1 = \langle e_2,\ e_5,\ e_8,\ e_9,\ e_{10}\rangle,$$
$$U_2 = \langle e_3,\ e_6,\ e_7,\ e_9,\ e_{10}\rangle,$$
$$U_3 = \langle e_4,\ e_5 + e_6,\ e_7,\ e_8,\ e_{10}\rangle.$$

Let us mark some of the edges of the lattice $\mathcal{L}(V; U_1, U_2, U_3)$, say that for the interval $U' \subset U''$, with an appropriate element $e \in U'' \setminus U'$.

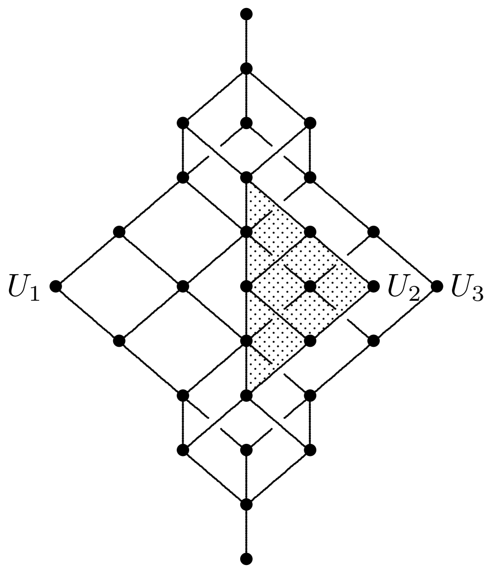


If we consider a direct sum of only some of the possible indecomposable subspace triples, then the lattice obviously will shrink: if the basis element e_i is missing, then the corresponding edge will turn out to be contracted to a point. On the other hand, if we use several copies, say t copies, of one of the indecomposable subspace triples as direct summands, the subspace lattice will not change (we only would have to replace a vector which had been used as a label for some edge by a set of t elements, or say by a t-dimensional complement C for the corresponding inclusion $U' \subset U''$.

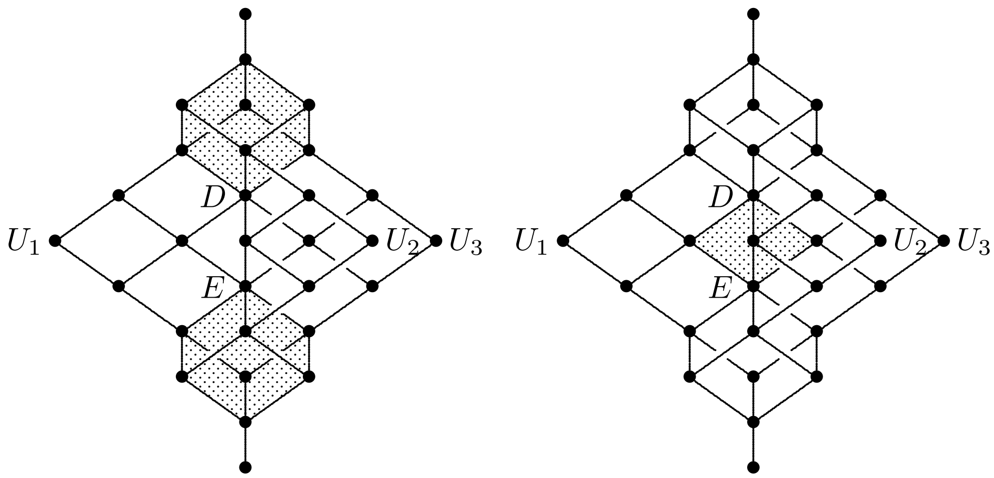
One can show that $\mathcal{L}_3$ is the "free modular lattice in 3 generators": the lattice generated freely by three elements subject to the modularity rule. The free modular lattice was determined first by Dedekind in 1900 (*Über die von drei Moduln erzeugte Dualgruppe*, Math. Ann. 53 (1900), 371-403).

Gian Carlo Rota wrote in 1997: *The free modular lattice with three generators (which has twenty-eight elements) is a beautiful construct that is presently exiled from textbooks of linear algebra. Too bad, because the elements of this lattice explicitly describe all projective invariants of three subspaces.* He writes "28 elements", since he does not take into account the zero subspace 0 and the total space V.

In order to stress the 3-fold symmetry of $\mathcal{L}_3$, it may be worthwhile to draw the attention to the three wings given by U_1, U_2, U_3. Since the wing for U_2 is somewhat hidden, some shading may be helpfull:

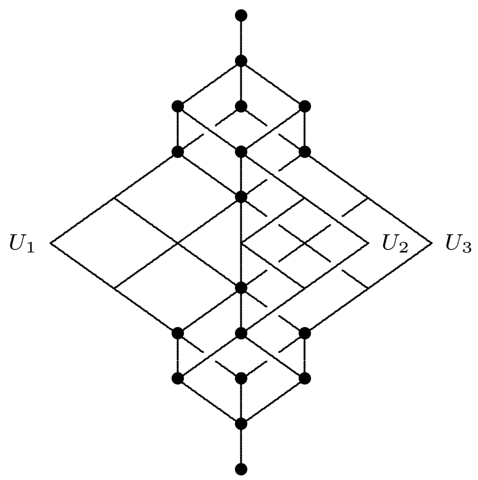


Fur a better understanding of $\mathcal{L}_3$, let us now shade also some other parts which deserve special attention.



On the left, we have shaded two parts which are genuine cubes (standing on a vertex), in particular, these are distributive sublattices. The upper cube corresponds to the direct sums of copies of $P(1), P(2), P(3)$, the lower cube to the direct sums of copies of $N(1), N(2), N(3)$. On the right, the central part has been shaded: the interval between E and D, this concerns the direct sums of copies of R. Note that the interval between E and D shows in an nutshell that $\mathcal{L}_3$ is not distributive.

In addition, let us mark the so called *perfect elements* in $\mathcal{L}_3$, these are the subspaces U of V such that $(U; U_1 \cap U, U_2 \cap U, U_3 \cap U)$ is a direct summand of $(V; U_1, U_2, U_3)$.



There have been attempts to describe also the free modular lattice $\mathcal{L}_4$ with four generators, for example by Gelfand and Ponomarev, but only very little is known about $\mathcal{L}_4$ at present. This is an infinite lattice with obviously a very complicated structure. Note for example that already $\mathcal{L}(k^3; k00, 0k0, 00k, \{(x,x,x) | x \in k\})$ turns out to be quite intricate: it is isomorphic to the lattice of **all** subspaces of a 3-dimensional vector space over the corresponding prime field k_0, thus infinite in case the characteristic of k is equal to zero. The perfect elements of $\mathcal{L}_3$ are known: they form a sublattice which still is infinite.