
THE HODGE AND TATE CONJECTURES FOR HYPER-KÄHLER SIXFOLDS OF GENERALIZED KUMMER TYPE

by

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Abstract. — We prove the conjectures of Hodge and Tate for any six-dimensional hyper-Kähler variety that is deformation equivalent to a generalized Kummer variety.

1. Introduction

The Hodge conjecture predicts that for any smooth and projective variety the subspace of the rational cohomology generated by the fundamental classes of subvarieties coincides with the space of Hodge classes. See [12] for the precise statement and some background. The conjecture was posed by Hodge at the ICM of 1950 as a generalization of Lefschetz (1,1) theorem, which is sufficient to treat curves, surfaces and threefolds. It is arguably one of the most outstanding problems in complex algebraic geometry.

General evidence supporting the Hodge conjecture is scarce, and it has become important to verify it in special cases. Known results are discussed in the book [29]. The conjecture holds for flag varieties, rationally connected 4-folds [6], some Fermat hypersurfaces [43], and in many degrees for unitary Shimura varieties [5].

The Hodge conjecture has been extensively studied for abelian varieties, see also the survey [21]. For instance, it has been proven for powers of elliptic curves, surfaces or threefolds [35], and simple abelian varieties of prime dimension [45]. Most of the results available deal with special abelian varieties on which all Hodge classes are obtained from those in degree 2 via cup-product. Starting from dimension 4, no locally complete family of abelian varieties satisfying the Hodge conjecture is known.

Another important class of varieties with trivial canonical bundle is that of hyper-Kähler varieties [3, 25]. In this text, a hyper-Kähler manifold is a compact and simply connected Kähler manifold whose $H^{2,0}$ is spanned by the class of a symplectic form. Hyper-Kähler manifolds have even dimension; in dimension 2, they are K3 surfaces. The other known

examples fall into the deformation types commonly referred to as $K3^{[n]}$, Kum^n , OG10, and OG6. The first two series, found by Beauville [3], provide $2n$ -dimensional examples for every $n \geq 2$, while the other two are O’Grady’s deformation types found in [37] and [38], in dimension 10 and 6, respectively.

Exciting progress on the Hodge conjecture for hyper-Kähler varieties came from the works of Markman [31], [32], [33], and Buskin [9]. They are able to deform certain algebraic cycles on these varieties using Verbitsky’s hyperholomorphic sheaves [50], obtaining remarkable cases of the Hodge conjecture. Related articles are [11] and [18].

The main result proven in the present paper is the following.

Theorem 1.1. — *Let K be a projective manifold of Kum^3 -type. Then the Hodge conjecture holds for K , i.e., any cohomology class in $H^{i,i}(K) \cap H^{2i}(K, \mathbb{Q})$ is a \mathbb{Q} -linear combination of fundamental classes of subvarieties of K , for any i .*

While finalizing this work, we realized that we can combine some of our ideas with recent results of Foster [18] and Varesco [48] to prove the Hodge conjecture for the fourfolds of Kum^2 -type as well. This will appear in a separate article.

We can construct examples of hyper-Kähler varieties satisfying the Hodge conjecture for each of the known deformation types, using [10], [17] and [16]. However, no locally complete family of hyper-Kähler varieties satisfying the conjecture was known besides $K3$ surfaces. In fact, it appears that locally complete families of (minimal, non-rigid) varieties with $h^{4,0} \neq 0$ for which the conjecture can be proven are very hard to find.

The Hodge conjecture is paralleled in the arithmetic setting by the Tate conjecture. Let k be a finitely generated field of characteristic 0 with algebraic closure \bar{k} , and let X/k be a smooth and projective variety. Given a prime number ℓ , the absolute Galois group of k acts on the ℓ -adic étale cohomology of $X_{\bar{k}}$, and the Tate conjecture predicts that the Galois invariants in $H_{\text{ét}}^{2i}(X_{\bar{k}}, \mathbb{Q}_{\ell}(i))$ are \mathbb{Q}_{ℓ} -linear combinations of fundamental classes of k -subvarieties of X . The strong Tate conjecture further includes the statement that these Galois representations are semisimple. See [46] for more information. The Tate conjecture is open already for divisors, in general. At least, it holds for divisors on abelian varieties [13], hyper-Kähler varieties [1] and many varieties with $h^{2,0} = 1$ [34].

Theorem 1.2. — *Let $k \subset \mathbb{C}$ be a finitely generated field, and assume that K is a smooth and projective variety over k whose base-change $K_{\mathbb{C}}$ is a hyper-Kähler variety of Kum^3 -type. Then the strong Tate conjecture holds for K , for any prime number ℓ .*

Theorem 1.2 will follow once Theorem 1.1 is proven. In fact, the conjectures of Hodge and Tate are equivalent for any hyper-Kähler variety of known deformation type, by previous work of the author [14], Soldatenkov [44] and the author with Fu and Zhang [17].

Outline. — The manifolds of Kum^n -type are by definition deformation equivalent to Beauville’s generalized Kummer variety $K^n(A)$ on an abelian surface A introduced in [3]. The Hodge conjecture holds for $K^n(A)$ by the results of de Cataldo-Migliorini [10] and because it holds for any power of the abelian surface. However, deformations of generalized Kummer varieties are no longer such, and, in general, are not related to an abelian surface.

In dimension 6, we have shown in [15] that any projective manifold K of Kum^3 -type is naturally associated with a K3 surface S_K . They are connected by the existence of a projective manifold Y_K of $\text{K3}^{[3]}$ -type which is at the same time a moduli space of sheaves on S_K and a crepant resolution of the quotient K/G , where $G \cong (\mathbb{Z}/2\mathbb{Z})^5$ is a group of involutions of K that act trivially on its second cohomology group.

We proved in [15, Corollary 5.8] the Hodge conjecture for all powers of the K3 surfaces S_K arising from the above construction. This was obtained as consequence of the ground-breaking work of O’Grady [40] and Markman [32], with complements by Voisin [52], on varieties of Kum^n -type, and using a decisive input due to Varesco [47].

