Representations of Quivers: First Steps.

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0. Recollections: Vector spaces.

Always, k is a fixed field, usually arbitrary (but later we sometimes may assume that k is algebraically closed, in order to avoid some complications). All the considerations in these lectures concern linear algebra problems, thus they require from the start that a field is given, but the actual structure of k will not play a role. Thus, one may stick to a field with which one feels comfortable, such as $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{Q}$. On the other hand, for actual calculations, it may be quite convenient to work say with the field $\mathbb{F}_2$ with two elements.

We consider vector spaces (that means k-spaces), usually they will be assumed to be finite dimensional. If V is a vector space, we will write $1_V : V \to V$ (or also just 1) for the identity map (this is the map which sends $v \in V$ to v itself, thus $1_V(v) = v$).

We assume that the following notions and constructions are known:

If U is a subspace of V, then one can form the factor space V/U.

If U, U' are subspaces, one can form $U \cap U'$ and $U + U'$. We sometimes will write $U \oplus U'$ instead of $U + U'$ provided $U \cap U' = 0$ (the symbol $\oplus$ is called *direct sum*).

Linear transformations (or just linear maps) $f : V \to W$. Given f, one may consider its kernel $\operatorname{Ker} f$ and its image $\operatorname{Im} f$, but also the *cokernel* $\operatorname{Cok} f = W/\operatorname{Im} f$.

The dual space V^* of V, dual map f^* of $f : V \to W$.

Basis of a vector space, matrix presentation of a linear map $f : V \to W$ (as soon as bases of V, W are chosen). As much as possible, we will avoid the use of bases, but in section 1 we will stress that sometimes nice bases may exist.

Dimension of a vector space.

Any basis of a subspace U of V can be extended to a basis of V. We will use the following notions:

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If $u_1, \dots, u_s$ is a basis of U and $v_1, \dots, v_t$ extends this to a basis of V, then we call $v_1, \dots, v_t$ a *complement basis for U in V* (it is just a basis of a complement for U in V; a complement C for U in V is by definition a subspace of V with $V = U \oplus C$.)

We call a basis $\mathcal{B}$ of V *compatible with the subspace* U of V provided $\mathcal{B} \cap U$ is a basis of U.

There are obvious relations between these notions: If $\mathcal{B}$ is a basis of V compatible with the subspace U, then $\mathcal{B} \setminus U$ is a complement basis for U in V. If we take the union of a basis of U and a complement basis for U in V, we obtain a basis of V which is compatible with U.

> Exercise 1: Show the following: If $\mathcal{B}$ is a basis of a finite-dimensional vector space V, then only finitely many subspaces of V are compatible with $\mathcal{B}$. Provide a formula for the number of such subspaces.

Intersection dimension formula. *If U, U' are subspaces of V, then*

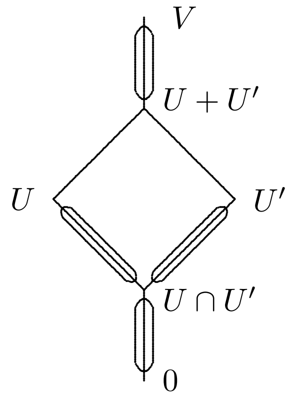
$$\dim(U \cap U') = \dim U + \dim U' - \dim(U + U').$$

We recall the essential step of the proof:

Important. *Let $v_1, \dots, v_r$ be a basis of $U \cap U'$, let $u_1, \dots, u_s$ be a complement basis for $U \cap U'$ in U, and let $u'_1, \dots, u'_{s'}$ be a complement basis for $U \cap U'$ in U', then the elements of the form v_i, u_i, u'_i are a basis for $U + U'$.*

Reformulation. *If U, U' are subspaces of V, then there is a basis $\mathcal{B}$ of V compatible both with U and U'.*

Proof: Extend the basis of $U + U'$ consisting of the elements of the form v_i, u_i, u'_i to a basis $\mathcal{B}$ of V.



We say that two subspaces U, U' are *comparable* provided $U \subseteq U'$ or $U' \subseteq U$.

A sequence

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_t$$

of subspaces U_i of a vector space V is called a *chain* of subspaces of V or also a *filtration* of V. Thus, a chain consists of a finite set of subspaces which are pairwise comparable (and conversely: a finite set of subspaces which are pairwise comparable can be labeled in such a way that we deal with a chain).

