Usually one restricts the attention to connected quivers: a quiver is *connected,* provided for any decomposition $Q_0 = Q_0' \cup Q_0''$, with $Q_0' \cap Q_0'' = \emptyset$ and both Q_0' and Q_0'' non-empty, there is an arrow between Q_0' and Q_0'' (that means: there is an arrow $\alpha\colon x \to y$ in Q_1 with $x \in Q_0'$ and $y \in Q_0''$ or else with $x \in Q_0''$ and $y \in Q_0'$).

> **Exercise 9.** Let Q be a quiver and assume that ther are given subquivers Q', Q'' of Q with $Q_0 = Q_0' \cup Q_0''$, and $Q_0' \cap Q_0'' = \emptyset$, and such that also $Q_1 = Q_1' \cup Q_1''$. Show that any representation of Q can be decomposed into a representation with support in Q_0', and a representation with support in Q_0''. (Here, we use the following terminology: The *support* of a representation M of Q is the set of all vertices $x \in Q_0$ such that $M_x \neq 0$.)

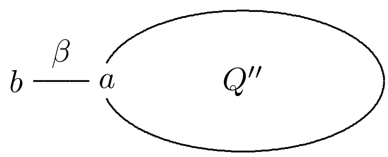
> **Exercise 10.** Show that a quiver with n vertices is a tree quiver if and only if it is connected and there are precisely $n-1$ arrows.

A representation M of Q is *thin,* provided any vector space M_x is at most 1-dimensional, for $x \in Q_0$.

> **Exercise 11.** Let M be a thin representation of a connected quiver Q and assume that all the maps M_α are non-zero, for all $\alpha \in Q_1$. Show that M is indecomposable.

Lemma. *Let Q be a tree quiver. If M is a thin indecomposable representations of Q, then there is an isomorphism $f\colon M \to M'$, such that $M_x' = k$, if $M_x \neq 0$, and such that $M_\alpha' = 1_k$ if $M_\alpha \neq 0$ (where $x \in Q_0$ and $\alpha \in Q_1$).* In particular, the isomorphism class of a thin indecomposable representation is uniquely determined by the support.

Proof. We use induction on the number of vertices of the tree quiver Q. If Q is of type $\mathbb{A}_1$, nothing has to be shown. Now assume that Q is obtained from a tree quiver Q'' with $n-1 \geq 1$ vertices by attaching an arm of the form $\mathbb{A}_2$ at the vertex a, say



Let M be a thin indecomposable representation of Q. We can assume that $M_\beta \neq 0$, since otherwise the support of M is either $\{b\}$ or it is contained in Q'', and in both cases the asserrtion follows by induction. By induction, we replace M by a representation M'' which has the required property for the vertices x and the arrows α belonging to Q''. We look now at the arrow β. If $\beta\colon b \to a$, then we replace M_b'' by the image of M_β'' and M_β'' by the inclusion map; if $\beta\colon a \to b$, then we replace M_b'' by $M_a''/\operatorname{Ker}(M_\beta'')$ and M_β'' by the corresponding projection map, as we know this yields a representation M' isomorphic to

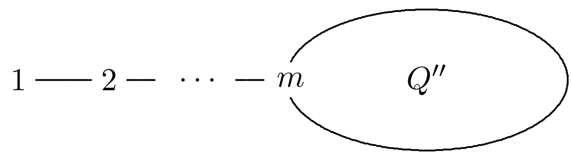
M'', thus to M, and actually we have now also $M'_b = k$, and $M'_\beta = 1$. This completes the proof.

If Q is not a tree, then there are additional indecomposable representations which are thin. Consider for example the *cyclic* quiver C_n with vertices $1, 2, \dots, n$ and arrows $\alpha_i \colon i \to i{+}1$ for $1 \le i \le n-1$ and $\alpha_n \colon n \to 1$ (this quiver is also called the quiver of type $\widetilde{\mathbb{A}}_{n-1}$ with cyclic orientation). Let us consider the thin indecomposable representations with $M_x = k$ for all vertices x. There are n such representations M such that $M_\alpha = 0$ for precisely one arrow α. For the remaining representations M, we may assume that $M_{\alpha_i} = 1_k$ for $i \le i \le n-1$. Then the map M_{α_n} can be an arbirtrary scalar multiplication $[c]$ with $c \in k \setminus \{0\}$, and these representations are pairwise non-isomorphic.