Exploiting the aforementioned relation between K and S_K , we deduce that the Hodge conjecture holds for the G -invariant part of the cohomology algebra of K . The remaining Hodge classes have the special property of staying Hodge on any deformation of K , and are called “canonical” Hodge classes (Definition 2.1). We prove in Theorem 7.1 that these are all obtained as linear combinations of fundamental classes of certain natural subvarieties arising from the study of the fixed loci of automorphisms in G . Thus, they are algebraic, and we complete the proof of the Hodge conjecture for K . Our study of canonical Hodge classes is influenced by the work of Hassett and Tschinkel on manifolds of Kum^2 -type [24].

Notation and conventions. — We work over the field of the complex numbers. A family of complex manifolds or algebraic varieties will be a smooth (submersive) and proper morphism $\mathcal{X} \rightarrow B$ over a smooth and connected base B . If X is a compact complex manifold, we say that a cohomology class $\alpha \in H^\bullet(X, \mathbb{Q})$ is analytic if it belongs to the subalgebra of $H^\bullet(X, \mathbb{Q})$ generated by Chern classes of coherent sheaves on X ; if X is projective, then analytic classes are the same as algebraic ones, see [51]. Tate twists in Hodge theory will be ignored throughout this text.

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2. Cohomology of Kum³-manifolds

In this section we recall several known properties of the cohomology of hyper-Kähler manifolds (of dimension > 2). We then discuss in detail the case of Kum³-type. As general references about hyper-Kähler manifolds we recommend [3] and [25].

Definition 2.1. — *Let X be a hyper-Kähler manifold. A canonical Hodge class on X is a cohomology class which stays Hodge on every deformation of X .*

Any polynomial expression in the Chern classes of X gives a canonical Hodge class. Another example comes from the Beauville-Bogomolov form q_X on $H^2(X, \mathbb{Q})$: identifying $H^2(X, \mathbb{Q})$ with its dual by means of this form, the pairing determines a class \bar{q}_X in $\text{Sym}^2(H^2(X, \mathbb{Q}))$; the latter space embeds in $H^4(X, \mathbb{Q})$ via cup-product by a theorem of Verbitsky [49], and we obtain a canonical Hodge class $\bar{q} = \bar{q}_X \in H^4(X, \mathbb{Q})$.

Remark 2.2. — If a_1, \dots, a_m is an orthogonal basis of $H^2(X, \mathbb{Q})$, we have

$$\bar{q}_X = \sum_{i=1}^m \frac{a_i^2}{q_X(a_i, a_i)} \in H^4(X, \mathbb{Q}).$$

The following result of Fujiki [19] is very useful: if $\omega \in H^{4k}(X, \mathbb{Q})$ is a canonical Hodge class, there exists $C(\omega) \in \mathbb{Q}$ (the generalized Fujiki constant of ω) such that

$$(1) \quad \int_X \omega \cdot \gamma^{\dim X - 2k} = C(\omega) \cdot q_X(\gamma, \gamma)^{\frac{1}{2} \dim X - k},$$

for any $\gamma \in H^2(X, \mathbb{Q})$. The Fujiki constant of X is by definition $C(1)$.

Another important tool is the action of the LLV-Lie algebra $\mathfrak{g}(X)$ on the cohomology of X , studied first by Verbitsky [49] and Looijenga-Lunts [30]. For $\gamma \in H^2(X, \mathbb{Q})$, consider the endomorphism e_γ of $H^\bullet(X, \mathbb{Q})$ of degree 2 defined by $e_\gamma(\beta) = \gamma \cdot \beta$. It turns out that, whenever $q_X(\gamma, \gamma) \neq 0$, there exists an adjoint endomorphism λ_γ of $H^\bullet(X, \mathbb{Q})$ of degree -2 such that $e_\gamma, \theta, \lambda_\gamma$ generate a copy of \mathfrak{sl}_2 inside $\mathfrak{gl}(H^\bullet(X, \mathbb{Q}))$, where the action of θ on each $H^k(X, \mathbb{Q})$ is multiplication by $k - \dim X$. Then $\mathfrak{g}(X)$ is defined as the Lie subalgebra of $\mathfrak{gl}(X, \mathbb{Q})$ generated by all \mathfrak{sl}_2 -triples as above. By definition, this Lie algebra only depends on the topology of X .

For any hyper-Kähler manifold X we have

$$\mathfrak{g}(X) = \mathfrak{so}(H^2(X, \mathbb{Q}) \oplus U),$$

where U denotes a hyperbolic plane; the lattice $H^2(X, \mathbb{Z}) \oplus U$ is called the Mukai lattice of X . The commutator $\mathfrak{g}^0(X)$ of θ is isomorphic to $\mathbb{Q} \cdot \theta \oplus \mathfrak{so}(H^2(X, \mathbb{Q}))$ and it acts on the cohomology preserving the degree. Moreover, the LLV-Lie algebra is compatible with Hodge theory, meaning that any $\mathfrak{so}(H^2(X, \mathbb{Q}))$ -submodule of the cohomology of X is

automatically a Hodge substructure; the canonical Hodge classes on X correspond to the trivial subrepresentations of $\mathfrak{so}(H^2(X, \mathbb{Q}))$ on the cohomology. They form a subalgebra, because the LLV-action of $\mathfrak{so}(H^2(X, \mathbb{Q}))$ on the cohomology algebra is via derivations.

To close this summary, we introduce the group

$$\mathrm{Aut}_0(X) := \{f \in \mathrm{Aut}(X) \mid f^*_{|H^2(X, \mathbb{Z})} = \mathrm{id}\}$$

of automorphisms of X acting trivially on its second cohomology. It is a finite group ([25, Proposition 9.1]) whose action on the cohomology commutes with the LLV-Lie algebra. The key property of $\mathrm{Aut}_0(X)$ is that it is deformation-invariant, by [24, Theorem 2.1].

