Remark. One may reconsider (5) and (5′) as follows: Assume that y is a sink of Q and let us look at the y-reduced representations M of Q, and at the y-reduced representations N of $\sigma_y^+ Q$. Up to isomorphism, it is enough to consider only those y-reduced representations M of Q such that the map $(\alpha_i)_i \colon \bigoplus_i M_{x_i} \to M_y$ is the projection onto a factor space of $\bigoplus_i M_{x_i}$, let us call them for "normalized", and similarly, it is enough to consider only those y-reduced representations N of $\sigma_y^+ Q$ such that the map $(\beta_i)_i \colon N_y \to \bigoplus_i M_{z_i}$ is the inclusion map of a subspace, let us call them again "normalized". Then we see: The reflection functors yield inverse bijections

$$\{M \mid \text{normalized } y\text{-reduced rep of } Q\} \underset{\sigma_y^-}{\overset{\sigma_y^+}{\rightleftarrows}} \{N \mid \text{normalized } y\text{-reduced rep of } \sigma_y^+ Q\}$$

since for normalized representations we really have $\sigma_y^- \sigma_y^+ M = M$ and $\sigma_y^+ \sigma_y^- N = N$ (equality, not only isomorphy).

> **Exercise 18.** Let y be a sink of Q and M a representation of Q. Show that there is a canonical monomorphism $f \colon \sigma_y^- \sigma_y^+ M \to M$ and that its image M' is a direct summand of M; thus, there is a direct decomposition $M = M' \oplus M''$ of representations of Q. What does one know about M'' ?
> Dually, if y is a source of Q and N a representation of Q, then show that there is a canonical epimorphism $g \colon N \to \sigma_y^+ \sigma_y^- N$, and that its kernel N'' is a direct summand. What does one know about N'' ?

There are two consequences of (5):

(6) *Let y be a sink for Q. If M is an indecomposable y-reduced representation of Q, then $\sigma_y^+ M$ is again indecomposable.*

Proof: Let M be an indecomposable y-reduced representation of Q, and $\sigma_y^+ M = N \oplus N'$ a direct decomposition. according to (5), M is isomorphic to

$$\sigma_y^- \sigma_y^+ M = \sigma_y^- (N \oplus N') = \sigma_y^- N \oplus \sigma_y^- N'$$

where we have used (4′). Since M is indecomposable, one of these direct summands, say $\sigma_y^- N'$ has to be zero. By (1), we know that $\sigma_y^+ M$ is y-reduced, thus also N' is y-reduced and therefore $\sigma_y^- N' = 0$ implies $N' = 0$.

(7) *Let y be a sink for Q. If M, M' are non-isomorphic y-reduced representations of Q, then also $\sigma_y^+ M$ and $\sigma_y^+ M'$ are non-isomorphic.*

Proof. Let M, M' be y-reduced representations of Q and assume that $\sigma_y^+ M$ and $\sigma_y^+ M'$ are isomorphic. According to (3′) and (5) it follows that the representations $M, \sigma_y^- \sigma_y^+ M, \sigma_y^- \sigma_y^+ M', M'$ are isomorphic.

Similarly, there are the dual assertions (with corresponding proofs):

(6′) *Let y be a source for Q. If N is an indecomposable y-reduced representation of Q, then $\sigma_y^- N$ is again indecomposable.*

(7′) *Let y be a source for Q. If N, N' are non-isomorphic y-reduced representations of Q, then also $\sigma_y^- N$ and $\sigma_y^- N'$ are non-isomorphic.*

Two further properties of the reflection functors should be added.

