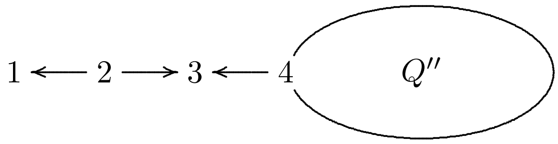
Example 4: Changing the orientation of an arm.

In section 3, we were looking at quivers Q with an arm attached at a vertex m, and we claimed that for classifying the indecomposable representations of Q with $M_m \neq 0$, one may restrict to deal with the subspace orientation of the arm. Using reflection functors, one obtains a proof of this claim, as follows:

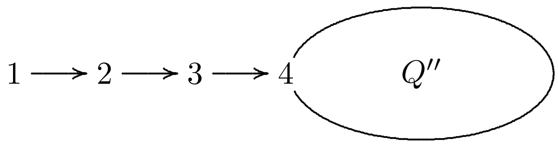
Let us consider the following arm:

Q



Using a sequence Σ of reflection functors σ_y^+ with $y \in \{1, 2, 3\}$, we want to change the orientation in order to deal with the subspace orientation

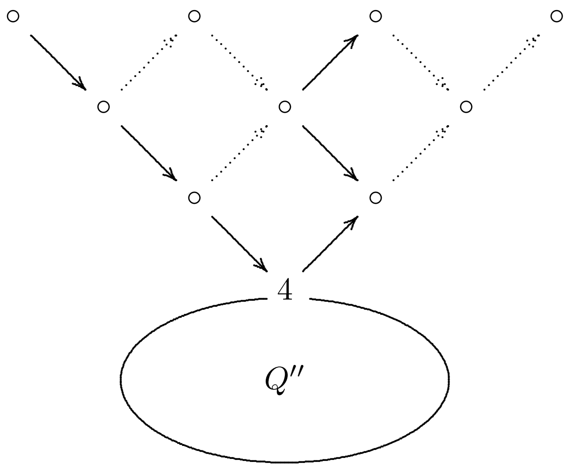
ΣQ



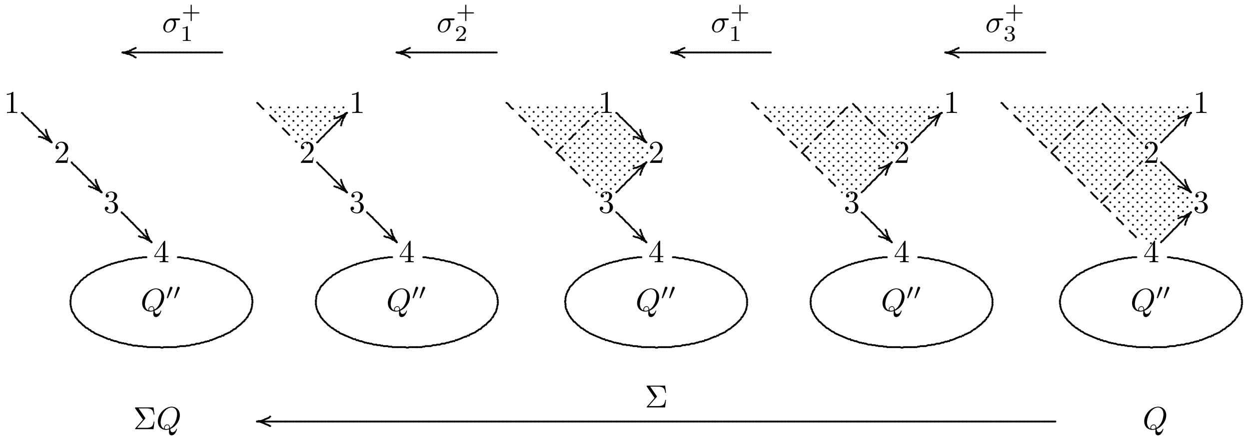
Obviously, we can take

$$\Sigma = \sigma_1^+ \sigma_2^+ \sigma_1^+ \sigma_3^+ ,$$

The effect of the various reflections is best envisioned by considering the quivers Q and ΣQ as subquivers of the following quiver



