

Insertion. Complement bases.

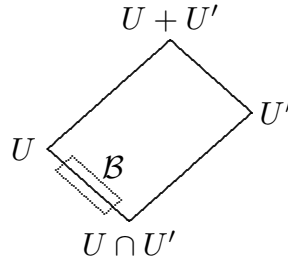
As in part I of the Shanghai Lectures, we need the notion of a *complement basis* for a subspace U of a vector space V ; these are elements $b_1, \dots, b_t$ such that the residue classes $\bar{b}_1, \dots, \bar{b}_t$ form a basis of V/U , or, equivalently, these elements form the basis of a subspace U' of V such that $V = U \oplus U'$.

Lemma (Second Noether isomorphism theorem). *Let U, U' be subspaces of a vector space V . Then:*

- (a) *(First formulation) the subfactors $U/(U \cap U')$ and $U + U'/U'$ are isomorphic.*
- (b) *(Second formulation) The assignment $u + (U \cap U') \mapsto u + U'$ yields an isomorphism*

$$U/(U \cap U') \longrightarrow U + U'/U'$$

- (c) *(Third formulation) Any complement basis for $U \cap U'$ in U is also a complement basis for U' in $U + U'$.*



Warning: The reverse assertion of (c) is usually not true: If $U' \not\subseteq U$, then there are complement bases of U' in $U + U'$ which are not complement bases for $U \cap U'$ in U . (Namely, take an element $u' \in U' \setminus U$, and a complement basis $b_1, \dots, b_t$ of $U \cap U'$ in U . Then $b_1 + u', \dots, b_t + u'$ is a complement basis for U' in $U + U'$, but the elements do not lie in U .)

There is the following **rule**: When we need to choose a complement basis, we will try to do this inside subspaces which are as small as possible.

As an immediate consequence one gets the so-called “butterfly lemma” which is usually used to prove the Schreier theorem (any two filtrations of a vector space have a common refinement), see for example Ringel-Schröer, section 8: Filtration of modules.

Butterfly Lemma. *Let $U \subseteq V$ and $U' \subseteq V'$ be subspaces of a vector space W . Then any complement basis of*

$$(U \cap V') + (V \cap U') \subseteq V \cap V'$$

is also a complement basis for the following two inclusions:

$$\begin{aligned} (U + U') \cap V &\subseteq (U + V') \cap V, \\ (U' + U) \cap V' &\subseteq (U' + V) \cap V'. \end{aligned}$$

In particular, all three subfactors are isomorphic:

$$\begin{aligned}
& ((U + V') \cap V) / ((U + U') \cap V) \\
& \simeq (V \cap V') / ((U \cap V') + (V \cap U')) \\
& \simeq ((U' + V) \cap V') / ((U' + U) \cap V').
\end{aligned}$$

Proof of the butterfly lemma: We deal with the following situation:

$$\begin{array}{ccc}
& & (U' + V) \cap V' \\
& \nearrow & \vdots \\
V \cap V' & & (U' + U) \cap V' \\
& \searrow & \\
& & (U \cap V') + (V \cap U')
\end{array}$$

and we have to show the following conditions:

$$\begin{aligned}
(V \cap V') + ((U' + U) \cap V') &= (U' + V) \cap V' \\
(V \cap V') \cap ((U' + U) \cap V') &= (U \cap V') + (V \cap U')
\end{aligned}$$

Proof: For the sum:

$$\begin{aligned}
& (V \cap V') + ((U' + U) \cap V') \\
&= (V \cap V') + U' + (U \cap V') \\
&= (V \cap V') + U' \\
&= U' + (V \cap V') \\
&= (U' + V) \cap V'.
\end{aligned}$$

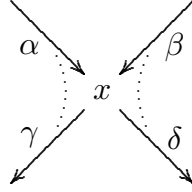
For the intersection:

$$\begin{aligned}
& (V \cap V') \cap ((U' + U) \cap V') \\
&= (U + U') \cap V \cap V' \\
&= (U + (U' \cap V)) \cap V' \\
&= ((U' \cap V) + U) \cap V' \\
&= (U' \cap V) + (U \cap V').
\end{aligned}$$

(Of course, we have used several times the modular law: If $U \subseteq W$, then $(U + V) \cap W = U + (V \cap W)$.)