2.1. The generalized Kummer sixfold. — By definition, Kum^3 -manifolds are deformations of the generalized Kummer sixfolds $K^3(A)$, where A is an abelian surface or a 2-dimensional complex torus. These are constructed as follows in [3, §7]. Let $A^{[4]}$ be the Douady space of lenght 4 subspaces of dimension 0 on A (if A is projective, $A^{[4]}$ is the Hilbert scheme of 0-dimensional subschemes of lenght 4 on A), which is a crepant resolution of the fourth symmetric product $A^{(4)}$ of A . The resolution map $\nu: A^{[4]} \rightarrow A^{(4)}$, commonly called the Hilbert-Chow morphism, sends $\xi \in A^{[4]}$ to its support. Consider the summation map $\Sigma: A^{(4)} \rightarrow A$, which sends $(a_1, a_2, a_3, a_4) \in A^{(4)}$ to $\sum_i a_i \in A$. Then the composition $\Sigma \circ \nu: A^{[4]} \rightarrow A$ is an isotrivial fibration and

$$K^3(A) := (\Sigma \circ \nu)^{-1}(0)$$

is a hyper-Kähler manifold, which is a crepant resolution of $A_0^{(4)} := \Sigma^{-1}(0)$.

The Hodge numbers of Kum^3 -manifolds have been determined in [22]:

$$(2) \quad \begin{array}{cccccccc} & & & & 1 & & & \\ & & & & & & & \\ & & & & 0 & & 0 & \\ & & & & & & & \\ & & & & 1 & & 5 & & 1 \\ & & & & 0 & & 4 & & 4 & & 0 \\ & & & & 1 & & 6 & & 37 & & 6 & & 1 \\ & & & & 0 & & 4 & & 24 & & 24 & & 4 & & 0 \\ & & & & 1 & & 5 & & 37 & & 372 & & 37 & & 5 & & 1 \end{array}$$

2.2. The LLV-decomposition. — Denoting by V the (rational) Mukai lattice of a Kum^3 -manifold K , the decomposition of the cohomology into isotypical $\mathfrak{so}(V)$ -components is given by Green-Kim-Laza-Robles in [23, Corollary 3.6]:

$$(3) \quad H^\bullet(K, \mathbb{Q}) = V_{(3)} \oplus V_{(1,1)} \oplus 16V \oplus 240\mathbb{Q} \oplus V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)},$$

where $V_{(3)}$ is the subalgebra generated by the second cohomology, the summand $V_{(1,1)}$ is isomorphic to $\bigwedge^2 V$, and the last summand is the odd cohomology, which is an irreducible

spinor representation. This is in fact a decomposition into submodules for the action of $H^2(K, \mathbb{Q})$ via cup-product. The components have ranks:

$$(4) \quad \begin{array}{|c|c|c|c|c|} \hline V_{(3)} & V_{(1,1)} & 16V & 240\mathbb{Q} & V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)} \\ \hline 156 & 36 & 144 & 240 & 128 \\ \hline \end{array}$$

Denoting by \bar{V} the summand $H^2(K, \mathbb{Q})$ of V , the $\mathfrak{so}(\bar{V})$ -representation on the cohomology is given by (we only display the cohomology in even degrees)

$$(5) \quad \begin{array}{|c|c|c|c|c|c|} \hline \text{deg} & V_{(3)} & V_{(1,1)} & 16V & 240\mathbb{R} & \text{rk} \\ \hline 0 & \mathbb{Q} & 0 & 0 & 0 & 1 \\ \hline 2 & \bar{V} & 0 & 0 & 0 & 7 \\ \hline 4 & \text{Sym}^2(\bar{V}) & \bar{V} & 16\mathbb{Q} & 0 & 51 \\ \hline 6 & \text{Sym}^3(\bar{V}) & \bigwedge^2 \bar{V} \oplus \mathbb{Q} & 16\bar{V} & 240\mathbb{Q} & 458 \\ \hline 8 & \text{Sym}^2(\bar{V}) & \bar{V} & 16\mathbb{Q} & 0 & 51 \\ \hline 10 & \bar{V} & 0 & 0 & 0 & 7 \\ \hline 12 & \mathbb{Q} & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

This also describes the Hodge structure on $H^{2\bullet}(K, \mathbb{Q})$ in terms of that on $H^2(K, \mathbb{Q})$.

2.3. Generalized Fujiki constants. — The following table shows the generalized Fujiki constants for the monomials in \bar{q} and the Chern classes on Kum³-manifolds.

$$(6) \quad \begin{array}{|c|c|c|c|} \hline C(1) = 60 & C(\bar{q}) = 132 & C(\bar{q}^2) = 396 & C(\bar{q}^3) = 2772 \\ \hline C(c_2) = 288 & C(\bar{q} \cdot c_2) = 864 & C(\bar{q}^2 \cdot c_2) = 6048 & — \\ \hline C(c_2^2) = 1920 & C(\bar{q} \cdot c_2^2) = 13440 & C(c_4) = 480 & C(\bar{q} \cdot c_4) = 3360 \\ \hline C(c_2^3) = 30208 & — & C(c_2 c_4) = 6784 & C(c_6) = 448 \\ \hline \end{array}$$

The constant $C(1)$ can be found in [42], while the generalized Fujiki constants for monomials in the Chern classes of degree at most 4 and their products with powers of \bar{q} can be calculated via the results of Beckmann and Song [4, Corollary 2.7, Proposition 2.4, Example 2.13]. The Chern numbers have been determined by Nieper-Wisskirchen in [36].

2.4. Canonical Hodge classes. — There are however many more canonical Hodge classes on Kum³-manifolds than those which are polynomials in \bar{q} and the Chern classes. By the LLV-decomposition (5), the space of canonical Hodge classes in degree 4 has dimension 17, spanned by \bar{q} and the copy of $16\mathbb{Q}$ coming from the third summand in (3), while canonical Hodge classes in the middle cohomology form a vector space of dimension 241, spanned by the unique (up to multiples) canonical Hodge class in the LLV-submodule $V_{(1,1)}$ and the summand $240\mathbb{Q}$ of (3). Canonical Hodge classes in degree 8 form another 17-dimensional subspace.

2.5. Automorphisms trivial on the second cohomology. — Let A_k denote the subset of points of order k of a 2-dimensional complex torus A . By [7], we have

$$\mathrm{Aut}_0(K^3(A)) = A_4 \rtimes \langle -1 \rangle,$$

where the second factor acts on the first as the inverse. The action on $K^3(A)$ is induced by the natural action on $A^{[4]}$; by [41], the involution -1 acts as multiplication by -1 on the third cohomology. There are two subgroups which will play an important role.