Exercise 12. Proof the last assertion.

3. Arms.

Let Q be a quiver which is obtained from a quiver Q'' by attaching an arm at the vertex m, say



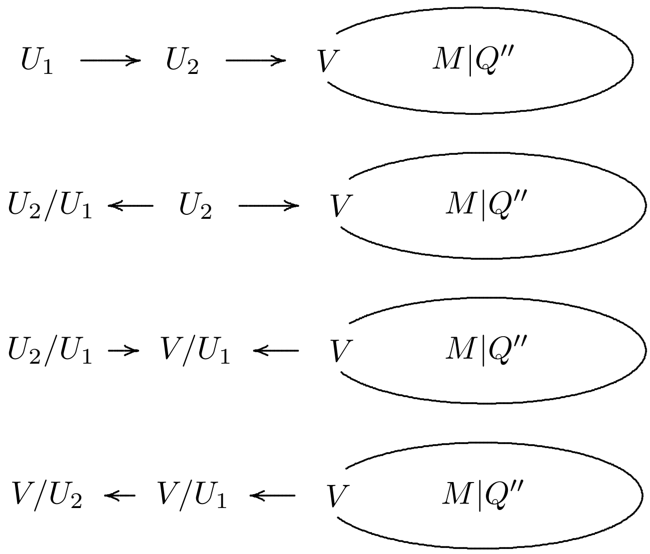
A representation M of Q is said to be *decreasing on the arm* provided the following holds for any arrow $\alpha \colon x \to y$ of the arm: If $x = i$ and $y = i{+}1$, then M_α is injective. If $x = i{+}1$ and $y = i$, then M_α is surjective. In particular, this means that

$$\dim M_1 \le \dim M_2 \le \cdots \le \dim M_m,$$

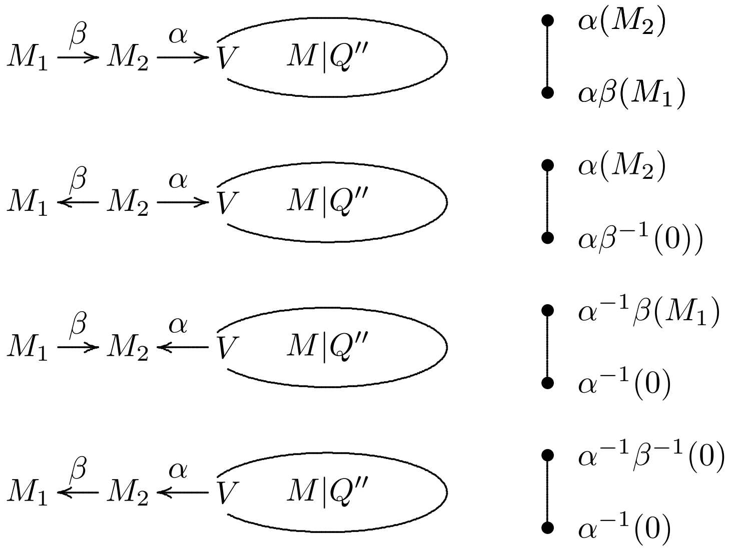
and that all the maps M_α have full rank.

If we deal with an arm attached at the vertex m, and if the arrows of the arm are of the form $x \to x{+}1$, for $1 \le x \le m{-}1$, then we say that this is an arm with *subspace orientation.* If a representation M is decreasing on an such an arm with subspace orientation, then up to isomorphism we can assume that the vector spaces M_x with $1 \le x \le m{-}1$ are subspaces of $V = M_m$, and that the maps on the arm are inclusion maps.