(8) *Let y be a sink. Let M be a representation of Q and M' a subrepresentation of M. Then $\sigma_y^+ M'$ is a subrepresentation of $\sigma_y^+ M$.*

Proof: The construction of $\sigma_y^+ M$ and $\sigma_y^+ M'$ and the inclusion maps $f_x \colon M'_x \to M_x$ yield the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\sigma_y^+ M')_y \arrow[r, "u"] & \bigoplus_i M'_{x_i} \arrow[r, "(\alpha_i)_i"] \arrow[d, "(f_{x_i})_i"'] & M'_y \arrow[d, "f_y"] \\
0 \arrow[r] & (\sigma_y^+ M)_y \arrow[r, "u"] & \bigoplus_i M_{x_i} \arrow[r, "(\alpha_i)_i"] & M_y
\end{tikzcd}$$

and this implies that $(\sigma_y^+ M')_y$ is mapped under $(f_{x_i})_i u$ into $(\sigma_y^+ M)_y$, say with inclusion map f_x^*:

$$\begin{tikzcd}
0 \arrow[r] & (\sigma_y^+ M')_y \arrow[r, "u"] \arrow[d, "f_y^*"'] & \bigoplus_i M'_{x_i} \arrow[r, "(\alpha_i)_i"] \arrow[d, "(f_{x_i})_i"'] & M'_y \arrow[d, "f_y"] \\
0 \arrow[r] & (\sigma_y^+ M)_y \arrow[r, "u"] & \bigoplus_i M_{x_i} \arrow[r, "(\alpha_i)_i"] & M_y
\end{tikzcd}$$

Since the left square commutes, we see that $\sigma_y^+ M'$ is a subrepresentation of $\sigma_y^+ M$.

(9) *Let y be a sink and M a y-reduced representation of Q. Let $\alpha_1, \ldots, \alpha_s$ be the arrows with $h(\alpha_i) = y$ and let $x_i = t(\alpha_i)$ for $1 \le i \le s$. Then*

$$\dim(\sigma_y^+ M)_y = -\dim M_y + \sum_{i=1}^{s} \dim M_{x_i}.$$

Proof. This follows directly from the exact sequence

$$0 \to (\sigma_y^+ M)_y \xrightarrow{(\alpha_i^*)_i} \bigoplus_i M_{x_i} \xrightarrow{(\alpha_i)_i} M_y \to 0.$$

For many assertions one needs to assume that one deals with a y-reduced representation. In case one considers indecomposable representations, there is the following numerical criterion for y-reducibility:

Lemma. *Let y be a sink of Q and let $\alpha_1, \ldots, \alpha_s$ be the arrows with $h(\alpha_i) = y$ and let $x_i = t(\alpha_i)$ for $1 \leq i \leq s$. Let M be an indecomposable representation of Q. Consider $d = -\dim M_x + \sum_{i=1}^{s} \dim M_x$. If $d = -1$, then M is isomorphic to $S(y)$, otherwise M is y-reduced and $d = \dim(\sigma_y^+ M)_y$.*

Proof. If $M = S(y)$, then $\dim M_y = 1$ and $\dim M_{x_i} = 0$ for $1 \leq i \leq s$, thus $d = -1$. Otherwise, M is y-reduced and according to (9) we know that $d = \dim(\sigma_y^+ M)_y$, in particular $d \geq 0$. This completes the proof.

> **Exercise 19.** Assume that y is a source. Formulate and prove the corresponding assertions ($8'$) and (9‘) as well as a numerical criterion for y-reducibility.

The chapter has been labeled reflection functors. What are functors? In the frame of this course, the reflection functors just have to be considered as construction, which allow to obtain new representations starting from given ones. Some properties of these constructions will be used, all will be shown without reference to what is called functoriality.

Whoever is familiar with the basic concepts of category theory will immediately realize that the so-called reflection functors are obviously functors, and indeed for y a sink in Q, the functor σ_y^+ is a functor from the category of representations of Q to the category of representations of $Q'=\sigma_y^+ Q$ which is right adjoint to the functor σ_y^-). These functors have been introduced by Bernstein-Gelfand-Ponomarev in 1982, this was the origin of a very fruitful development leading to what now is known as the general tilting theory.