- Let $\Gamma := A_4 \subset \mathrm{Aut}_0(K^3(A))$; it is the subgroup of automorphisms which act trivially on $H^3(K^3(A), \mathbb{Z})$ (and hence on the whole odd cohomology, since it is an irreducible LLV-module). We have $\Gamma \cong (\mathbb{Z}/4\mathbb{Z})^4$.
- Let $G := A_2 \times \langle -1 \rangle \subset \mathrm{Aut}_0(K^3(A))$; note that $G \cong (\mathbb{Z}/2\mathbb{Z})^5$ is abelian and intersects Γ in A_2 .

Thanks to the deformation-invariance of the automorphisms trivial on the second cohomology, for any K of Kum³-type we have $\mathrm{Aut}_0(K) \cong (\mathbb{Z}/4\mathbb{Z})^4 \rtimes \mathbb{Z}/2\mathbb{Z}$, and the subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^5$ and $\Gamma \cong (\mathbb{Z}/4\mathbb{Z})^4$.

Remark 2.3. — For an arbitrary K of Kum³-type, the group G is canonically defined as the subgroup of $\mathrm{Aut}_0(K)$ generated by the automorphisms g such that the fixed locus K^g has a 4-dimensional component, see [15, §2]. We will eventually prove that $G \subset \mathrm{Aut}_0(K)$ is the subgroup of automorphisms acting trivially on $H^4(K, \mathbb{Z})$, see Remark 6.2.

Lemma 2.4. — *For any K of Kum³-type, the space of Γ -invariants in $H^\bullet(K, \mathbb{Q})$ is a LLV-submodule*

$$(H^\bullet(K, \mathbb{Q}))^\Gamma \cong V_{(3)} \oplus V_{(1,1)} \oplus V \oplus V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}.$$

Moreover:

- (i) the space of canonical Hodge classes in $H^4(K, \mathbb{Q})^\Gamma$ is two-dimensional, spanned by the linearly independent classes c_2 and \bar{q}_K ;
- (ii) the space of canonical Hodge classes in $H^6(K, \mathbb{Q})^\Gamma$ is one-dimensional, and it is contained in the LLV-summand $V_{(1,1)}$.

Proof. — It suffices to work with the generalized Kummer variety $K^3(A)$ on an abelian surface A . Consider the LLV-decomposition (3):

$$H^\bullet(K^3(A), \mathbb{Q}) = V_{(3)} \oplus V_{(1,1)} \oplus 16V \oplus 240\mathbb{Q} \oplus V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}.$$

The group Γ acts trivially on $V_{(3)}$ and $V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}$. In degree 4, the decomposition becomes

$$(7) \quad H^4(K^3(A), \mathbb{Q}) = \mathrm{Sym}^2(H^2(K^3(A), \mathbb{Q})) \oplus H^2(A, \mathbb{Q}) \oplus 16\mathbb{Q},$$

and Γ acts trivially on the first summand. By [3, pp. 769], the quotient of $K^3(A) \times A$ by the diagonal action of $\Gamma = A_4$ is $A^{[4]}$. Since the group Γ acts trivially on the cohomology of A as well as on the cohomology groups $H^2(K^3(A), \mathbb{Q})$ and $H^3(K^3(A), \mathbb{Q})$, we have

$$\begin{aligned} H^4(A^{[4]}, \mathbb{Q}) = & H^4(K^3(A), \mathbb{Q})^\Gamma \oplus (H^3(K^3(A), \mathbb{Q}) \otimes H^1(A, \mathbb{Q})) \\ & \oplus (H^2(K^3(A), \mathbb{Q}) \otimes H^2(A, \mathbb{Q})) \oplus H^4(A, \mathbb{Q}). \end{aligned}$$

By [22], the Hodge numbers of the left hand side are

$$h^{4,0}(A^{[4]}) = 2, \quad h^{3,1}(A^{[4]}) = 23, \quad h^{2,2}(A^{[4]}) = 61.$$

Calculating the Hodge numbers of the last 3 summands at the right hand side with (2), we deduce that

$$H^4(K^3(A), \mathbb{Q})^\Gamma = \text{Sym}^2(H^2(K, \mathbb{Q})) \oplus T$$

for an effective Hodge structure T of weight 4 with Hodge numbers $(0, 1, 6, 1, 0)$. Since Γ respects the LLV-decomposition (7), we find that $T \cong H^2(K, \mathbb{Q}) \oplus \mathbb{Q}$. This implies that the summand $V_{(1,1)}$ of (3) is contained in $H^\bullet(K^3(A), \mathbb{Q})^\Gamma$, and that $(16V)^\Gamma \cong V$.

Next, the results of [22] give $b_6(A^{[4]}) = 592$ and we write

$$H^6(A^{[4]}, \mathbb{Q}) = H^6(K^3(A), \mathbb{Q})^\Gamma \oplus R$$

where

$$\begin{aligned} R = & (H^5(K^3(A), \mathbb{Q}) \otimes H^1(A, \mathbb{Q})) \oplus (H^4(K^3(A), \mathbb{Q})^\Gamma \otimes H^2(A, \mathbb{Q})) \\ & \oplus (H^3(K^3(A), \mathbb{Q}) \otimes H^3(A, \mathbb{Q})) \oplus H^2(K^3(A), \mathbb{Q}). \end{aligned}$$

We have just determined the invariants for the action of Γ on $H^4(K^3(A), \mathbb{Q})$, and we calculate that $\dim_{\mathbb{Q}}(R) = 479$, so that $H^6(K^3(A), \mathbb{Q})^\Gamma$ has rank 113. Table (5) now gives

$$H^6(K^3(A), \mathbb{Q})^\Gamma \cong \text{Sym}^3(H^2(K^3(A), \mathbb{Q})) \oplus \left(\bigwedge^2 H^2(K^3(A), \mathbb{Q}) \oplus \mathbb{Q} \right) \oplus H^2(K^3(A), \mathbb{Q}) \oplus S,$$

where S is the space of Γ -invariants on the LLV-trivial summand $240\mathbb{Q}$. But then

$$\dim_{\mathbb{Q}}(S) = 113 - 84 - 21 - 1 - 7 = 0.$$

We have thus shown that

$$(H^\bullet(K, \mathbb{Q}))^\Gamma \cong V_{(3)} \oplus V_{(1,1)} \oplus V \oplus V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}.$$