Let us discuss now all the possible orientations of an arm of the form $\mathbb{A}_m$, say for $m = 3$. There are $4 = 2^{m-1}$ different orientations. Assume that there is given a representation M which is decreasing on the arm, then (up to isomorphism) the vector spaces V_1 and V_2 on the arm are given by a chain of two subspaces $U_1 \subseteq U_2$ of the vector space $V = M_3$ as follows:



We also should mention the converse procedure, starting with a representation M of Q which is decreasing on an arm attached at the vertex m. How does one describe the subspace chain in $V = M_m$? Again we consider the case $m = 3$. Given an arrow γ, we write γ instead of M_γ.



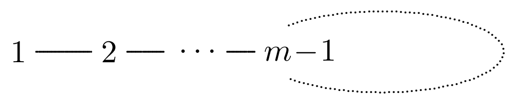
One sees in this way that dealing with a representation M which is decreasing on an arm attached at the vertex m, the relevant information concerning the vector spaces M_x on the arm is a chain $(U_i)_i$ of subspaces of the vector space M_m. Looking at representations which are decreasing on an arm attached at m, we may assume that we actually deal with a corresponding chain of subspaces, thus that we consider the subspace orientation of the arm. Such changes of orientation of parts of a quiver will be discussed later in more detail, under the name of "reflection functors".

Theorem 2. *Let Q be a quiver which is obtained from a subquiver by attaching an arm at the vertex m. Let M be an indecomposable representation of Q with $M_m \neq 0$. Then M is decreasing on the arm.*

The proof will use Theorem 1 with $t' = 1$.

Proof, using induction on m.

First, consider the case that the last arrow of the arm is of the form $\alpha\colon m{-}1 \to m$. If $M_{m-1} = 0$, then the indecomposability implies that $M_x = 0$ for all $x \leq m{-}1$, thus nothing has to be shown. Thus we assume that $M_{m-1} \neq 0$ and use induction: we see that M is decreasing on the arm



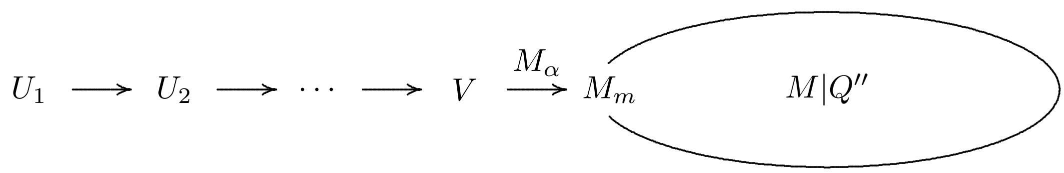
and, as we have seen, the relevant information is a chain of subspaces U_i of $V = M_{m-1}$. Thus, we deal with a chain of subspaces U_i, for $1 \leq i \leq m-2$, of V and in addition we have to consider a further subspace of V, namely $U' = \operatorname{Ker} M_\alpha$. We can apply Theorem 1 to these subspaces and conclude that there exists a basis $\mathcal{B}$ of V which is compatible with all the subspaces $U_1, \dots, U_{m-2}$ as well as U'. For $1 \leq i \leq m-1$, let $\mathcal{B}_i$ be the set of elements of $\mathcal{B}$ which belong to U_i and not to U', and let $\mathcal{B}'_i$ be the set of elements of $\mathcal{B}$ which belong to U_i as well as to U'. Then, for $1 \leq i \leq m-1$, we have a direct sum decomposition of U_i, namely

$$U_i = \langle \mathcal{B}_i \rangle \oplus \langle \mathcal{B}'_i \rangle$$

and of course, the inclusion maps $U_i \to U_{i+1}$ yield inclusions

$$\langle \mathcal{B}_i \rangle \subseteq \langle \mathcal{B}_{i+1} \rangle \qquad \text{and} \qquad \langle \mathcal{B}'_i \rangle \subseteq \langle \mathcal{B}'_{i+1} \rangle.$$

Observe that $\langle \mathcal{B}'_{m-1} \rangle$ is just the kernel of M_α. Thus, looking at the subspace orientation of the arm, we deal with



and we obtain a direct decomposition of this representation into the following two representations:

$$\langle\mathcal{B}_1\rangle \longrightarrow \langle\mathcal{B}_2\rangle \longrightarrow \cdots \longrightarrow \langle\mathcal{B}_{m-1}\rangle \longrightarrow M_m \quad M|Q''$$

$$\langle\mathcal{B}'_1\rangle \longrightarrow \langle\mathcal{B}'_2\rangle \longrightarrow \cdots \longrightarrow \langle\mathcal{B}'_{m-1}\rangle \longrightarrow 0 \quad \text{zero}$$