The basic feature of the reflection functor σ_y^+ is the following. Let us denote by $\mathcal{F}$ the category of all representations of Q which are direct sums of copies of $S(y)$ and by $\mathcal{G}$ the category of all y-reduced representations of Q. Similarly, we denote by $\mathcal{X}$ the category of all representations of Q' which are direct sums of copies of $S(y)$ and by $\mathcal{Y}$ the category of all y-reduced representations of Q' (following the usual convention now when dealing with tilting functors). The functor σ_y^+ is an equivalence from the category $\mathcal{G}$ onto the category $\mathcal{Y}$, with inverse σ_y^-. The functor σ_y^+ can be written in the form $\mathrm{Hom}(T,-)$, where T is a tilting module, and the corresponding derived functor $\mathrm{Ext}^1(T,-)$ furnishes an equivalence from $\mathcal{F}$ onto $\mathcal{X}$. Also, $(\mathcal{F},\mathcal{G})$ is a torsion pair in the category of representations of Q, and $(\mathcal{Y},\mathcal{X})$ is a torsion pair in the category of representations of Q', in this case both torsion pairs are split. But be aware that looking at the torsion pairs (and the equivalences of categories mentioned) the order is surprising: it is the torsion class $\mathcal{G}$ of the first torsion pair is equivalent to the torsionfree call $\mathcal{F}$ of the second torsion pair, the torsionfree class $\mathcal{F}$ of the first torsion pair is equivalent to the torsion class $\mathcal{X}$ of the second torsion pair. It is this flip-flop which is one of the reasons for using the name "tilting".

5. Iteration.

Some of the observations in the last section can be formulated in the following way:

Proposition. *Let y be a sink of the quiver Q and let $M^{(1)}, \ldots, M^{(m)}$ be a set of pairwise non-isomorphic indecomposable representations of Q. Then at most one of the* *representations $\sigma_y^+ M^{(i)}$ is zero, the remaining ones are indecomposable and pairwise non-isomorphic.*

Proof: If all the $M^{(i)}$ are y-reduced, then the representations $\sigma_y^+ M^{(1)}, \ldots, \sigma_y^+ M^{(m)}$ are indecomposable and pairwise non-isomorphic. Otherwise, one of the representations $M^{(i)}$, say $M^{(m)}$ will be isomorphic to $S(y)$, but the remaining ones are y-reduced. Then $\sigma_y^+ M^{(m)} = 0$, and $\sigma_y^+ M^{(1)}, \ldots, \sigma_y^+ M^{(m-1)}$ are indecomposable and pairwise non-isomorphic.

Of course, there is the corresponding assertions also for y a source.

A *sink sequence* $y_1, \ldots, y_r$ (or a $(+)$-*admissible* sequence) is a sequence of vertices of Q which starts with a sink y_1 and such that for all $i \geq 2$ the vertex y_i is a sink for the quiver $\sigma_{y_{i-1}}^+ \cdots \sigma_{y_1}^+ Q$ (that means: we start with a sink y_1, change the orientation of all the arrows ending in y_1, take a vertex y_2 which now is a sink, change now also the orientation of all the arrows ending in y_2, look again for a sink, and so on.)

Corollary. *Let $y_1, \ldots, y_r$ be a sink sequence for the quiver Q and let $\Sigma = \sigma_{y_r}^+ \cdots \sigma_{y_1}^+$. Let $M^{(1)}, \ldots, M^{(m)}$ be a set of pairwise non-isomorphic indecomposable representations of Q. Then at most r of the representations $\Sigma M^{(i)}$ are zero, the remaining ones are indecomposable and pairwise non-isomorphic.*