Therefore, the only (up to multiples) Γ -invariant canonical Hodge class in $H^6(K^3(A), \mathbb{Q})$ is the one which comes from the summand $V_{(1,1)}$ in the LLV-decomposition, which proves (ii). The canonical Hodge classes in $\text{Sym}^2(H^2(K^3(A), \mathbb{Q}))$ are all multiples of $\bar{q}_{K^3(A)}$. It follows that the Γ -invariant canonical Hodge classes in $H^4(K^3(A), \mathbb{Q})$ form

a 2-dimensional vector space, which clearly contains $\bar{q}_{K^3(A)}$ and c_2 . These classes are linearly independent by [4, Example 2.13], and (i) follows. \square

2.6. A basis for Γ -invariant canonical Hodge classes. — It will be convenient for our computations to define

$$(8) \quad z := c_2 - \frac{C(c_2)}{C(\bar{q})}\bar{q} = c_2 - \frac{24}{11}\bar{q} \in H^4(K, \mathbb{Q}).$$

The point is that z is a canonical Hodge class and $C(z) = 0$; it follows that $\bar{q}^2 \cdot z = 0$. The classes \bar{q} and z form a basis of Γ -invariant canonical Hodge classes in degree 4.

Lemma 2.5. — *The classes \bar{q}^2 and $\bar{q} \cdot z$ are linearly independent and give a basis of Γ -invariant canonical Hodge classes in $H^8(K, \mathbb{Q})$. We have:*

$C(z^2) = \frac{384}{11}$	$C(\bar{q} \cdot z^2) = \frac{2688}{11}$	$C(z^3) = -\frac{22016}{121}$
$z^2 = \frac{32}{363}\bar{q}^2 - \frac{172}{231}\bar{q} \cdot z$	$c_2^2 = \frac{160}{33}\bar{q}^2 + \frac{76}{21}\bar{q} \cdot z$	$c_4 = \frac{40}{33}\bar{q}^2 - \frac{47}{21}\bar{q} \cdot z$

Proof. — Using the constants of table (6), we find the equations

$$13440 = \bar{q} \cdot c_2^2 = \bar{q} \cdot \left(\frac{24^2}{11^2}\bar{q}^2 + \frac{48}{11}\bar{q} \cdot z + z^2 \right) = \frac{24^2}{11^2}\bar{q}^3 + \bar{q} \cdot z^2$$

and

$$30208 = c_2^3 = \frac{24^3}{11^3}\bar{q}^3 + 3 \cdot \frac{24}{11}\bar{q} \cdot z^2 + z^3,$$

which give $\bar{q} \cdot z^2 = \frac{2688}{11}$ and $z^3 = -\frac{22016}{121}$. These numbers imply that $\bar{q} \cdot z$ is not zero, and hence $\bar{q}^2, \bar{q} \cdot z$ form a basis for Γ -invariant canonical Hodge classes in $H^8(K, \mathbb{Q})$. By [4, Proposition 2.4], we have $C(\bar{q} \cdot z^2) = 7C(z^2)$ and we find $C(z^2) = \frac{384}{11}$. We have

$$z^2 = \frac{C(z^2)}{C(\bar{q}^2)}\bar{q}^2 + \lambda\bar{q} \cdot z = \frac{32}{363}\bar{q}^2 + \lambda\bar{q} \cdot z,$$

for some $\lambda \in \mathbb{Q}$. Taking the cup-product with z we find $z^3 = \lambda\bar{q} \cdot z^2$, and so $\lambda = -\frac{172}{231}$. Similarly, writing $c_2^2 = \frac{C(c_2^2)}{C(\bar{q}^2)}\bar{q}^2 + a\bar{q} \cdot z = \frac{160}{33}\bar{q}^2 + a\bar{q} \cdot z$ we obtain

$$30208 = c_2^3 = \frac{24}{11} \cdot \frac{160}{33}\bar{q}^3 + a\bar{q} \cdot z^2,$$

so that $a = \frac{76}{21}$, while writing $c_4 = \frac{C(c_4)}{C(\bar{q}^2)}\bar{q}^2 + b\bar{q} \cdot z = \frac{40}{33}\bar{q}^2 + b\bar{q} \cdot z$ we obtain

$$6784 = c_2c_4 = \frac{24}{11} \cdot \frac{480}{396}\bar{q}^3 + b\bar{q} \cdot z^2,$$

from which we find $b = -\frac{47}{21}$. \square

3. Geometry of generalized Kummer sixfolds

The following construction is of fundamental importance for the present article.

Theorem 3.1 ([15]). — *Let K be any Kum^3 -manifold. There exists a crepant resolution*

$$Y_K \rightarrow K/G$$

with Y_K a manifold of $\text{K3}^{[3]}$ -type. Moreover, $W := \bigcup_{1 \neq g \in G} K^g$ is the union of 16 transversal fourfolds of $\text{K3}^{[2]}$ -type, and, letting $\tilde{K} := \text{Bl}_W(K)$, we have $Y_K = \tilde{K}/G$.

If K is projective, then there exists a uniquely determined (up to isomorphism) K3 surface S_K such that Y_K is a moduli space of stable sheaves on S_K .

In the above statement, $G \cong (\mathbb{Z}/2\mathbb{Z})^5$ is the subgroup of $\text{Aut}_0(K)$ defined in §2.5. Taking moduli spaces of stable sheaves on K3 surfaces is a prominent source of construction of hyper-Kähler varieties of $\text{K3}^{[n]}$ -type, see [27] and [2].

Remark 3.2. — Choosing any ordering W_i , $i = 1, \dots, 16$, of the irreducible components of W , the blow-up $\tilde{K} \rightarrow K$ may be performed as the successive blow-up of the strict transforms of the W_i . This can be checked via a local computation or applying the general [28, Proposition 2.10]. We thus find a sequence

$$\tilde{K} = \tilde{K}_{16} \xrightarrow{p_{16}} \tilde{K}_{15} \xrightarrow{p_{15}} \dots \xrightarrow{p_2} \tilde{K}_1 \xrightarrow{p_1} \tilde{K}_0 = K,$$

where p_i is the blow-up of \tilde{K}_{i-1} in a smooth submanifold $\bar{W}_i \subset \tilde{K}_{i-1}$; each \bar{W}_i is the iterated blow-up of the $\text{K3}^{[2]}$ -manifold W_i along smooth surfaces.