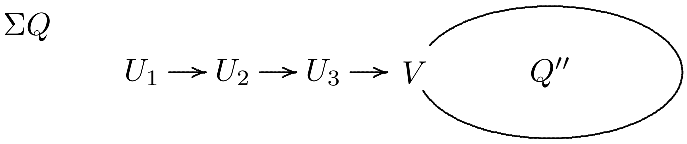
Going from right to left, we decrease in any step the shaded area:



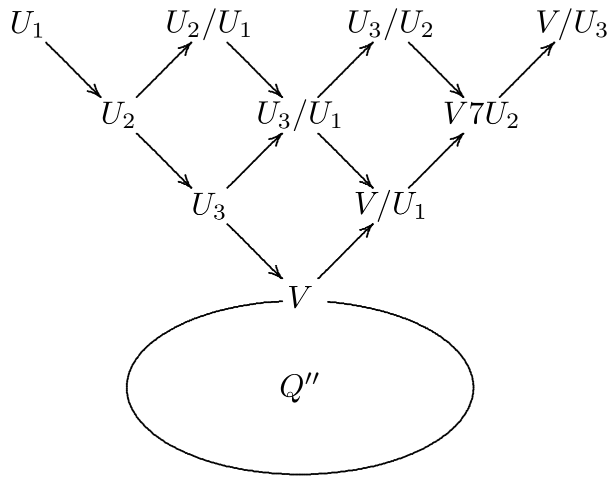
Σ provides a bijection between the indecomposable Q-modules M with $M_4 \neq 0$ and the indecomposable ΣQ-modules N with $N_4 \neq 0$.

Proof: We only have to observe that the representations we are dealing with are indecomposable and non-zero at the vertex x, thus they are y-reduced, for any sink y in question (the sinks y we are working with, are either 1, 2 or 3).

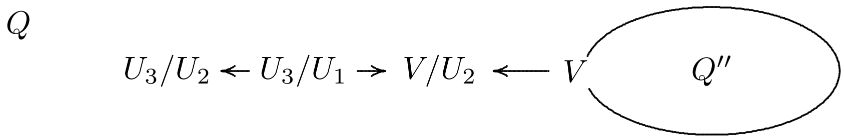
Let us provide more details on this correspondence. Now we start with ΣQ, thus with the subspace orientation of the arm, and we assume that the subspaces $U_1 \subseteq U_2 \subseteq U_3$ of $V = M_4$ have been given:



Here are the vector spaces which we have to use if we apply a sequence of reflection functors σ_y^- for a source sequence using only vertices $y \in \{1, 2, 3\}$ (thus reversing what for example the reflection functor Σ did). Note the maps are inclusion and projection maps with an appropriate choice of signs ± 1 in order to obtain the exact sequences which are needed; but observe that finally the signs do not matter.

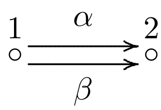


For the quiver Q we started with, we obtain in this way



7. The Kronecker quiver.

This is the quiver with two vertices, say labeled 1 and 2 and two arrows $1 \to 2$, say labeled α and β.



The representations of the Kronecker quiver are often called *Kronecker modules.* Let us repeat the basic definitions for this special case:

A Kronecker module is of the form $M = (M_1, M_2; M_\alpha, M_\beta)$ where M_1 and M_2 are k-spaces, whereas M_α and M_β are k-linear maps $M_1 \to M_2$, instead of M_α and M_β, we usually will write just α and β, respectively. The *dimension vector* $\mathbf{dim}\, M$ of M is by definition the pair $\mathbf{dim}\, M = (\dim M_1, \dim M_2)$. There is the *zero* Kronecker module $0 = (0, 0; 0, 0)$, and there are the simple Kronecker modules $S(1) = (k, 0; 0, 0)$ and $S(2) = (0, k; 0, 0)$. Besides $R_0 = (k, k; 1, 0)$ and $R_\infty = (k, k; 0, 1)$ (which correspond in some sense to the arrows) there are further indecomposable Kronecker modules with dimension vector $(1, 1)$, namely $R_c = (k, k; 1, c)$ with $0 \neq c \in k$, and all the R_c with $c \in k \cup \{\infty\}$ are pairwise non-isomorphic.