Proof, by induction on r. The case $r = 1$ is just the proposition. Let $r \geq 2$, and let $\Sigma' = \sigma_{y_r}^+ \cdots \sigma_{y_2}^+$, so that $\Sigma M = \Sigma' \sigma_y^+ M$ for any representation M of Q. If none of the representations $\sigma_y^+ M^{(i)}$ is zero, then by the proposition, the representations $\sigma_y^+ M^{(1)}, \ldots, \sigma_y^+ M^{(m)}$ are indecomposable and pairwise non-isomorphic. By induction at most $r-1$ of the representations $\Sigma' \sigma_y^+ M^{(1)}, \ldots, \Sigma' \sigma_y^+ M^{(m)}$ are zero, the remaining ones are indecomposable and pairwise non-isomorphic. If one of the representations $\sigma_y^+ M^{(i)}$ is zero, say $\sigma_y^+ M^{(m)} = 0$, then the representations $\sigma_y^+ M^{(1)}, \ldots, \sigma_y^+ M^{(m-1)}$ are indecomposable and pairwise non-isomorphic and we apply Σ' to these representations.

We should add a remark concerning quivers with several sinks (or several sources). If y_1, y_2 are different sinks of Q, then $\sigma_{y_1}^+ \sigma_{y_2}^+ = \sigma_{y_2}^+ \sigma_{y_1}^+$ for quivers as well as representations. Indeed, $\sigma_{y_1}^+ \sigma_{y_2}^+ Q = \sigma_{y_2}^+ \sigma_{y_1}^+ Q$ is obtained from Q by changing the orientation of all the arrows α with head y_1 or y_2, this we can do in one step. Similarly, for M a representation of Q, we have

$$\sigma_{y_1}^+ \sigma_{y_2}^+ M = \sigma_{y_2}^+ \sigma_{y_1}^+ M,$$

again the replacement of the vector spaces M_{y_1} and M_{y_2} as well as the corresponding maps can be done simultaneously.

Often it will be reasonable to consider all the sinks at the same time and to invoke the reflection functors for all the sinks at once (see for example the discussion of the 3-subspace quiver Q in the next section: we denote by 0 its sink, by $1, 2, 3$ its sources. First, we apply σ_0^+, thus changing the direction of all the arrows, we obtain in this way the quiver $Q' = \sigma_0^+ Q$

with 0 now being a source and with three sinks $1, 2, 3$. Under the reflection functor σ_0^+, the simple representation $S(0)$ of Q is sent to zero, whereas the 0-reduced representations of Q correspond bijectively to the 0-reduced representations of Q'. Now we look at Q' and its representation. As we have mentioned, Q' has the three sinks $1, 2, 3$ and we will apply $\sigma_3^+\sigma_2^+\sigma_1^+$ to the quiver Q' as well as to the representations of Q'. Applying $\sigma_3^+\sigma_2^+\sigma_1^+$ to the quiver Q', we again change the direction of all the arrows and obtain Q back. Under the reflection functor $\sigma_3^+\sigma_2^+\sigma_1^+$, the simple representations $S(1), S(2), S(3)$ of Q' are sent to zero, whereas the representations of Q' which are i-reduced for $i = 1, 2, 3$ correspond bijectively to the representations of $\sigma_3^+\sigma_2^+\sigma_1^+Q' = Q$ which are i-reduced for $i = 1, 2, 3$.

6. Star quivers with subspace orientation.

Let Q be a star quiver with center c and with subspace orientation. Let $I = I(Q)$ be the representation with $I(0)_x = k$ for all vertices x and $I_\alpha = 1$ for all arrows α. Thus I is thin, indecomposable and has full support (its support is Q_0). Conversely, any thin indecomposable representation with full support is isomorphic to Q.

Theorem. *Let Q be a star quiver with center c and with subspace orientation. Let $I = I(Q)$. If there is a sink sequence $y_1, \dots, y_r$ such that $\sigma_{y_r}^+ \cdots \sigma_{y_1}^+ I = 0$, then Q has at most r isomorphism classes of indecomposable subspace representations.*

Proof: First, let us observe that the direct sum I^n of n copies of I is the representation with $(I^n)_x = k^n$ (or a fixed n-dimensional vector space V) for all vertices x and $(I^n)_\alpha$ the identity map for all arrows α.

