Before we continue, some remarks concerning the previous considerations should be added.

First of all, we have used very often complement bases. How does one find a complement basis say for the inclusion $U_1 \subseteq U_2$ ? Just in the same way as one would construct a basis of $U_2$ (and a basis of $U_2$ is of course a complement basis for 0 in $U_2$): One starts with a set of vectors which generate together with $U_1$ the subspace $U_2$, and looks whether any non-trivial linear combination of these elements belongs to $U_1$. If there is such a linear combination, then one can delete one of the elements (occurring with non-zero coefficient in the linear combination) and still deals with a set of vectors which generate together with $U_1$ the subspace $U_2$. If there is no non-trivial linear combination which belongs to $U_1$, then we deal already with a complement basis for $U_1$ in $U_2$.

Second, let us analyze our proof of theorem 1 in the case $t = 3,\ t' = 1$.

(1) We have chosen 8 different complement bases. What is the meaning of the number 8 ?

(2) Is there a unified formulation for which pairs of subspaces we have to choose complement bases?

Let us recall which pairs we have used:

$$\begin{array}{lcl@{\qquad\qquad}lcl} 0 & \subseteq & U_1 \cap U & U_1 \cap U & \subseteq & U_1 \\ U_1 \cap U & \subseteq & U_2 \cap U & U_1 + (U_2 \cap U) & \subseteq & U_2 \\ U_2 \cap U & \subseteq & U_3 \cap U & U_2 + (U_3 \cap U) & \subseteq & U_3 \\ U_3 \cap U & \subseteq & U & U_3 + U & \subseteq & V \end{array}$$

Let us write as before $U = U'_1$, and also $U_0 = U'_0 = 0$, $U_4 = U'_2 = V$. Then we can say: the left pairs are all concerned with the inclusions $U_{i-1} \subseteq U_i$ for $1 \le i \le t+1$, and the inclusion $U'_0 \subseteq U'_1$, whereas the right side deals with the inclusions $U_{i-1} \subseteq U_i$ for $1 \le i \le t+1$, and the inclusion $U'_1 \subseteq U'_2$. We can rewrite the list above as follows:

$$\begin{array}{lcl@{\qquad}lcl} (U_0 \cap U'_1) + (U_1 \cap U'_0) & \subseteq & U_1 \cap U'_1 & (U_0 \cap U'_2) + (U_1 \cap U'_1) & \subseteq & U_1 \cap U'_2 \\ (U_1 \cap U'_1) + (U_2 \cap U'_0) & \subseteq & U_2 \cap U'_1 & (U_1 \cap U'_2) + (U_2 \cap U'_1) & \subseteq & U_2 \cap U'_2 \\ (U_2 \cap U'_1) + (U_3 \cap U'_0) & \subseteq & U_3 \cap U'_1 & (U_2 \cap U'_2) + (U_3 \cap U'_1) & \subseteq & U_3 \cap U'_2 \\ (U_3 \cap U'_1) + (U_4 \cap U'_0) & \subseteq & U_4 \cap U'_1 & (U_3 \cap U'_2) + (U_4 \cap U'_1) & \subseteq & U_4 \cap U'_2 \end{array}$$