Two Kronecker modules M, M' are *isomorphic,* provided there are isomorphisms

$$f_1\ M_1 \to M_1' \quad \text{and} \quad f_2 \colon M_2 \to M_2'$$

of vector spaces such that the following two equalities hold:

$$f_2 M_\alpha = M'_\alpha f_1, \quad \text{and} \quad f_2 M_\beta = M'_\beta f_1.$$

In this case, one calls $f = (f_1, f_2)\colon M \to M'$ an isomorphism (of Kronecker modules) and if such an isomorphism exists, then we write $M \cong M'$.

If two Kronecker modules M, M' are given, the *direct sum* $M \oplus M'$ is defined as follows:

$$M \oplus M' = (M_1 \oplus M'_1, M_2 \oplus M'_2; M_\alpha \oplus M'_\alpha, M_\beta \oplus M'_\beta),$$

If $M^{(i)}$ with $1 \leq i \leq t$ are Kronecker modules, then we write $M^{(1)} \oplus \cdots \oplus M^{(t)}$ or also $\bigoplus_{i=1}^t M^{(i)}$ for the direct sum of these Kronecker modules.

A Kronecker module M is said to be *indecomposable* provided it is non-zero and if for any isomorphism $M \cong M' \oplus M''$ one of M', M'' is zero.

The aim of this section is to classify the indecomposable Kronecker modules, at least in case k is an algebraically closed field.

We will use reflection functors, but here we are in a very special situation. In contrast to most other quivers, the quivers $\sigma_1 Q$ and $\sigma_2 Q$ have the same shape as Q, thus, after renaming the vertices, we can identify them with Q. Also, Q has precisely one sink and precisely one source, thus dealing with the reflection functors, we do not have to mention the vertex used. The definition is as follows:

Let M be a Kronecker module. Define $\sigma^+ M$ by $(\sigma^+ M)_2 = M_1$ and by $(\sigma^+ M)_1$ being given as the kernel appearing in the following exact sequence

$$0 \to (\sigma^+ M)_1 \xrightarrow{\binom{\alpha}{\beta}} M_1 \oplus M_1 \xrightarrow{(\alpha\ \beta)} M_2.$$

Similarly, we define $\sigma^- M$ by $(\sigma^- M)_1 = M_2$ and by $(\sigma^- M)_2$ being exhibited as the cokernel given by the exact sequence

$$M_1 \xrightarrow{\binom{\alpha}{\beta}} M_2 \oplus M_2 \xrightarrow{(\alpha\ \beta)} (\sigma^+ M)_2 \to 0.$$

If we are using these reflection functors iteratively, we write $\sigma^{+t} M = (\sigma^+)^t M$ and $\sigma^{-t} M = (\sigma^-)^t M$.

Proposition 1. *For every natural number $n \geq 0$, there is a unique indecomposable Kronecker module with dimension vector $(n, n+1)$, namely $P_n = \sigma^{-n} S(2)$ and a unique indecomposable Kronecker module with dimension vector $(n+1, n)$, namely $Q_n = \sigma^{+n} S(1)$. The remaining indecomposable Kronecker modules have a dimension vector of the form (n, n) with $n \geq 1$.*

The Kronecker modules P_n are called *preprojective,* the Kronecker modules Q_n *preinjective.* An indecomposable Kronecker module M will be said to be *regular* provided $\dim M_1 = \dim M_2$. In general, direct sums of indecomposable regular Kronecker modules will be said to be regular.

We introduce the *defect* δM of a Kronecker module M as

$$\delta M = \dim M_1 - \dim M_2.$$

Thus, regular Kronecker modules always have defect 0, but the converse is true only for indecomposable Kronecker modules. A typical Kronecker module with zero defect but not being regular is $S(1) \oplus S(2) = (k, k; 0, 0)$.