The components W_i of W can be described very explicitly when $K = K^3(A)$ is the generalized Kummer sixfold on a complex torus A .

3.1. Notation. — We denote by $A_n \subset A$ the subgroup of points of order n ; we have $A_n \cong (\mathbb{Z}/n\mathbb{Z})^4$. Given a point $\tau \in A_2$ we will denote by $A_{2,\tau} \subset A_4$ the subset of those a such that $2a = \tau$ (equivalently, such that $a = -a + \tau$). Translation by any $\tau' \in A_2$ maps $A_{2,\tau}$ to itself, showing that $A_{2,\tau}$ is a torsor under A_2 . If $\tau \in A_2$, by abuse of notation we will denote by τ (resp. by $(\tau, -1)$) the automorphism of A given by $\tau(a) = a + \tau$ (resp. by $(\tau, -1)(a) = -a + \tau$). For any $\tau \in A_2$, the quotient $A/\langle(\tau, -1)\rangle$ is a singular surface with 16 double points corresponding to the points of $A_{2,\tau}$. Blowing-up these points gives a K3 surface $\text{Km}^\tau(A)$, isomorphic to the Kummer surface associated to A .

Definition 3.3. — (i) For any $\tau \in A_2$, we let $W_\tau \subset K^3(A)$ be the strict transform of

$$\{(a, b, -a + \tau, -b + \tau), a, b \in A\} \subset A_0^{(4)}$$

under the Hilbert-Chow morphism $\nu: K^3(A) \rightarrow A_0^{(4)}$.

(ii) For any $\tau \neq \tau' \in A_2$, we let $V_{\tau, \tau'} \subset K^3(A)$ be the strict transform of

$$\{(a, a + \tau + \tau', -a + \tau, -b + \tau'), a \in A\} \subset A_0^{(4)}$$

under the Hilbert-Chow morphism $\nu: K^3(A) \rightarrow A_0^{(4)}$. Note that $V_{\tau, \tau'} = V_{\tau', \tau}$.

The W_τ , $\tau \in A_2$, are the 16 components of the locus $W = \bigcup_{1 \neq g \in G} (K^3(A))^g \subset K^3(A)$ appearing in Theorem 3.1 in the special case $K = K^3(A)$. We summarize in the next two Propositions several results obtained in [15, §2] (where the notation is slightly different).

Proposition 3.4. — (i) Let $\tau \in A_2$. Then W_τ is a hyper-Kähler manifold of K3^[2]-type, isomorphic to $\text{Km}^\tau(A)^{[2]}$. The Hilbert-Chow morphism $\nu: K^3(A) \rightarrow A^{(4)}$ restricts to a resolution

$$W_\tau = \text{Km}^\tau(A)^{[2]} \xrightarrow{\nu} (A/\langle(\tau, -1)\rangle)^{(2)}.$$

(ii) Let $\tau \neq \tau' \in A_2$. Then $V_{\tau, \tau'}$ is a K3 surface isomorphic to $\text{Km}^\tau(A/\langle\tau + \tau'\rangle)$, and ν restricts to a resolution

$$V_{\tau, \tau'} = \text{Km}^\tau(A/\langle\tau + \tau'\rangle) \xrightarrow{\nu} A/\langle(\tau, -1), (\tau', -1)\rangle.$$

(iii) For τ varying in A_2 we obtain 16 distinct codimension 2 submanifolds W_τ of $K^3(A)$. For any $\tau \neq \tau'$, the submanifolds W_τ and $W_{\tau'}$ intersect transversely in $V_{\tau, \tau'}$. For τ, τ', τ'' pairwise distinct, $W_\tau, W_{\tau'}, W_{\tau''}$ meet transversely in 4 points, while 4 or more distinct components W_τ have empty intersection.

Consider the group $\text{Aut}_0(K^3(A)) = A_4 \rtimes \langle -1 \rangle$, with its subgroups $G = A_2 \times \langle -1 \rangle$ and $\Gamma = A_4$ as in §2.5. The submanifolds introduced above arise from the study of the fixed loci of these automorphisms.

Proposition 3.5. — (i) Any automorphism $g \in G$ restricts to an automorphism of each of the W_τ and of the $V_{\tau, \tau'}$. An automorphism $\gamma_\epsilon \in \Gamma$ induced by translation by $\epsilon \in A_4$ restricts to isomorphisms

$$W_\tau \xrightarrow{\sim} W_{\tau+2\epsilon} \quad \text{and} \quad V_{\tau, \tau'} \xrightarrow{\sim} V_{\tau+2\epsilon, \tau'+2\epsilon}.$$

(ii) The manifold W_τ is the unique positive dimensional component of the fixed locus of the automorphism $(\tau, -1) \in G$. The induced action of $G/\langle(\tau, -1)\rangle \cong (\mathbb{Z}/2\mathbb{Z})^4$ on W_τ is identified with the natural action of A_2 on $\text{Km}^\tau(A)^{[2]} = W_\tau$.

(iii) For $\tau \neq 0 \in A_2$, the fixed locus of the automorphism $\tau \in G$ consists of the disjoint union of the 8 K3 surfaces $V_{\tau-\theta, \theta}$, with $\theta \in A_2$. The stabilizer in G of $V_{\tau, \tau'}$ is the subgroup $\langle(\tau, -1), (\tau', -1)\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Remark 3.6. — The submanifolds $\iota_\tau: W_\tau \rightarrow K^3(A)$ deform with $K^3(A)$. In fact, let $\mathcal{K} \rightarrow B$ be a family of manifolds of Kum³-type with $0 \in B$ such that $\mathcal{K}_0 = K^3(A)$ for some 2-dimensional complex torus A . The groups $\text{Aut}_0(\mathcal{K}_b)$ yield a local system over B ; up to a finite étale base-change, we have a well-defined fibrewise action of $\text{Aut}_0(K^3(A))$ on $\mathcal{K} \rightarrow B$. For each $\tau \in A_2$, a component of the fixed locus $\mathcal{K}^{(\tau, -1)}$ is a subfamily $\mathcal{W}_\tau \subset \mathcal{K}$ of manifolds of K3^[2]-type over B , with $\mathcal{W}_{\tau,0} = W_\tau$.