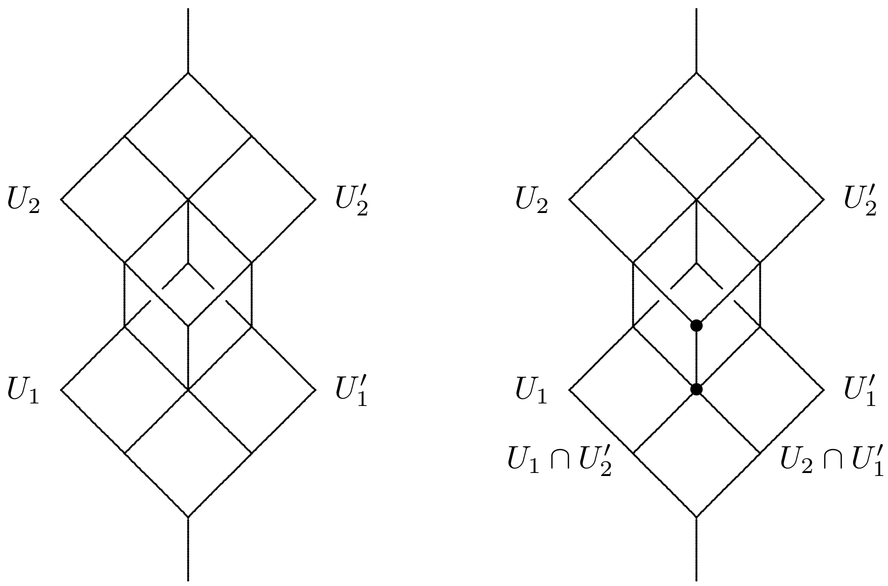
(in order to improve the comparison with the list above, we have shaded some parts which provide no information: this concerns, on the left, the addition of 0, whereas, on the right, two columns concern the intersection with $V$).

In this way, we see that the number 8 has to be considered as $8 = t \times 2 = (t+1)(t'+1)$.

In general, the subspace pairs to be considered are those of the form

$$(U_{i-1} \cap U'_j) + (U_i \cap U'_{j-1}) \ \subseteq\ U_i \cap U'_j, \quad 1 \le i \le t+1,\ 1 \le j \le t'+1.$$

In the case $t = 3,\ t' = 1$ no term looks as complicated as the general term. The first situation where the general term does not shrink occurs for $t = t' = 2$; in that case, the "subspace lattice" generated by the subspaces $U_1 \subseteq U_2$ and $U'_1 \subseteq U'_2$ looks as follows:




On the right, the bullets mark the two relevant subspaces

$$(U_1 \cap U'_2) + (U_2 \cap U'_1) \ \subseteq \ U_2 \cap U'_2.$$

In general, theorem 1 can be strengthened as follows:

**Theorem 1′.** *Given two chains $U_i$, $0 \le i \le t+1$, and $U'_j$, $0 \le j \le t'+1$, of subspaces of $V$, with $U_0 = U'_0 = 0$ and $U_{t+1} = U'_{t'+1} = V$, then the union of complement bases for*

$$(U_{i-1} \cap U'_j) + (U_i \cap U'_{j-1}) \ \subseteq \ U_i \cap U'_j, \quad 1 \le i \le t+1,\ 1 \le j \le t'+1$$

*is a basis of $V$ which is compatible with all the subspaces $U_i, U'_j$.*

There are several ways to prove this result.

- One may work in the same way as we did when looking at the case $t = 3,\ t' = 1$.
- In this course, we will obtain the result by looking at representations of quivers of type $\mathbb{A}_n$, more precisely at $\mathbb{A}_{t+t'+1}$.
- One may also use the "butterfly lemma" as in the proof of the Schreier theorem which asserts that any two filtrations of a vector space have a common refinement (see for example Ringel-Schröer, section 8: Filtration of modules).

## 2. Quivers and representations of quivers.

A *quiver* $Q$ (sometimes also called a directed graph) consists of vertices and oriented edges (arrows): loops and multiple arrows are allowed. An arrow goes from some vertex (its tail) to some vertex (its head), if we denote the tail of the arrow $\alpha$ by $t(\alpha)$, the head by $h(\alpha)$, we see that we deal with two set-theoretical maps