The theorem asserts, in particular, that $|\delta M| \leq 1$ *for all indecomposable Kronecker modules* and that *the indecomposable Kronecker modules of non-zero defect are uniquely determined by their dimension vectors.*

Proof of proposition 1. Instead of 2-reduced, we say sink-reduced, instead of 1-reduced, we say source-reduced. For a sink-reduced Kronecker module, there is the following exact sequence

$$0 \to (\sigma^+ M)_1 \xrightarrow{\binom{\alpha}{\beta}} M_1 \oplus M_1 \xrightarrow{(\alpha\ \beta)} M_2 \to 0,$$

thus we get a formula for the dimension vectors:

$$\begin{aligned}\mathbf{dim}\, \sigma^+ M &= (\dim(\sigma^+ M)_1, \dim(\sigma^+ M)_2 \\ &= (2 \dim M_1 - \dim M_2, \dim M_1) \\ &= (\dim M_2 + \delta M, \dim M_2 + \delta M) \\ &= \mathbf{dim}\, M + (\delta M)(1, 1)\end{aligned}$$

and similarly, for a sink-reduced Kronecker module M, we get

(a) $$\mathbf{dim}\, \sigma^- M = \mathbf{dim}\, M - (\delta M)(1, 1)$$

Note that this implies

(b) $$\delta(\sigma^+ M) = \delta M = \delta(\sigma^- M)$$

Also, (a) shows: If M is regular (thus automatically sink-reduced and source-reduced), then

$$\mathbf{dim}\, \sigma^+ M = \mathbf{dim}\, M = \mathbf{dim}\, \sigma^- M.$$

Now assume that M is indecomposable and has negative defect. If all the Kronecker modules $\sigma^{+t} M$ with $t \geq 0$ would be sink-reduced, the formula would yield

$$\mathbf{dim}\, \sigma^{+t} M = \mathbf{dim}\, M + t(\delta M)(1, 1)$$

for all t, but this is impossible for $\delta < 0$, since dimension vectors have non-negative coordinates. It follows that there is some minimal t such that $\sigma^{+t}M$ is not sink-reduced, and therefore isomorphic to $S(2)$. But then M is isomorphic to $\sigma^{-t}\sigma^{+t}M = \sigma^{-t}S(2)$. Now $\mathbf{dim}\, S(2) = (0,1)$ and $\delta S(2) = -1$, thus

$$\mathbf{dim}\, M = \mathbf{dim}\, \sigma^{-t}S(2) = \mathbf{dim}\, S(2) - t\delta S(2)(1,1) = (0,1) + (t,t) = (t, t+1).$$

Similarly, if M is indecomposable and has positive defect, then M has to be isomorphic to $\sigma^{+t}S(1)$ for some t and therefore $\mathbf{dim}\, M = (1,0) + t(1,1) = (t+1, t)$.

In this way, we have found all indecomposable Kronecker modules with non-negative defect. The remaining ones are regular, by definition.

Recall that the Jordan blocks are $(n \times n)$-matrices of the form

$$J(\lambda, m) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \dots & & \lambda \end{bmatrix},$$

this is a square matrix with only eigenvalue λ. If $\lambda = 0$, we obtain the nilpotent matrix $N(m) = J(0, m)$.

Proposition 2. *Let M be an indecomposable Kronecker module which is regular. Then either M is isomorphic to $R_\infty[m] = (k^m, k^m; N(m), 1)$ or else M_α is bijective.*

Before we present the proof, some remarks may be of interest.

The essential assertion of Proposition 2 is the following: *If M is an indecomposable Kronecker module which is regular, then at least one of the maps M_α or M_β is bijective.*

There is the following Lemma:

Lemma. *Let $M = (M_1, M_2; M_\alpha, M_\beta)$ be a Kronecker module with M_α bijective. Then M is isomorphic to $M' = (M_1, M_1; 1, M_\alpha^{-1}M_\beta)$;* here, the map $M'_\alpha = 1$ is the identity map of M_1.

Proof of Lemma: Define an isomorphism $f = (f_1, f_2) : M' \to M$ by taking $f_1 = 1$, the identity map of M_1, and $f_2 = M_\alpha$. Note that

$$f_2 M'_\alpha = M_\alpha = M_\alpha f_1, \quad \text{and} \quad f_2 M'_\beta = M_\alpha M_\alpha^{-1} M_\beta = M_\beta = M_\beta f_1.$$

Recall that two endomorphisms ϕ, ϕ' of a vector space V are called *similar,* provided there is an automorphism f of V such that $f\phi = \phi' f$.