Since M is indecomposable and $M_m \neq 0$, it follows that the lower representation is zero, thus $B'_{m-1} = \emptyset$ and therefore $\mathrm{Ker}(M_\alpha) = 0$. This shows that M_α is injective, as we wanted to show. (If we want to consider the other orientations explicitly, we have to consider corresponding subfactors of $V = M_{m-1}$.)

As second case, we have to assume that the last arrow is $\beta\colon m \to m-1$. As above, we deal with a chain of subspaces $U_i = M_1$, for $1 \leq i \leq m-2$, and in addition we consider the subspace $U' = \mathrm{Im}(M_\beta)$. As before, Theorem 1 yields a basis $\mathcal{B}$ compatible with these subspaces. For $1 \leq i \leq m-1$, let $\mathcal{B}_i$ be the set of elements of $\mathcal{B}$ which belong to U_i as well as to U', and let $\mathcal{B}'_i$ be the set of elements of $\mathcal{B}$ which belong to U_i and not to U'. We obtain a direct decomposition into the following two representations:

$$\langle\mathcal{B}_1\rangle \longrightarrow \langle\mathcal{B}_2\rangle \longrightarrow \cdots \longrightarrow \langle\mathcal{B}_{m-1}\rangle \xleftarrow{M_\beta} M_m \quad M|Q''$$

$$\langle\mathcal{B}'_1\rangle \longrightarrow \langle\mathcal{B}'_2\rangle \longrightarrow \cdots \longrightarrow \langle\mathcal{B}'_{m-1}\rangle \longleftarrow 0 \quad \text{zero}$$

As in the first case, the indecomposability of M implies that one of the summands is zero, and again it has to be the second. It follows that M_β is surjective, and this is what we wanted to prove.

Theorem 3 (Classification of the indecomposable representations of an $\mathbb{A}_n$-quiver). *Every indecomposaable representation of a quiver of type $\mathbb{A}_n$ is thin.*

Proof: We consider a quiver Q with underlying graph

$$1 \text{ --- } 2 \text{ --- } \cdots \text{ --- } n$$

and an indecomposable representation M of Q. We can assume that $M_1 \neq 0$ and $M_n \neq 0$ (otherwise we replace Q by a suitable subquiver). We can apply theorem 2 both for $m = n$ as well as for $m = 1$ and conclude that all the maps M_α are bijective, in particular $\dim M_1 = \cdots = \dim M_n = t$ for some $t \geq 1$. But if $t > 1$, then we choose a basis of M_1 and use the maps M_α (or their inverses) in order to create corresponding bases in the vector spaces M_i with $2 \leq i \leq n$. This leads to a direct decomposition of M into t thin indecomposable representations. Our assumption that M is indecomposable implies that $t = 1$.

To be more precise, let us exhibit a complete list of representatives for the isomorphism classes of the indecoomposable representations of Q, where Q is a quiver of type $\mathbb{A}_n$, say with underlying graph

$$1 \,—\!\!—\, 2 \,—\, \cdots \,—\, n\ .$$

For $1 \leq i \leq j \leq n$, define a representation $M[i,j]$ of Q as we did for the special case of linear orientation: We put $M[i,j]_x = k$ provided $i \leq x \leq j$, and zero otherwise, and we use as maps $M[i,j]_\alpha$ the identity map of k, whenever this is possible. Then:

Theorem 3′. *Let Q be a quiver of type $\mathbb{A}_n$. The representations $M[i,j]$ with $1 \leq i \leq j \leq n$ form a complete set of representatives of the indecomposable representations of Q.*

(This means: these representations are indecomposable, they are pairwise non-isomorphic, and any indecomposable representation is isomorphic to one of these representations.)

As an immediate consequence of Theorem 3, we obtain a proof of theorem 1, or even theorem 1′, looking at the quivers of type $\mathbb{A}_n$.