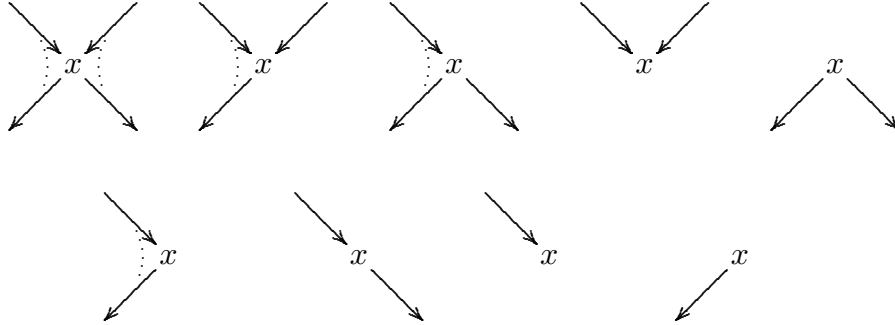
2. Complete gentle algebras.

An algebra is said to be *complete gentle* provided it is given by a quiver and monomial relations of length two with the following property: for every vertex x , there are precisely two arrows ending in x and two arrows starting in x , and for every arrow α there is precisely one arrow α' and precisely one arrow α'' such that $\alpha\alpha'$ and $\alpha''\alpha$ are relations. Thus, if x is a vertex and α, β are the arrows ending in x , then the arrows starting in x can be labeled γ, δ such that $\gamma\alpha$ and $\delta\beta$ are relations and these are the only relations starting with α or β . This is the picture which one should remember: the local structure of a complete gentle algebra (it concerns a vertex x , the arrows starting or ending in x and the relations passing the vertex) looks as follows:



It follows: if n is the number of vertices, then there are precisely $2n$ arrows, as well as $2n$ relations.

Deleting some arrows and the relations which involve these arrows, we obtain the *gentle algebras*. Locally, the gentle algebras look as follows:



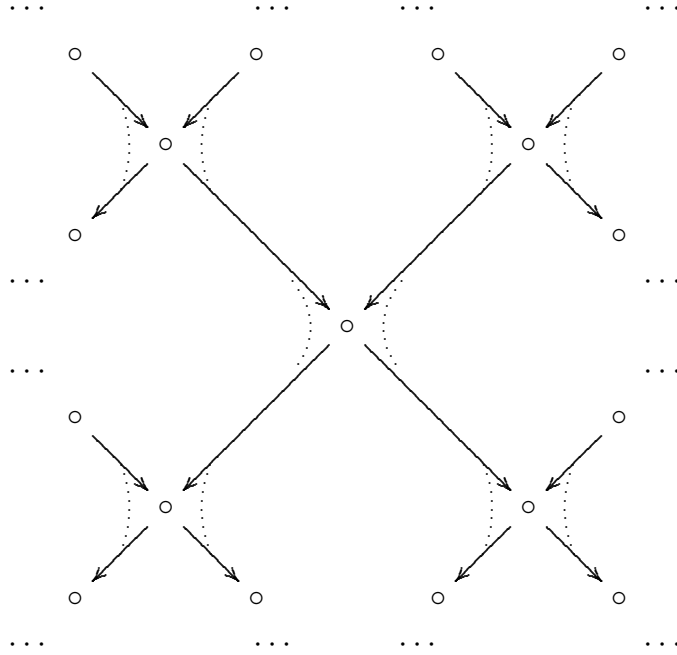
Conversely, assume that there is given an algebra A with a finite number, say n vertices which locally has such a shape. Let m be the number of arrows. Now either $m = 2n$, then necessarily A is a complete gentle algebra. Or else, $m < 2n$. Then there exist a vertex x which is the starting point of at most one arrow, and a vertex y which is the endpoint of at most one arrow. We may add an arrow $x \rightarrow y$ and if necessary, also suitable zero relations involving this arrow, in order to obtain an algebra which locally again has this shape, with the same number n of vertices but with $m + 1$ arrows. By induction, we see that we can add arrows and relations in order to obtain a complete gentle algebra.