$$t, h\colon Q_1 \to Q_0,$$

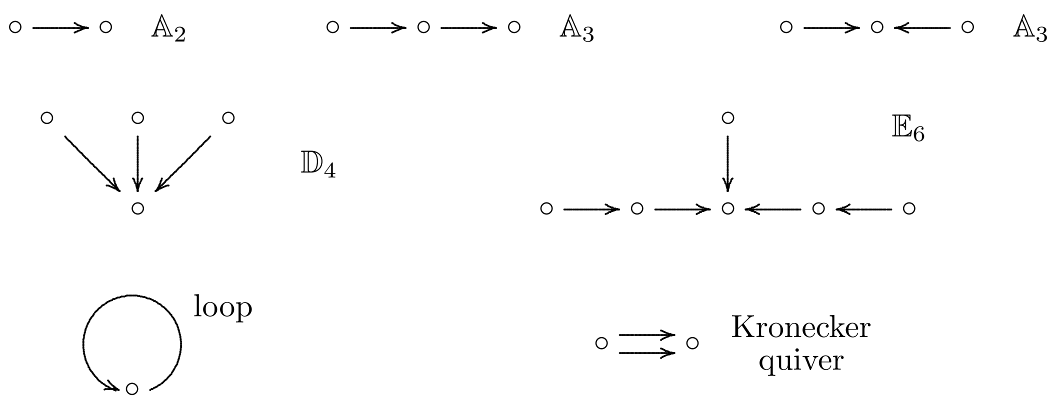
where $Q_0$ denotes the set of vertices, $Q_1$ the set of arrows. Here is the formal definition of a *quiver* $Q = (Q_0, Q_1, t, h)$: there are given two sets $Q_0, Q_1$ and two maps $h, t\colon Q_1 \to Q_0$, the elements of $Q_0$ are called *vertices,* the elements of $Q_1$ are called *arrows,* and for every arrow $\alpha \in Q_1$, there is defined its *tail* $t(\alpha)$ and its *head* $h(\alpha)$. One depicts this in the usual way:

$$t(\alpha) \xrightarrow{\alpha} h(\alpha). \quad \text{or also} \quad \alpha\colon t(\alpha) \to h(\alpha).$$

Given a quiver $Q$, one may delete the orientation of the arrows and obtains in this way the *underlying graph* $\overline{Q}$, this is the triple consisting of the two sets $Q_0, Q_1$ and the functions which attaches to $\alpha \in Q_1$ the set $\{t(\alpha), h(\alpha)\}$ (this means that one does no longer distinguish which one of the vertices is the head and which one is the tail. The reverse process will be called *choosing an orientation.*

> The wording was chosen by Gabriel (1972): "quiver" means literally a box for holding arrows. Before Gabriel, quivers were called "diagram schemes" by Grothendieck.

Here is a collection of typical quivers, with the names which are now usually attached, often these names refer just to the underlying graph.




Of course, one may consider much more complicated quivers, say with 1000 vertices and 7000 arrows, but the representation theory already of quite small quivers usually turns out to be quite complicated. There are quivers with many edges which we will deal with, for example



$\mathbb{A}_n$

with $n$ vertices, usually labeled $1, 2, \ldots, n$, and with $n-1$ arrows $\alpha_i$ with $\{t(\alpha_i), h(\alpha_i)\} = \{i, i+1\}$.

A *representation* of the quiver $Q$ is of the form $M = (M_x, M_\alpha)_{x,\alpha}$, where $M_x$ is a vector space, for every vertex $x \in Q_0$, and $M_\alpha \colon M_{t(\alpha)} \to M_{h(\alpha)}$ is a linear map, for every $\alpha \in Q_1$; instead of $M_\alpha$ one often writes just $\alpha$. Thus, representations of quivers are nothing else than collections of vector spaces and linear maps between these vector spaces.

> Why do we use the letter $M$ for a representation of a quiver? The representations of a quiver $M$ may be considered as the "**m**odules" over the "path algebra" of $Q$.

Of course, for any quiver there is defined the corresponding *zero representation* (or "trivial" representation) with all the vector spaces being zero (and all the maps being zero maps). The zero representation is usually just denoted by 0.

Looking back at section 1, we observe that there we have implicitly dealt with some non-trivial representations. For example, the 3-subspace warning concerns the following representation of a quiver of type $\mathbb{D}_4$:


with $\Delta = \{(x, x) \mid x \in k\}$ and all the maps being the corresponding inclusion maps. The calculation presented there asserts that "this is an indecomposable representation" (but we did not yet define what means "indecomposable").