We start with the following lemma.

Lemma. *Let Q be a star quiver with center c and with subspace orientation. A representation M of Q is a subspace representation if and only if M is a subrepresentation of a direct sum of copies of $I(Q)$.*

Proof: If M is a subrepresentation of I^n for some n, then all the maps M_α are restrictions of identity maps, thus they are inclusion maps. This means that all the vector spaces M_x must be subspaces of M_c (and M_c is a subspace of $(I^n)_c$), in particular, M is a subspace representation.

Conversely, assume that M is a subspace representation of Q, thus all the vector spaces M_x are subspaces of M_c and the maps M_α are corresponding inclusion maps. Let us denote by $u_x\colon M_x \to M_c$ the given inclusion map. Let M' be the representation of Q with $M'_x = M_c$ for all vertices x (thus M_x is a fixed vector space for all x), and M'_α the identity map for all arrows α. Then M' is a direct sum of copies of I and $u = (u_x)_x$ $M \to M'$ is an embedding of representations, thus M is a subrepresentation of M'.

Proof of Theorem. Let $y_1, \dots, y_r$ such that $\Sigma I = 0$, where we write $\Sigma = \sigma_{y_m}^+ \cdots \sigma_{y_1}^+$.

If M is a subrepresentation of I^n for some n, then ΣM is a subrepresentation of $\Sigma(I^n) = (\Sigma I)^n = 0$, according to properties (8) and (4) for the reflection functors, but this means that $\Sigma M = 0$.

According to the corollary in section 5, we know that there are at most r isomorphism classes of indecomposable representations M of Q with $\Sigma M = 0$. This completes the proof.

Example 1: The 3-subspace quiver $\mathbb{D}_4$.

The 3-subspace quiver Q is the following quiver (of type $\mathbb{D}_4$), and we use the labels $0, 1, 2, 3$ for the vertices.

$$\begin{tikzcd} \overset{1}{\circ} \arrow[dr] & \overset{2}{\circ} \arrow[d] & \overset{3}{\circ} \arrow[dl] \\ & \underset{0}{\circ} & \end{tikzcd}$$

Dealing with a subspace-representation M of Q, we just write $M = (M_0; M_1, M_2, M_3)$.

Proposition. *There are, up to isomorphism, precisely* 9 *indecomposable subspace representations of* Q*, namely the following:*

	$P(1) =$ $(k; k, 0, 0)$		$N(1) =$ $(k; 0, k, k)$	
$S(0) =$ $(k; 0, 0, 0)$	$P(2) =$ $(k; 0, k, 0)$	$R =$ $(k^2; k0, 0k, \Delta)$	$N(2) =$ $(k; k, 0, k)$	$I =$ $(k; k, k, k)$
	$P(3) =$ $(k; 0, k, 0)$		$N(3) =$ $(k; k, k, 0)$	

here, $\Delta = \{(x, x) \mid x \in k\} \subset k^2$.

Before we prove the proposition, let us mention the following consequence:

Corollary. *Any quiver of type* $\mathbb{D}_4$ *has precisely* 12 *isomorphism classes of indecomposable representations.*

Proof. First we deal with the subspace orientation as considered above. Besides the subspace representations we also have to take into account the indecomposable representations M with $M_0 = 0$, thus those living at one of the three arms. Such a representation M is simple, namely one of the representations $S(1), S(2), S(3)$. Any other orientation is obtained by changing the orientation of the arms. But such a change of orientation does not change the number of isomorphism classes of indecomposable representations.

Proof of proposition: It is clear that the representations listed are pairwise non-isomorphic. The indecomposability is clear for the thin representations, and the 3-subspace warning just asserts that also R is indecomposable.