Adding relations to gentle algebras, we obtain the *special biserial algebras*, adding only monomial relations, one obtains the *string algebras*. It follows: The special biserial algebras (in particular, the string algebras) are factor algebras of complete gentle algebras (thus, the module category of a special biserial algebra is a full exact subcategory of the module category of a complete gentle algebra). We see: *In order to classify the indecomposable*

representations of the special biserial algebras, it is sufficient to do this for the complete gentle algebras.

Examples. Always we will assume that we deal with a connected quiver.

(1) The 4-regular tree becomes a complete gentle algebra by choosing the appropriate orientation as well as corresponding relations; up to symmetry, there is only one possible choice. This is the only gentle algebra which is a tree, and it is the universal cover of all complete gentle algebras.



For the complete gentle algebra with underlying graph the 4-regular tree we will see: *All indecomposable representations are thin.*

(2) One vertex: There are two different algebras, namely

$$k[X, Y]/(XY), \quad k\langle X, Y \rangle/(X^2, Y^2).$$

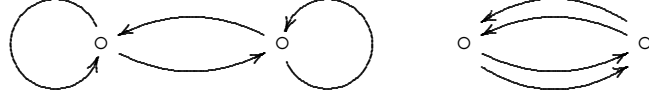
The first one was considered by Gelfand and Ponomarev, its representations are given by a vector space V and a pair of annihilating operators f, g (i.e., $f, g: V \rightarrow V$ are linear transformations with $fg = gf = 0$), or, equivalently, by an A -module, where $A = k[X, Y]/(XY)$ (the factor algebra of the polynomial ring in two variables X, Y modulo the ideal generated by XY).

The second algebras arise in the modular representation theory of dihedral groups: let k be a field of characteristic 2 and G a dihedral groups say of order $2n$. Then G is generated by two elements g, h with $g^2 = h^2 = (gh)^n = 1$. A representation of G is given by a vector space V and two linear transformations g, h satisfying these relations. Instead of g, h we also may look at the transformations $g - 1$ and $h - 1$. But

$$(g - 1)^2 = g^2 - 2g - 1 = g^2 - 1 = 0,$$

since we assume that we work over a field of characteristic 2. Similarly, also $(h - 1)^2 = 0$. This shows that we deal with a representations of the quiver with two loops labeled X and Y satisfying the relations $X^2 = 0 = Y^2$. Equivalently, we can say that kG is a factor algebra of the algebra $k\langle X, Y \rangle / (X^2, Y^2)$.

(3) Two vertices: *there are three different algebras with loops, and two algebras without loops*. The possible quivers are

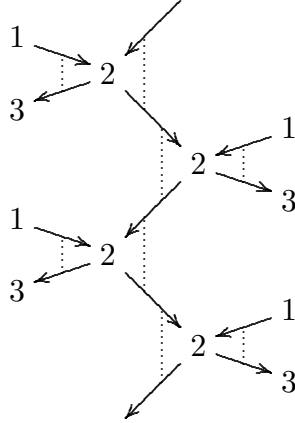


(4) For dealing with algebras of finite representation type, the following algebra algebra has turned out to be quite important:

$$\begin{array}{ccc}
 & \sigma & \\
 & \circ & \\
 3 \longleftarrow & 2 & \longleftarrow 1 \\
 & \beta & \alpha
 \end{array}
 \quad
 \begin{array}{l}
 \beta\alpha = 0 \\
 \sigma^2 = 0
 \end{array}$$

(Note that the indecomposable projective module with top 1 is of length 4, it is the serial string module $M(\beta\sigma\alpha)$).