Up to isomorphism, the Kronecker modules M in $\mathcal{R}'$ are of the form $(V, V, 1, \phi)$, where $\phi\colon V \to V$ is an endomorphism.

> **Exercise 22.** Let V be a vector space, and ϕ, ϕ' endomorphisms of V. Then $(V, V, 1, \phi)$ is isomorphic to $(V, V, 1, \phi')$ if and only if ϕ and ϕ' are similar.

If k is algebraically closed, then the indecomposable endomorphisms of a vector space V, say $V = k^n$, are classified by the Jordan normal forms. Thus we see:

Proposition 2′. *Let k be an algebraically closed field. Let M be an indecomposable Kronecker module which is regular. Then either M is isomorphic to $R_\infty[m] = (k^m, k^m; N(m), 1)$ with $m \geq 1$, or else to a Kronecker module of the form $(k^n, k^n; 1, J(\lambda, m))$ with $\lambda \in k$ and $m \geq 1$.*

Note that we have to distinguish the Kronecker modules $R_\infty[m] = (k^m, k^m; N(m), 1)$ and $(k^m, k^m; 1, N(m))$; not that they are **not** isomorphic.

In the proof of Proposition 2, we will deal with submodules of regular Kronecker modules. We will need the following criterion:

Submodule characterization of the regular Kronecker modules. *A Kronecker module M with zero defect is regular if and only if $\delta N \leq 0$ for any submodule N of M.*

Proof. First, consider the case of M being regular. If N is a submodule of M with positive defect, Then $(\sigma^+)^r N$ is a submodule of $(\sigma^+)^r M$ for all r. However the dimension vector of $(\sigma^+)^r N$ properly increases with r, whereas $\mathbf{dim}(\sigma^+)^r M = \mathbf{dim}\, M$ for all r.

The reverse implication is trivial: If we assume that all submodules N of M satisfy $\delta N \leq 0$, and N is a direct summand of M, say $M \cong N \oplus N'$, then $0 = \delta M = \delta N + \delta N'$ implies that both $\delta N = 0 = \delta N'$.

Corollary. *Let M be a regular Kronecker module. Any submodule N of M of defect zero is regular.*

Proof of proposition 2. We denote by $\mathcal{R}'$ the class of Kronecker modules M with M_α being bijective; and by $\mathcal{R}_\infty$ the class of Kronecker modules isomorphic to direct sums of Kronecker modules of the form $R_\infty[m]$, these are the indecomposable Kronecker modules M with M_β bijective and $(M_\beta)^{-1} M_\alpha$ nilpotent. Of course, all the Kronecker modules in $\mathcal{R}'$ as well as in $\mathcal{R}_\infty$ are regular. We may reformulate Proposition 2 as follows:

Proposition 2. *Any regular Kronecker module M is the direct sum of a Kronecker module in $\mathcal{R}'$ and a Kronecker module in $\mathcal{R}_\infty$.*

The proof is by induction on the dimension of $M = (M_1, M_2, \alpha, \beta)$. If $\dim M = 0$, nothing has to be shown, since the zero module belongs both to $\mathcal{R}_\infty$ as well as to $\mathcal{R}'$. Thus assume that M is not the zero module. If M_α is invertible, then M belongs to $\mathcal{R}'$. Thus we assume that α is not invertible. Since $\dim M_2 = \dim M_1$, this means that α is not surjective. Let N_2 be a subspace of codimension 1 of M_2 which contains the image of M_α. Let $N_1 = \beta^{-1}(N_2)$, this is a subspace of M_1. Obviously, the map $\overline{\beta}$ induces a map

$$\overline{\beta}\colon M_1/N_1 \to M_2/N_2,$$

and β is injective, since $N_1 = \beta^{-1}(N_2)$. Now, by assumption, $\dim M_2/N_2 = 1$. Note that M is sink-reduced, thus $\alpha M_1 + \beta M_1 = M_2$, in particular βM_1 is not contained in N_2 and therefore $\overline{\beta} \neq 0$. This shows that $\dim M_1/N_1 = 1$. Therefore $\dim N_1 = \dim N_2$. Since by construction $\alpha N_1 \subseteq N_2, \beta N_2 \subseteq N_2$, we have constructed a submodule $N = (N_1, N_2)$ of M of defect zero. By the corollary above, N itself is regular.