It follows that the pull-back map $\iota_\tau^*: H^2(K^3(A), \mathbb{Z}) \rightarrow H^2(W_\tau, \mathbb{Z})$ is injective. Indeed, considering a deformation $\mathcal{K} \rightarrow B$ as above with general element of Picard rank 0, we see that the pull-back ι_τ^* must be either zero or injective, since $H^2(\mathcal{K}_b, \mathbb{Q})$ is an irreducible Hodge structure for general b and the integral second cohomology of Kum³-manifolds is torsion-free. But this map cannot be zero, as the restriction of a symplectic form on $K^3(A)$ gives a symplectic form on W_τ .

3.2. Divisors on the canonical submanifolds. — For $\tau \neq \tau' \in A_2$ we have the identification $V_{\tau, \tau'} = \text{Km}^\tau(A/\langle \tau + \tau' \rangle)$; thus $V_{\tau, \tau'}$ contains 16 disjoint rational curves.

Definition 3.7. — For $\beta \in (A_{2, \tau} \sqcup A_{2, \tau'})/\langle \tau + \tau' \rangle$, we let $R_{\tau, \tau', \beta} \subset V_{\tau, \tau'}$ be the rational curve lying over the node β of $A/\langle (\tau, -1), (\tau', -1) \rangle$. We let $r_{\tau, \tau', \beta} \in H^2(V_{\tau, \tau'}, \mathbb{Z})$ be the fundamental class of $R_{\tau, \tau', \beta}$.

Let $\tau \in A_2$. By Proposition 3.4 we have $W_\tau = \text{Km}^\tau(A)^{[2]}$ and ν factors through

$$\bar{\nu}: \text{Km}^\tau(A)^{[2]} \rightarrow (A/\langle (\tau, -1) \rangle)^{(2)},$$

where the latter is immersed in $A_0^{(4)}$ via $(a, b) \mapsto (a, b, -a + \tau, -b + \tau)$. The morphism $\bar{\nu}$ introduces 17 exceptional divisors on W_τ .

Definition 3.8. — Fix $\tau \in A_2$.

- For any $\alpha \in A_{2, \tau}$ we let $S_{\tau, \alpha} \subset W_\tau$ be the exceptional component of $\bar{\nu}$ lying over $\{(\alpha, b), b \in A\} \subset (A/\langle (\tau, -1) \rangle)^{(2)}$; we let $s_{\tau, \alpha} \in H^2(W_\tau, \mathbb{Z})$ be its fundamental class.
- We let $D_\tau \subset W_\tau$ be the divisor of non-reduced subschemes, which is the exceptional component of $\bar{\nu}$ over $\{(a, a), a \in A\} \subset (A/\langle (\tau, -1) \rangle)^{(2)}$. We let $\delta_\tau \in H^2(W_\tau, \mathbb{Z})$ be half the fundamental class of D_τ .

By [3, Proposition 6], the class δ_τ is integral. Moreover, denoting by $\pi^\tau: A \dashrightarrow \text{Km}^\tau(A)$ the natural rational map of degree 2, we have (up to finite index, see [26, Chapter 3, §2.5])

$$H^2(W_\tau, \mathbb{Z}) = \pi_*^\tau(H^2(A, \mathbb{Z})) \oplus \bigoplus_{\alpha \in A_{2, \tau}} \mathbb{Z} \cdot s_{\tau, \alpha} \oplus \mathbb{Z} \cdot \delta_\tau.$$

With respect to the Beauville-Bogomolov form on W_τ , the classes δ_τ and $s_{\tau, \alpha}$, $\alpha \in A_{2, \tau}$, are pairwise orthogonal and have square -2 .

Remark 3.9. — Let $\epsilon \in A_4$. Translation by ϵ on $K^3(A)$ sends D_τ to $D_{\tau+2\epsilon}$ and $S_{\tau,\alpha}$ to $S_{\tau+2\epsilon,\alpha+\epsilon}$, for any $\tau \in A_2$ and $\alpha \in A_{2,\tau}$.

We will now make some further observations about the geometry of the submanifolds introduced above.

Proposition 3.10. — Let $\tau \neq \tau' \in A_2$ and consider $V_{\tau,\tau'} \subset W_\tau$.

(i) Restriction from W_τ to $V_{\tau,\tau'}$ gives in $H^2(V_{\tau,\tau'}, \mathbb{Z})$:

$$\delta_{\tau|_{V_{\tau,\tau'}}} = \frac{1}{2} \sum_{\gamma \in A_{\tau'}/\langle \tau+\tau' \rangle} r_{\tau,\tau',\gamma}, \quad \text{and} \quad s_{\tau,\alpha|_{V_{\tau,\tau'}}} = r_{\tau,\tau',\bar{\alpha}} \quad \text{for any } \alpha \in A_{2,\tau},$$

where $\bar{\alpha} \in A_{2,\tau}/\langle \tau + \tau' \rangle$ denotes the image of α .

(ii) Let $N_{V_{\tau,\tau'}|W_\tau}$ be the normal bundle of $V_{\tau,\tau'} \subset W_\tau$. We have

$$\deg \left(c_2(N_{V_{\tau,\tau'}|W_\tau}) \right) = 12.$$

Proof. — It is well-known that there is a commutative diagram

$$\begin{array}{ccc} \text{Bl}_\Delta(\text{Km}^\tau(A)^2) & \xrightarrow{\pi} & \text{Km}^\tau(A)^2 \\ \rho \downarrow & & \downarrow \\ \text{Km}^\tau(A)^{[2]} & \xrightarrow{\bar{\pi}} & \text{Km}^\tau(A)^{(2)} \end{array}$$

where $\Delta \subset \text{Km}^\tau(A)^2$ is the diagonal, π is the blow-up map, and ρ is a double cover branched over the exceptional divisor of π . The lower horizontal map $\bar{\pi}$ is the blow-up of the image $\bar{\Delta}$ of the diagonal in $\text{Km}^\tau(A)^{(2)}$. There is a natural action of A_2 on $\text{Km}^\tau(A)$ and hence on $\text{Km}^\tau(A)^{[2]}$; the inclusion of $V_{\tau,\tau'}$ into W_τ is induced by the morphism

$$\psi: \text{Km}^\tau(A) \rightarrow \text{Km}^\tau(A)^2$$

given by $s \mapsto (s, (\tau + \tau')(s))$. We denote by $Z_{\tau+\tau'}$ the image, i.e., the graph of the automorphism of $\text{Km}^\tau(A)$ induced by translation by $\tau + \tau'$ on A .