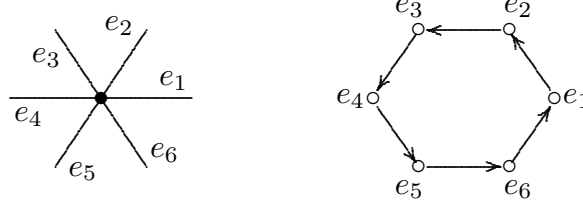
The universal covering looks as follows:



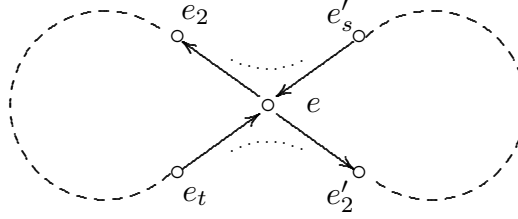
If we add an arrow $1 \rightarrow 3$, then we deal with a minimal non-distributive algebra (which again is special biserial).

(5) Planar graphs. Any planar graph Γ (without loops) with a given embedding into the plane yields a complete gentle algebra $A(\Gamma)$ as follows: As vertices of the quiver $Q(\Gamma)$ of $A(\Gamma)$ take the edges of the graph. If x is a vertex of Γ , and $e_1, \dots, e_t$ are all the edges having x as an endpoint, then we can assume that they are cyclically ordered, say

counter clockwise, according to the given embedding into the plane and we define arrows $e_1 \rightarrow e_2 \rightarrow \dots \rightarrow e_t \rightarrow e_1$.



we obtain in this way for every vertex x in Γ with t neighbors a cycle in $Q(\Gamma)$ with t vertices (thus a loop in case $t = 1$). Any vertex e of $Q(\Gamma)$ belongs to two such cycles, say to $e = e_1, e_2, \dots, e_t$ and to $e = e'_1, e'_2, \dots, e'_s$, then we use as relations the paths $e'_s \rightarrow e \rightarrow e_2$ and $e_t \rightarrow e \rightarrow e'_2$:



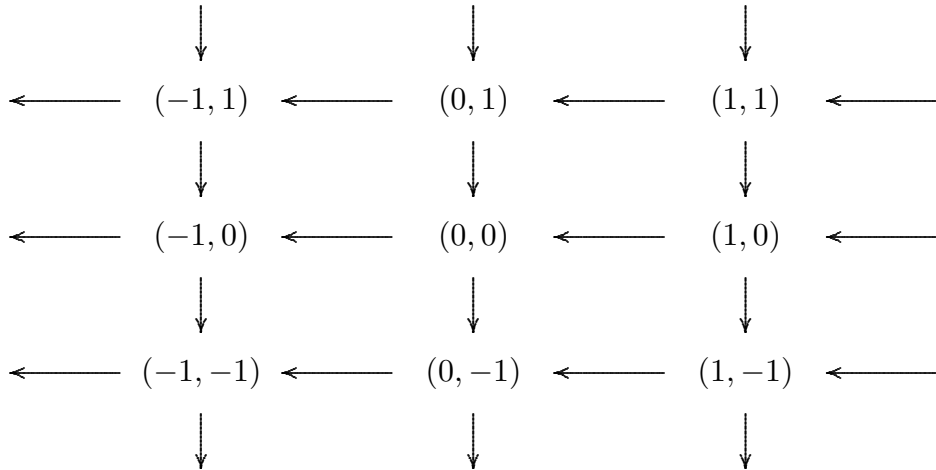
The blocks of group algebras with cyclic defect group are obtained in this way, they are factor algebras of $Q(\Gamma)$, where Γ is a suitable tree, the so-called Brauer tree.

More generally, one may start with a graph Γ embedded into an oriented surface, in particular, we could start with a finite triangulation of a compact oriented surface and take as Γ the 0-simplices and the 1-simplices.

(6) Double complexes. A double complex is by definition of the form $(M_{ij}, d_{ij}^h, d_{ij}^v)$ with index set $\mathbb{Z} \times \mathbb{Z}$, such that all M_{ij} are vector spaces, $d_{ij}^h: M_{ij} \rightarrow M_{i-1,j}$ and $d_{ij}^v: M_{ij} \rightarrow M_{i,j-1}$ are linear transformations and the following relations are satisfied:

$$d^h \circ d^h = 0, \quad d^v \circ d^v = 0, \quad d^h \circ d^v + d^v \circ d^h = 0.$$

If we delete the last kind of relations, we see that we deal with the representations of the complete gentle algebra with the following quiver



with horizontal and vertical relations.