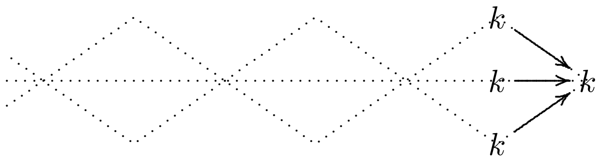
It remains to show that there are no other indecomposable subspace representations. According to the Theorem, it is sufficient to look at the representation $I = I(Q)$ Since 0 is a sink for Q and $1, 2, 3$ are sources, it is obvious that the sequence $0, 1, 2, 3, 0, 1, 2, 3, 0$ is a sink-sequence.

Claim:

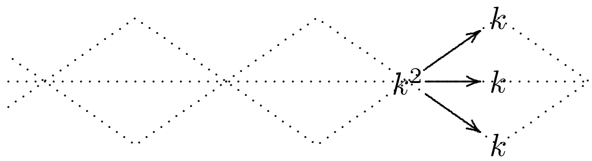
$$\sigma_0^+ \sigma_3^+ \sigma_2^+ \sigma_1^+ \sigma_0^+ \sigma_3^+ \sigma_2^+ \sigma_1^+ \sigma_0^+ I = 0.$$

For this calculation we propose to use a special arrangement of vector spaces and linear maps (and as we will note below, one actually only has to look at the dimensions of the vector spaces, thus at natural numbers). We start with the given representation I on the right, and form inductively kernels of appropriate linear maps. Let us draw a sequence of pictures which shows step by step some representations obtained by applying reflection functors; in practice one will just produce a single picture, working from right to left (we will exhibit it below).

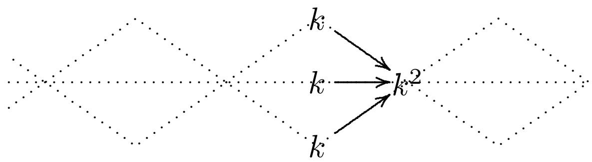
$I^{(0)} = I$



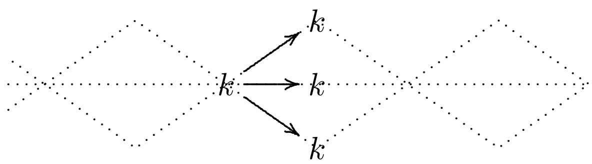
$I^{(1)} = \sigma_0^+ I^{(0)}$



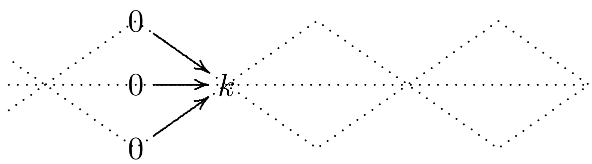
$I^{(2)} = \sigma_3^+ \sigma_2^+ \sigma_1^+ I^{(1)}$



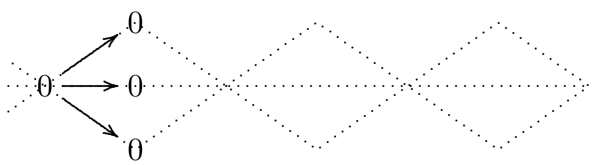
$I^{(3)} = \sigma_0^+ I^{(2)}$



$I^{(4)} = \sigma_3^+ \sigma_2^+ \sigma_1^+ I^{(3)}$



$I^{(5)} = \sigma_0^+ I^{(4)}$



We exhibit in this way the representations $I^{(0)}, I^{(2)}, I^{(4)}$ of Q as well as the representations $I^{(1)}, I^{(3)}, I^{(5)}$ of $\sigma_0^+ Q$.

Note that here the individual maps which have to be used have not been specified; it may be sometimes quite cumbersome, to write such maps explicitly, but for our purpose one is not forced to do so. As we already have mentioned, the important information are the various vector space dimensions due to property (9).