By induction, we write $N = N' \oplus N''$ with $N' \in \mathcal{R}'$ and $N'' \in \mathcal{R}_\infty$. We claim that there exists $x \in M_1 \setminus N_1$ such that $\alpha x \in N_2''$. Namely, choose $x \in M_1 \setminus N_1$ and consider αx. Now $\alpha x \in N_2 = N_2' + N_2''$, thus we write $\alpha x = y' + y''$ with $y' \in N_2'$ and $y'' \in N_2''$. Since α is bijective for M', there exists $x' \in N_1'$ with $\alpha(x') = y'$ and therefore $\alpha(x - x') = y'' \in N_2''$. Note that with x also $x - x'$ belongs to $M_1 \setminus N_1$. Thus replace x by $x - x'$.

Now, starting with an element $x \in M_1 \setminus N_1$ such that $\alpha x \in N_2''$, we define a submodule $M'' = (M_1'', M_2'')$ of M as follows:

$$M_1'' = N_1'' + kx, \quad \text{and} \quad M_2'' = N_2'' + k(\beta x)$$

(it is obvious that this is a submodule). Note that

$$(*) \qquad N_1' + M_1'' = N_1' + N_1'' + kx = N_1 + kx = M_1.$$

It follows that $\beta x \notin N_2$, since otherwise (M_1, N_2) would be a submodule of positive defect. In particular, $\beta x \notin N_2''$, and therefore $\beta\colon M_1'' \to M_2''$ is invertible. Also, since $\beta^{-1}\alpha(x) \in N_1''$ and $\beta^{-1}\alpha$ is nilpotent on N_1'', we see that $\beta^{-1}\alpha$ is nilpotent on M_1''. This shows that M'' belongs to $\mathcal{R}_\infty$.

If follows from $(*)$ and $\dim M_1 = \dim N_1' + \dim M_1''$, that $N_1' \cap M_1'' = 0$. Similar, we have

$$N_2' + M_2'' = N_2' + N_2'' + k(\beta x) = N_2 + k(\beta x) = M_2$$

and $\dim M_2 = \dim N_2' + \dim M_2''$, thus $N_2' \cap M_2'' = 0$.

Altogether we see that $M = N' \oplus M''$, where $N' \in \mathcal{R}'$ and $M'' \in \mathcal{R}_\infty$. This completes the proof.

Proposition 2 is an essential part of our discussion of the Kronecker modules. However, the rather clumsy proof presented here disguises some very clear assertions concerning the structure of the category $\mathcal{R}$ of all regular Kronecker modules. First of all, $\mathcal{R}$ is (considered as a category in its own right) an “abelian” category (this is an immediate consequence of the submodule characterization of regular Kronecker modules), so that the simple objects of $\mathcal{R}$ have to be of interest. Note that these are the non-zero regular Kronecker modules with zero as the only proper regular submodule, typical examples are the Kronecker modules R_c with $c \in k \cup \infty$. What is shown in our proof is mainly the following: If S is a simple object in $\mathcal{R}$, and not isomorphic to R_∞, then $\mathrm{Ext}^1(R_\infty, S) = 0 = \mathrm{Ext}^1(S, R_\infty)$. It is an obvious consequence of the vanishing of these Ext-groups that any regular Kronecker modules decomposes into the direct sum of a Kronecker module with a filtration with all factors being R_∞ (this part belongs to $\mathcal{R}_\infty$) and a Kronecker module with a filtration where all factors are simple regular Kronecker modules and none is isomorphic to R_∞ (this part belongs to $\mathcal{R}'$).

Proposition 3. *The Kronecker module P_n is isomorphic to $(k^n, k^{n+1}, \alpha, \beta)$, where α, β are obtained by adding to the $(n \times n)$-identity matrix one additional zero row: for α the additional row is added as the last row, for β as the first row:*

$$\alpha = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

The Kronecker module Q_n is isomorphic to $(k^{n+1}, k^n, \alpha, \beta)$, where

$$\alpha = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}$$

(here, α, β are obtained by adding to the $(n \times n)$-identity matrix one additional zero column: for α the additional column is added as the first column, for β as the last column).