Also, in section 1 we were considering a vector space $V$ with 4 subspaces $U_1$, $U_2$, $U_3$, $U$ such that $U_1 \subseteq U_2 \subseteq U_3$. Such a system can be considered as a representation of the following quiver of type $\mathbb{A}_5$

$$\circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longleftarrow \circ \qquad \mathbb{A}_5$$

namely as

$$U_1 \to U_2 \to U_3 \to V \leftarrow U$$

where again all the maps are the inclusion maps.

Given a representation $M$ of a quiver $Q$, a *direct sum decomposition* of $M$ is of the following form: for every $x \in Q_0$, there is given a direct sum $M_x = M'_x \oplus M''_x$ and for every $\alpha \colon x \to y$, one has $M_\alpha(M'_x) \subseteq M'_y$ and $M_\alpha(M''_x) \subseteq M''_y$. One may denote the restriction of $M_\alpha$ to $M'_x$ by $M'_\alpha \colon M'_x \to M'_y$, and similarly, the restriction of $M_\alpha$ to $M''_x$ by $M''_\alpha \colon M''_x \to M''_y$. One obtains in this way representations $M' = (M'_x, M'_\alpha)_{x,\alpha}$ and $M'' = (M''_x, M''_\alpha)_{x,\alpha}$ and one writes $M = M' \oplus M''$.

The representation theory of quivers is concerned with the following question: given a representation $M$ of some quiver $Q$, is it possible to decompose the representation? If there is no non-trivial decomposition and $M$ is non-zero, then $M$ is said to be indecomposable: To repeat: $M$ is *indecomposable* if and only if $M \neq 0$ and for any decomposition $M = M' \oplus M''$, either $M' = 0$ or $M'' = 0$.

Of course, there is the corresponding question: describe all the indecomposable representations of a given quiver. For some (quite small) quivers, this will be possible (and indeed for all the examples exhibited above), but in general it seems to be impossible (there is a notion of "wildness": nearly all the large quiver are wild and one does not expect that there is a decent way to classify all the indecomposable representations of any wild quiver).

Let us consider the quiver $\mathbb{A}_2$, we label the vertices 1 and 2 so that the unique arrow is $\alpha\colon 1 \to 2$. The representations of $Q$ are of the form $M = (M_1, M_2, M_\alpha)$, where $M_1, M_2$ are vector spaces and $M_\alpha\colon M_1 \to M_2$ is a linear map, we will denote $M$ just by writing $M = (M_\alpha\colon M_1 \to M_2)$. There are three indecomposable representations of $V$ which are easy to describe:

$$(0 \to k), \quad (k \to 0), \quad (1_k\colon k \to k).$$

(and later it will turn out that these are the only indecomposable representations "up to isomorphism" — but at the moment the notion of an isomorphism of representations of a quiver has not yet been defined). Why are these representations indecomposable? This should be clear for the first two representations, thus let us look at the third one: write it as $M = (M_\alpha\colon M_1 \to M_2)$ with $M_1 = M_2 = k$ and $M_\alpha$ the identity map. What is important is only that $M_\alpha \neq 0$. Assume we have given a direct decomposition $M = M' \oplus M''$, thus $M_1 = M_1' \oplus M_1''$, $M_2 = M_2' \oplus M_2''$, such that $M_\alpha(M_1') \subseteq M_2'$ and $M_\alpha(M_1'') \subseteq M_2''$. Since $M_1 = k$ is one-dimensional, we must have $M_1' = 0$ or $M_1'' = 0$. Without loss of generality, we can assume that $M_1'' = 0$, thus $M_1' = M_1$. Now $M_\alpha$ is non-zero and maps $M_1'$ into $M_2'$, therefore also $M_2' \neq 0$. Since $M_2 = M_2' \oplus M_2''$ is one-dimensional and $M_2' \neq 0$, it follows that $M_2'' = 0$. Thus $M'' = 0$.