Here is the combined picture, with all the zeros deleted:

$$\begin{tikzcd}
 & k \arrow[dr] & & k \arrow[dr] & \\
k \arrow[ur] \arrow[r] \arrow[dr] & k \arrow[r] & k^2 \arrow[ur] \arrow[r] \arrow[dr] & k \arrow[r] & k \\
 & k \arrow[ur] & & k \arrow[ur] &
\end{tikzcd}$$

As we have mentioned, it is sufficient to record the dimension of the vector spaces, thus to produce the following picture:

$$\begin{tikzcd}
 & 1 \arrow[dr] & & 1 \arrow[dr] & \\
1 \arrow[ur] \arrow[r] \arrow[dr] & 1 \arrow[r] & 2 \arrow[ur] \arrow[r] \arrow[dr] & 1 \arrow[r] & 1 \\
 & 1 \arrow[ur] & & 1 \arrow[ur] &
\end{tikzcd}$$

This picture is called a "hammock".

Hammocks have been introduced by Sheila Brenner in 1986, and have to be considered as an important combinatorial stucture arising in representation theory. Note: it is not an accident that the hammock picture which we have constructed here resembles the arrangement of the 9 indecomposable subspace representations. In fact, the so-called Auslander-Reiten quiver of the category of subspace representations of Q looks as follows:

$$\begin{tikzcd}
 & P(1) \arrow[dr] & & N(1) \arrow[dr] & \\
S(0) \arrow[ur] \arrow[r] \arrow[dr] & P(2) \arrow[r] & R \arrow[ur] \arrow[r] \arrow[dr] & N(2) \arrow[r] & I \\
 & P(3) \arrow[ur] & & N(3) \arrow[ur] &
\end{tikzcd}$$

and the numbers 1 and 2 are just the numbers $M_0 = \dim \operatorname{Hom}(S(0), M)$ for any vertex labeled by the representation M.

Example 2: The subspace quivers of type $\mathbb{A}_n$.

Let us deal with the star quivers $\mathbb{T}_{t_1,t_2}$ with subspace orientation.

Let us consider the special case $t_1 = 3, t_2 = 4$. We use the following labels for the vertices:

$$\begin{tikzcd}
\underset{\circ}{1} \arrow[r] & \underset{\circ}{2} \arrow[r] & \underset{\circ}{3} & \underset{\circ}{4} \arrow[l] & \underset{\circ}{5} \arrow[l] & \underset{\circ}{6} \arrow[l]
\end{tikzcd}$$

thus the branching vertex $c = 3$ is the unique sink.

Claim:

$$\sigma_4^+\sigma_5^+\sigma_6^+\sigma_3^+\sigma_4^+\sigma_5^+\sigma_2^+\sigma_3^+\sigma_4^+\sigma_1^+\sigma_2^+\sigma_3^+ I(Q) = 0.$$

Exercise 20. Proof this assertion, by constructing the following "hammock":

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & 1 \arrow[dr] & & \\
 & & 1 \arrow[ur] \arrow[dr] & & 1 \arrow[dr] & \\
 & 1 \arrow[ur] \arrow[dr] & & 1 \arrow[ur] \arrow[dr] & & 1 \\
1 \arrow[ur] \arrow[dr] & & 1 \arrow[ur] \arrow[dr] & & 1 \arrow[ur] & \\
 & 1 \arrow[ur] \arrow[dr] & & 1 \arrow[ur] & & \\
 & & 1 \arrow[ur] & & &
\end{tikzcd}$$

It follows that Q can have, up to isomorphism, at most 12 indecomposable subspace representations. Of course it is easy to exhibit 12 thin indecomposable subspace representations.

More generally one shows:

Proposition. *Let Q be the quiver of type $\mathbb{T}_{t_1,t_2}$ with subspace orientation. Then there is a sink sequence $y_1, \dots, y_r$ with $r = t_1t_2$ such that*

$$\sigma_{y_r}^+ \cdots \sigma_{y_1}^+ I(Q) = 0.$$

Since Q has t_1t_2 isomorphism classes of thin indecomposable representations, it follows that *any indecomposable subspace representation of Q is thin.*

This provides a new proof of Theorem 1: *Given a vector space V with two chains of subspaces, there is a basis of V which is compatible with all the subspaces.*

Example 3: The 4-subspace quiver.