Proof of Theorem 1 with t, t' arbitrary.

Proof: We assume that there are given two chains of subspace of a vector space V:

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_t \qquad \text{and} \qquad U'_1 \subseteq U'_2 \subseteq \cdots \subseteq U'_{t'},$$

we want to show that there exists a basis $\mathcal{B}$ of V which is compatible with all these subspaces. Consider the quiver Q of type $\mathbb{A}_n$ with $n = t + t' + 1$ and with the following orientation:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow t \longrightarrow t{+}1 \longleftarrow t{+}2 \longleftarrow \cdots \longleftarrow t{+}t' \leftarrow t{+}t'{+}1\ .$$

The given vectorspace V, the various subspaces U_i, U'_j and the inclusion maps yield a representation M of this quiver:

$$U_1 \longrightarrow U_2 \longrightarrow \cdots \longrightarrow U_t \longrightarrow V \longleftarrow U'_{t'} \longleftarrow \cdots \longleftarrow U'_2 \leftarrow U'_1$$

write this representation as a direct sum of indecomposable representations $M[i,j]$, say

$$M = M^{(1)} \oplus \cdots \oplus M^{(s)}$$

where any $M^{(r)}$ is of the form $M[i, j]$ for some pair $i \leq j$ (depending on r), and actually we must have $i \leq t+1 \leq j$ (since $M[i, j]_{t+1} \neq 0$). Now choose a non-zero element b_r in $(M^{(r)})_{t+1}$, for $1 \leq r \leq s$. Then $\mathcal{B} = \{b_1, \dots, b_s\}$ is the required basis.

Actually, a more detailed analysis of this direct decomposition

$$M = M^{(1)} \oplus \cdots \oplus M^{(s)}$$

with $M^{(r)}$ isomorphic to $M[i_r, j_r]$ provides a proof of Theorem 1′. Namely, we can use this decomposition in order to derive a rule how to choose the various complement bases needed: the rule formulated in Theorem 1′.

We should mention that Theorem 3 above is not only a consequence of Theorem 2, but also that **Theorem 3 implies Theorem 2:**

Proof: If Q' is an arm of Q attached at the vertex m, and M is an indecomposable representation of Q with $M_m \neq 0$, then we consider the restriction M' of M to the subquiver Q'. According to Theorem 3, we may decompose M' into a direct sum of thin indecomposables, and we consider the question whether the vertex m belongs to the support of such an indecmposable or not. Write $M' = X' \oplus Y'$, where X' is a direct sum of thin indecomposaables $N^{(i)}$ such that $(N^{(i)})_m \neq 0$, whereas $Y'_m = 0$. It follows that $M = X \oplus Y$, where the restriction of X to Q' is X', the restriction of Y to Q' is Y', and the restriction of X' to Q'' is the same as that of M to Q'' (and the restriction of Y to Q'' is zero). Since $X \neq 0$ and M is indecomposable, it follows that $Y = 0$. It remains to look at X', this is the direct sum of indecomposable representations Z of Q' which are thin and satisfy $Z_m \neq 0$. Clearly, for these representations Z, we know: If $x = i$ and $y = i+1$, then Z_α is injective, if $x = i+1$ and $y = i$, then Z_α is surjective. Since X' is the direct sum of such representations, we see: If $x = i$ and $y = i+1$, then $X'_\alpha = M_\alpha$ is injective, if $x = i+1$ and $y = i$, then $X'_\alpha = M_\alpha$ is surjective. This completes the proof.

Remark. The representation theory of quivers of type $\mathbb{A}$ (which has been the main target of our considerations up to now) is one of the basic topics of the representation theory of quivers: The indecomposable representations of such a quiver are easy to write down, and there is only a finite number of isomorphism classes of indecomposable representations (therefore, these quivers are said to be *representation-finite*).

We have derived Theorem 3 from the the special case of Theorem 1 (namely dealing with a chain of subspaces and one additional subspace), whereas Theorem 1 may be (and should be) considered as a useful application of Theorem 3, as we have pointed out.

Many different proofs of Theorem 3 are known, all seem to be of interest in their own and shed some light on this situation.