1. Vector spaces with two chains of subspaces.

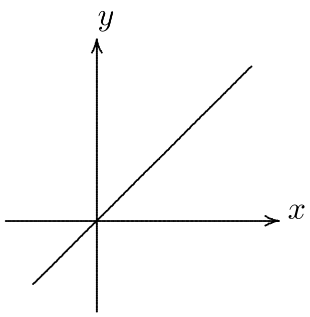
Theorem 1. *Given two chains U_i, $1 \le i \le t$ and U'_i, $1 \le i \le t'$ of subspaces of V, then there exists a basis of V which is compatible with all these subspaces.*

We will give here the proof for $t' = 1$ (and arbitrary t), the general case will be considered later. Before we start with the proof, let us mention a warning, a proposition and a general observation.

The 3-subspace warning. *Let $U = k^2$, $U_1 = k0$, $U_2 = 0k$, $U_3 = \{(x,x) \mid x \in k\}$, then there is no basis of V compatible with U_1, U_2, U_3.*

Proof: Let $\mathcal{B}$ be a basis of V compatible with U_1, U_2. Then $\mathcal{B} \cap U_1$ consists of a single element, say b_1, with $0 \neq b_1 \in k0$. Similarly, $\mathcal{B} \cap U_2$ consists of a single element, say b_2, with $0 \neq b_2 \in 0k$. In particular, $b_1 \neq b_2$. Since $\dim V = 2$, we see that $\mathcal{B} = \{b_1, b_2\}$. It follows that $\mathcal{B} \cap U_3 = \emptyset$, thus it is not a basis of U_3.

Picture in case $k = \mathbb{R}$. We deal with the real plane, the coordinate axes and the diagonal:



Exercise 2. Given a basis $\mathcal{B}$ of a vector space V of dimension at least 2, exhibit explicitly a subspace of V which is not compatible with this basis.

Proposition. *Given a finite set of subspaces of U, then either there are three of these subspaces which are pairwise incomparable, or else the subspaces can be indexed in such a way that they form at most two chains.*

Proof, by induction on the number s of given subspaces. If $s \leq 2$, nothing has to be shown.

Let $s \geq 3$ and assume that there is given a family $\mathcal{S}$ of s subspaces of V such that in any triple of these subspaces, two of them are comparable.

Choose an element $U \in \mathcal{S}$ of maximal dimension.

Apply induction to the remaining elements of $\mathcal{S}$. Thus, we index them in such a way that we deal with at most two chains. If it is a single chain, then we consider U as a second chain, and we are done. Thus, we assume that $\mathcal{S} \setminus \{U\}$ is given by the two chains

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_t, \qquad U'_1 \subseteq U'_2 \subseteq \cdots \subseteq U'_{t'}$$

If $U_t, U'_{t'}$ are incomparable, then at least one of them, say U_t has to be comparable with U, and since U has maximal dimension, $U_t \subseteq U$. Thus we extend the first chain by U

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_t \subseteq U$$

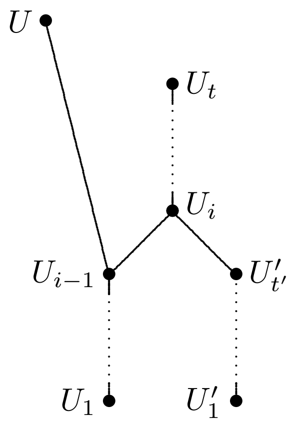
and see that also $\mathcal{S}$ consists of two chains.

Thus, we can assume that $U_t, U'_{t'}$ are comparable, say $U'_{t'} \subseteq U_t$. We may assume that t is as large as possible (if one of the elements U'_j is comparable to all the subspaces $U_1, \ldots, U_t$, we add U'_j to the first chain).

In particular, $U'_{t'}$ is not comparable with all the subspaces $U_1, \ldots, U_t$, and we choose i be minimal with $U'_{t'} \subseteq U_i$. Note that $i > 1$, since otherwise $U'_{t'}$ would be comparable with all the U_i.

We consider the tripel $U,\ U_i,\ U'_{t-1}$. The subspaces $U_{i-1}, U'_{t'}$ are incomparable (by construction of i, we know that $U'_{t'} \not\subseteq U_i$, and $U_i \subseteq U'_{t'}$ would imply that $U'_{t'}$ is comparable with the whole first chain). Also the subspaces $U, U'_{t'}$ are incomparable, it follows that U, U_{i-1} are comparable, and by the maximality of the dimension of U, we see that $U_{i-1} \subseteq U$. Thus, there are the following two chains:

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_{i-1} \subseteq U, \qquad U'_1 \subseteq U'_2 \subseteq \cdots \subseteq U'_{t'} \subseteq U_i \subseteq \cdots \subseteq U_t.$$



This completes the proof of the proposition.