Proof of proposition 3, using induction. We start with $(k^{t-1}, k^t; \alpha, \beta)$ with α, β being matrices as specified above, thus

$$\begin{bmatrix} \alpha \\ \cdots\cdots \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \\ \hdashline 0 & \cdots & 0 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

(here, the upper and the lower blocks both are the $(t-1) \times (t-1)$-identity matrix, altogether there are $2t$ rows and $t-1$ columns. We have to determine the cokernel q of the corresponding map $k^{t-1} \to k^{2t}$ given by this matrix. The cokernel q is the map $k^{2t} \to k^{t+1}$

with the following matrix

$$\begin{bmatrix} 0 & \cdots & 0 & -1 & & 0 \\ 1 & & 0 & & \ddots & \\ & \ddots & & 0 & & -1 \\ 0 & & 1 & 0 & & 0 \end{bmatrix}$$

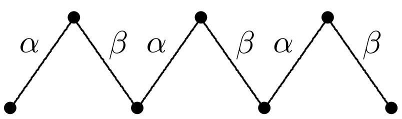
(here, on the left side one deals with a zero row above the $t \times t$-identity matrix, on the right side one deals with a zero row below the negative of the $t \times t$-identity matrix). It is sufficient to check that the composition of the matrices is zero and that the new matrix has rank $t+1$ (the latter is seen by looking just at the first $t+1$ columns).

Now the matrix for the cokernel is not yet what we want — we have to construct new bases of k^t and k^{t+1} so that the maps $\alpha = qi_1, \beta = qi_2$ are given by the required matrices: we have to renumber the given bases and multiply any second element with -1. Here are the corresponding matrix calculations:

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \\ \iddots & & & \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \\ \iddots & & & \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & \end{bmatrix} \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \\ 0 & \cdots & 0 \end{bmatrix}$$

String modules and band modules. Some Kronecker modules can be visualized quite well, namely the so called string modules. For example, consider the following labeled graph:



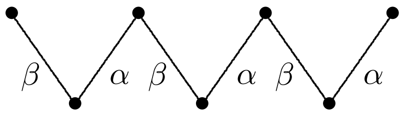
P_3

This means the following: The upper bullets symbolize the basis vectors of the vector space V, the lower ones the basis vectors of the vector space W, the edges are considered as arrows pointing downwards. There is just one arrow labeled α starting at the first upper bullet: this means that this vector is sent under α to the basis vector represented by the bullet where the arrow ends (here: the first bullet in the lower row). If we label the basis vectors of V and of W from left to right, we see that we obtain the following matrices:

$$\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This Kronecker module is just $P_3 = (k^3, k^4, \alpha, \beta)$.

Similarly, consider the following labeled graph:

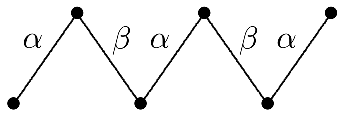


Q_3

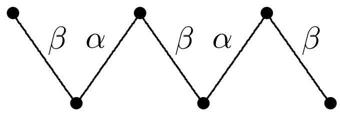
Again, the upper bullets symbolize the basis vectors of the vector space V, the lower ones the basis vectors of the vector space W. Since there is no arrow labeled α starting at the first upper bullet, the corresponding vector is sent under α to zero, and so on. Thus, here we deal with the Kronecker module

$$Q_3 = (k^4, k^3; \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}).$$

There are two other sequences of string modules, those of the form $R_0[m]$ and $R_\infty[m]$, and with $m \geq 1$. For example, for $m = 3$, these are given by the graphs



$R_0[3]$



$R_\infty[3]$

The general definition is as follows:

$$R_0[m] = (k^m, k^m, I_m, N(m)), \quad R_\infty[m] = (k^m, k^m, N(m), I_m),$$

where I_m is the $(m \times m)$-identity matrix and $N(m)$ is the $(m \times m)$-Jordan block with eigenvalue 0.