If $M, M'$ are representations of the quiver $Q$, a *homomorphism* $f\colon M \to M'$ is of the form $f = (f_x)_x$ with linear maps $f_x\colon M_x \to M_x'$ for all $x \in Q_0$ such that the following diagrams commute:

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & M_x' \\ {\scriptstyle M_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle M_\alpha'} \\ M_y & \xrightarrow{f_y} & M_y' \end{array}$$

(the "commutation" of this square means that the equality $M_\alpha' f_x = f_y M_\alpha$ holds). Of course, given a representation $M$, there is always the identity homomorphism $1_M\colon M \to M$ with $(1_M)_x$ the identity map of $M_x$. Also, for any pair $M$, $M'$ of representations, there is the zero homomorphism $0\colon M \to M'$ (with $0_x\colon M_x \to M_x'$ being the zero map).

Consider the three representations

$$(0 \to k), \quad (k \to 0), \quad (1_k\colon k \to k),$$

and let us determine whether there are non-zero homomorphisms $M \to M'$ or not. Of course, If $M = (0 \to k)$ and $M' = (k \to 0)$, there cannot be a non-zero homomorphism $f\colon M \to M'$, since $f = (f_1, f_2)$ and for $f_1\colon M_1 \to M_1'$ and for $f_2\colon M_2 \to M_2'$ there only exist the zero maps. Now let $M = (0 \to k)$ and $M' = (1\colon k \to k)$, and look for pairs $f = (f_1, f_2)$ with $f_1\colon M_1 \to M_1'$ and $f_2\colon M_2 \to M_2'$. For $f_1$ the only possibility is the zero map, whereas for $f_2\colon k \to k$ we may try to take any scalar multiplication, say take the multiplication by $c \in k$ (as a map $k \to k$). But of course, we have to check whether the following diagram is commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & k \\ \big\downarrow & & \big\downarrow{\scriptstyle 1} \\ k & \xrightarrow{\ c\ } & k \end{array}$$

it always is, thus there are non-zero homomorphisms $(0 \to k) \longrightarrow (1\colon k \to k)$. (Note that in this square, as well as in the following ones, the vertical maps are those of the form $M_\alpha, M'_\alpha$, whereas the horizontal ones are those of the form $f_1$ and $f_2$.) On the other hand, if we are looking for homomorphisms $(1\colon k \to k) \longrightarrow (0 \to k)$, we have to deal with the diagram

$$\begin{array}{ccc} k & \longrightarrow & 0 \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow \\ k & \xrightarrow{\ c\ } & k \end{array}$$

and here it turns out that the diagram commutes only in case $c = 0$, thus there is no non-zero homomorphism $(1\colon k \to k) \longrightarrow (0 \to k)$.

In a similar way, one deals with homomorphisms between $(k \to 0)$ and $(1\colon k \to k)$. The only homomorphism $(k \to 0) \longrightarrow (1\colon k \to k)$ is the zero homomorphism, since the following diagram on the left commutes only for $c = 0$.

$$\begin{array}{ccc} k & \xrightarrow{\ c\ } & k \\ \big\downarrow & & \big\downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & k \end{array} \qquad\qquad \begin{array}{ccc} k & \xrightarrow{\ c\ } & k \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow \\ k & \longrightarrow & 0 \end{array}$$

On the other hand, the above diagram on the right commutes for all $c$, thus thus any $c \in k$ defines a homomorphism $(1\colon k \to k) \longrightarrow (k \to 0)$.

Exercise 6. Consider the following quiver of type $\mathbb{A}_5$

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

For every pair of integers $i, j$ with $1 \le i \le j \le 5$ define a representation $M[i,j]$ with $M[i,j]_x = k$ if $i \le x \le j$ and $M[i,j]_x = 0$ otherwise, and such that $M[i,j]_\alpha$ is the identity map whenever possible.
(a) Show that all representations $M[i,j]$ are indecomposable.
(b) Determine the pairs $(i,j)$ and $(i',j')$ such that there is a non-zero homomorphism

$$M[i,j] \to M[i',j'].$$