Finally, we insert the following general observation:

Modular law. *Let U, U_1, U_2 be subspaces of V with $U_1 \subseteq U_2$, then*

$$U_1 + (U \cap U_2) = (U_1 + U) \cap U_2.$$

Proof: The inclusion $\subseteq$ is trivial, since $U_1 \subseteq U_2$. The other inclusion is really interesting, but has to be calculated: Take an element of the right side, it is of the form $u_1 + u = u_2$, with $u_1 \in U_1, u \in U, u_2 \in U_2$. Now $u = u_2 - u_1$ belongs not only to U, but also to U_2, since u_1, u_2 are in U_2, thus $u \in U \cap U_2$ and therefore $u_1 + u \in U_1 + (U \cap U_2)$.

Note that, without the assumption $U_1 \subset U_2$, the assertion would not be true. Example: the 3-subspace warning! We have

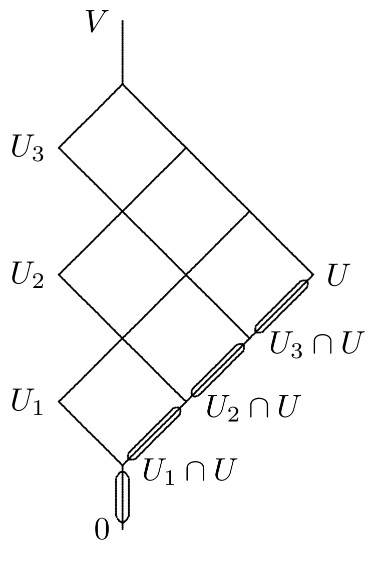
$$U_1 + (U_2 \cap U_3) = U_1 + 0 = U_1$$
$$(U_1 + U_2) \cap U_3 = V \cap U_3 = U_3.$$

Now we provide a **proof of Theorem 1** under the assumption that $t' = 1$. We write U instead of U'_1, thus we deal with a single subspace as well as a chain of subspaces. The case of general t' will be considered later. Actually, instead of looking at a general t, we deal with the case $t = 3$.

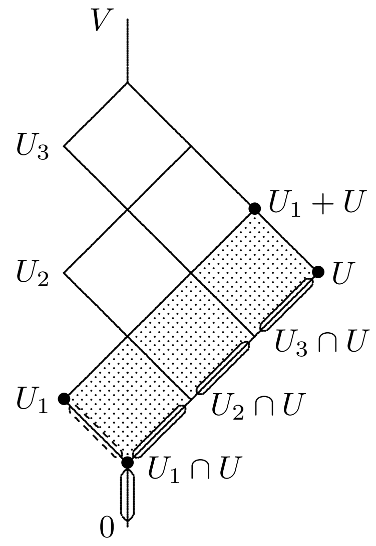
1) We consider first the filtration

$$0 \ \subseteq \ U_1 \cap U \ \subseteq \ U_2 \cap U \ \subseteq \ U_3 \cap U \ \subseteq \ U$$

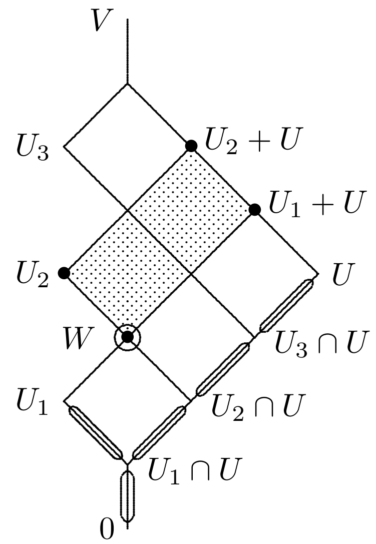
and choose a basis of U compatible with this filtration.



In particular, we have given in this way a basis of U compatible with $U_1 \cap U$. Taking a complement basis for U_1 in $U_1 \cap U$, we obtain a basis of $U_1 + U$ which is compatible with U_1 (as well as with $U_1 \cap U,\ U_2 \cap U,\ U_3 \cap U,\ U$).