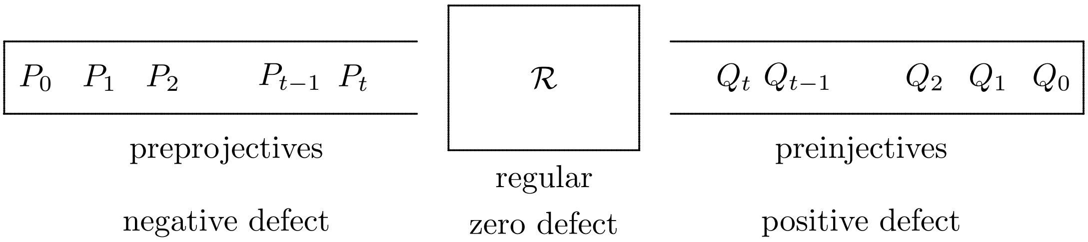
In the structure theorem above, the string modules $R_0[m]$ did not play a role, they are just special elements of $\mathcal{R}'$, those with nilpotent Jordan blocks.

Here is the formal definition: An indecomposable Kronecker module M is said to be a *string module* provided M it is isomorphic to $P_n, Q_n, R_0[m]$ of $R_\infty[m]$ for some $n \geq 0$ or $m \geq 1$. An Kronecker module M is called a *band module* provided both maps M_α, M_β are bijective. The essential result concerning Kronecker modules is the following:

Any indecomposable Kronecker module is either a string module or a band module.

Exercise 23. Here is the outline of a proof: As before, let $\mathcal{R}_\infty$ be the class of Kronecker modules isomorphic to direct sums of Kronecker modules of the form $R_\infty[m]$. Similarly, let $\mathcal{R}_0$ be the class of Kronecker modules isomorphic to direct sums of Kronecker modules of the form $R_0[m]$. Show that any regular Kronecker module is the direct sum of Kronecker modules in $\mathcal{R}_\infty$ and in $\mathcal{R}_0$ and a Kronecker module M which is a band module.

The structure of the category of Kronecker modules: We denote the class of regular Kronecker modules by $\mathcal{R}$. The picture which one always has to have in the mind, is the following:



here, the action of σ^+ on the preprojective part as well as on the preinjective part is the shift to the left, thus

$$\begin{aligned} \sigma^+(P_0) &= 0, \\ \sigma^+(P_t) &= P_{t-1} \quad \text{for} \quad t \geq 1 \\ \sigma^+(Q_t) &= Q_{t+1} \quad \text{for} \quad t \geq 0 \end{aligned}$$

whereas σ^- is the corresponding shift to the right. On the regular part, both σ^+ and σ^- provide permutations of the isomorphism classes.

Actually, this picture describes the global structure of the category of Kronecker modules: non-zero homomorphisms go from left to right. To be precise: there are no non-zero homomorphisms from a regular or a preinjective Kronecker module to a preprojective Kronecker module, and also none from a regular to a preprojective. Also inside the preprojective part, as well as inside the preinjective part, non-zero homomorphisms only go from left to right: If $\operatorname{Hom}(P_n, P_m) \neq 0$, then $n \leq m$, if $\operatorname{Hom}(Q_n, Q_m) \neq 0$, then $n \geq m$.

Linear relations on a vector space. As we have mentioned, the concept of a "relation" is very basic in mathematics. Modern mathematics is usually formulated in terms of sets and maps between sets, but actually the (set-theoretical) maps are defined as special relations. Recall that a *relation* between two sets W_1 and W_2 is just a subset of $W_1 \times W_2$, and the graph $\Gamma(f)$ of a (set-theoretical) map $f\colon W_1 \to W_2$ is such a relation. Of course, special attention deserve endo-maps (these are such maps with $W_1 = W_2$), the graph of an endo-map $f\colon W \to W$ is a subset of $W \times W$, and an arbitrary subset of $W \times W$ (with W a set) is called an endo-relation, or just a *relation on the set* W.

Similarly, in the linear world, we should look not only at linear transformations, but more generally at "linear relations", a *linear relation* between two vector spaces W_1, W_2 is by definition a subspace U of $W_1 \oplus W_2$. And a *linear relation on a vector space* W is by definition just a subspace of $V \oplus V$.

Proposition. *The linear relations on vector spaces are nothing else than the source-reduced Kronecker modules.*