Let Q be the quiver of type $\mathbb{T}_{2,2,2,2}$ with subspace orientation (it is called the 4-subspace quiver). its subspace representations are written in the form $(V; U_1, U_2, U_3, U_4)$ and are called subspace-quadruples (here, $U_1, \dots, U_4$ are subspaces of the vector space V).

Proposition. *For any natural number n there is one (and only one) indecomposable subspace-quadruples $(V; U_1, U_2, U_3, U_4)$ with $\dim V = 2n+1$ and $\dim U_i = n+1$ for $1 \leq i \leq 4$.*

Proof, by induction on n. For $n = 0$, we take the representation $M(0) = I(Q)$. Of course, this is the only thin indecomposable representation with full support. Now assume, we have already constructed $M(n-1) = (V; U_1, U_2, U_3, U_4)$ with $\dim V = 2n-1$ and $\dim U_i = n$ for $1 \le i \le 4$, for some $n \ge 1$. Let

$$M(n) = \sigma_4^+\sigma_3^+\sigma_2^+\sigma_1^+\sigma_0^+ M(n-1).$$

Then

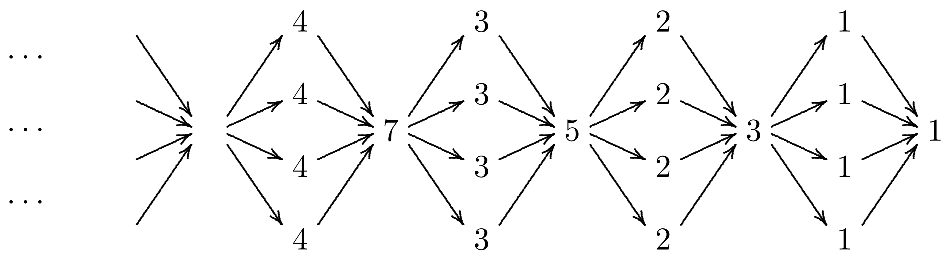
$$\begin{aligned}\dim M(n)_0 &= \dim(\sigma_4^+\sigma_3^+\sigma_2^+\sigma_1^+\sigma_0 M(n-1))_0\\ &= \dim(\sigma_0 M(n-1))_0\\ &= -\dim M(n-1)_0 + \sum\nolimits_{i=1}^{4} \dim M(n-1)_i\\ &= -(2n-1) + 4n = 2n+1,\end{aligned}$$

and

$$\begin{aligned}\dim M(n)_1 &= \dim(\sigma_4^+\sigma_3^+\sigma_2^+\sigma_1^+\sigma_0 M(n-1))_1\\ &= \dim(\sigma_1^+\sigma_0 M(n-1))_1\\ &= -\dim(\sigma_0 M(n-1))_1 + (\sigma_0 M(n-1))_0\\ &= -n + (2n+1) = n+1.\end{aligned}$$

and similarly for $i = 2, 3, 4$.

Remark. As before, these calculations are best remembered using the following picture:



> **Exercise 21.** Why are all indecomposable representations M of Q with $\dim M_0 = 2n+1$ and $\dim M_i = n+1$ for $1 \le i \le 4$ isomorphic? Use induction and the reflection functors!

In particular, we see:

Proposition. *The 4-subspace quiver is representation-infinite.*

A quiver Q is said to be *representation-finite* provided there are only finitely many isomorphism classes of indecomposable representations, otherwise it is called *representation-infinite.*

As we have seen earlier, the quivers of type $\mathbb{A}_n$ as well as those of type $\mathbb{D}_4$ are representation-finite.