2) Next, we want to extend this basis to a basis of $U_2 + U$ which is compatible with U_2, thus we want to deal with the following part:



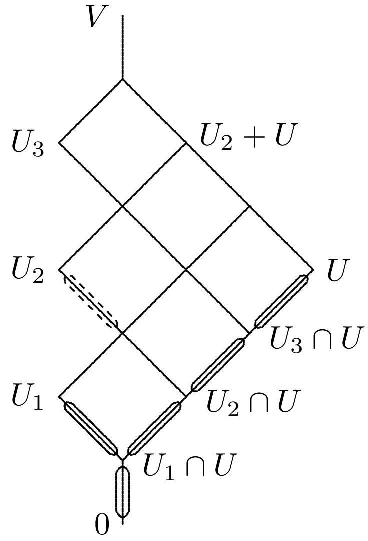
We need to know that the given basis of $U_1 + U$ is compatible with $W = U_2 \cap (U_1 + U)$ (the encircled bullet), because then we can take a complement basis of W in U_2.

The picture suggests that this is the case, however any picture may be misleading. What we know is that our basis of $U_1 + U$ is compatible with U_1 and with $U_2 \cap U$ and thus with $U_1 + (U \cap U_2)$.

Fortunately, the modularity asserts:

$$W = (U_1 + U) \cap U_2 = U_1 + (U \cap U_2).$$

Thus, we take a complement basis for W in U_2 and obtain a basis of $U + U_2$ compatible with $U_1+U,\ U_2+U,\ U_3+U,\ U,\ U_1,\ U_2$.

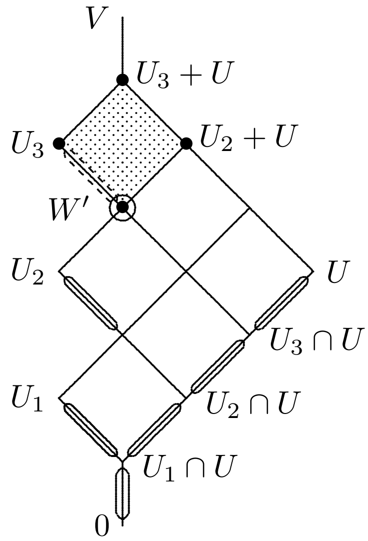


3) The third step should be clear: we want to extend this partial basis in order to obtain also a basis of U_3. This time, we have to look at

$$W' = (U_2+U)\cap U_3 = U_2 + (U\cap U_3).$$

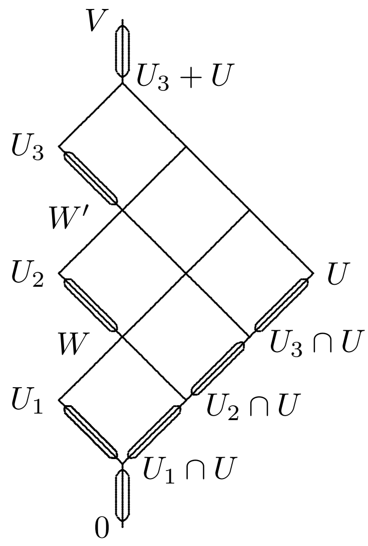
Here, the second equality sign holds again according to the modular law.

Thus, we take a complement basis for W' in U_3 and obtain a basis of $U+U_3$ compatible with U_3 (as well as with $U,\ U_1,\ U_2$).

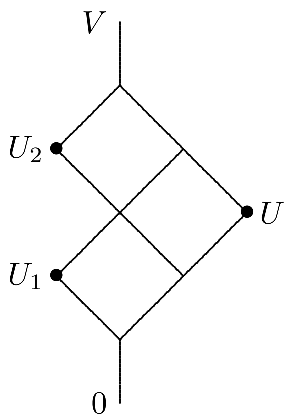


4) It remains to add a complement basis for U_3+U in V. This completes the proof.

Altogether we use the following complement bases:



Exercise 3. a) Explain the modular law for the subspaces U and $U_1 \subseteq U_2$ of V using the following illustration:



b) Given the subspaces U and $U_1 \subseteq U_2$, use the operations $+$ and $\cap$ as often as possible. How many subspaces of V can be obtained in this way?

Exercise 4*, for courageous students: Given three arbitrary subspaces $U_1,\ U_2,\ U_3$, use the operations $+$ and $\cap$ as often as possible. How many subspaces of V can be obtained in this way? Draw a corresponding picture.

Exercise 5*, for courageous students: Provide a proof of Theorem 1 for the case $t = 2,\ t' = 2$ along the lines of the proof for $t = 3,\ t' = 1$ given above.