We prove (i). Let $\bar{Z}_{\tau+\tau'}$ be the image of $Z_{\tau+\tau'}$ in $\text{Km}^\tau(A)^{(2)}$. Then the restriction $\bar{\pi}: V_{\tau,\tau'} \rightarrow \bar{Z}_{\tau+\tau'}$ is the blow-up of the intersection of $\bar{Z}_{\tau+\tau'}$ with $\bar{\Delta}$, which consists of 8 points parametrized by $A_{2,\tau'}/\langle \tau + \tau' \rangle$. Since $D_\tau \subset W_\tau$ is the exceptional divisor of π , we deduce that it restricts to $\sum_{\beta \in A_{2,\tau'}/\langle \tau+\tau' \rangle} r_{\tau,\tau',\beta}$; as $[D_\tau] = 2\delta_\tau$, this gives the first assertion. For the second assertion of (i), note that $S_{\tau,\alpha} \subset \text{Km}^\tau(A)^{[2]}$ parametrizes the subschemes whose support intersects the rational curve of $\text{Km}^\tau(A)$ lying over the node corresponding to $\alpha \in A_2$. The intersection of $\bar{\pi}(S_{\tau,\alpha})$ with $\bar{Z}_{\tau+\tau'}$ is disjoint from $\bar{\Delta}$, and it is easy to see in $\text{Km}^\tau(A)^{(2)}$ that the restriction of $\bar{\pi}(S_{\tau,\alpha})$ to $\bar{\pi}(V_{\tau,\tau'})$ is the curve $\bar{\pi}(R_{\tau,\tau',\bar{\alpha}})$.

We now prove (ii). The normal bundle of the graph $Z_{\tau+\tau'} \subset \mathrm{Km}^\tau(A)^2$ is isomorphic to the tangent bundle of the K3 surface $\mathrm{Km}^\tau(A)$ and hence, by [20, Theorem 6.3],

$$\int_{\mathrm{Km}^\tau(A)^2} [Z_{\tau+\tau'}]^2 = \deg \left(c_2(N_{Z_{\tau+\tau'}|\mathrm{Km}^\tau(A)^2}) \right) = 24.$$

Note that $Z_{\tau+\tau'}$ is transversal to Δ and it is stabilized by the involution of $\mathrm{Km}^\tau(A)^2$ which switches the two factors. By [20, Theorem 6.7], we have $\rho^*([V_{\tau,\tau'}]) = \pi^*([Z_{\tau+\tau'}])$ in the cohomology of $\mathrm{Bl}_\Delta(\mathrm{Km}^\tau(A)^2)$, and therefore

$$\int_{W_\tau} [V_{\tau,\tau'}]^2 = \frac{1}{2} \int_{\mathrm{Bl}_\Delta(\mathrm{Km}^\tau(A)^2)} \rho^*([V_{\tau,\tau'}])^2 = \frac{1}{2} \int_{\mathrm{Bl}_\Delta(\mathrm{Km}^\tau(A)^2)} \pi^*([Z_{\tau+\tau'}])^2 = 12;$$

this self-intersection number coincides with the degree of $c_2(N_{V_{\tau,\tau'}|W_\tau})$. \square

By [3, Proposition 8], we have

$$H^2(K^3(A), \mathbb{Z}) = H^2(A, \mathbb{Z}) \oplus \mathbb{Z} \cdot \xi,$$

where ξ has self-intersection -8 and $2\xi = [E]$ is the fundamental class of the exceptional divisor E of the Hilbert-Chow morphism $\nu: K^3(A) \rightarrow A_0^{(4)}$.

Proposition 3.11. — *Let $\tau \neq \tau' \in A_2$ and consider $V_{\tau,\tau'} \subset W_\tau \subset K^3(A)$. We have:*

$$\begin{aligned} H^2(W_\tau, \mathbb{Z}) \ni \xi|_{W_\tau} &= 2\delta_\tau + \frac{1}{2} \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha}, \\ H^2(V_{\tau,\tau'}, \mathbb{Z}) \ni \xi|_{V_{\tau,\tau'}} &= \sum_{\beta \in A_{2,\tau}/\langle \tau+\tau' \rangle} r_{\tau,\tau',\beta} + \sum_{\gamma \in A_{2,\tau'}/\langle \tau+\tau' \rangle} r_{\tau,\tau',\gamma}. \end{aligned}$$

Proof. — The intersection $E \cap W_\tau$ is supported on the union of the divisors $S_{\tau,\alpha}$ and D_τ . First, we observe that the restriction of E to W_τ has multiplicity 1 along each $S_{\tau,\alpha}$. Indeed, each $S_{\tau,\alpha} \subset W_\tau$ is generically a \mathbb{P}^1 -bundle over $\{(\alpha, b, \alpha, -b + \tau), b \in A\} \subset A_0^{(4)}$, an open subset of which is contained in the locus U of those $(a_1, a_2, a_3, a_4) \in A_0^{(4)}$ with at least three distinct points in their support. The restriction $\nu: \nu^{-1}(U) \rightarrow U$ is the blow-up of the big diagonal. Since $\nu(D_\tau) \cap U = \emptyset$, the restriction of ν to $\nu^{-1}(U) \cap W_\tau$ is a blow-up with exceptional divisors the $S_{\tau,\alpha}$. Hence, $E|_{W_\tau} = kD_\tau + \sum_{\alpha \in A_{2,\tau}} S_{\tau,\alpha}$, for some k .

We now compute the restriction of E to $V_{\tau,\tau'}$. It is supported over the 16 curves $R_{\tau,\tau',\beta}$, for $\beta \in (A_{2,\tau} \sqcup A_{2,\tau'})/\langle \tau + \tau' \rangle$. By the above and Proposition 3.10.(i), the restriction of E to $V_{\tau,\tau'}$ has multiplicity 2 at the curve $R_{\tau,\tau',\bar{\alpha}}$ for any $\bar{\alpha} \in A_{2,\tau}/\langle \tau + \tau' \rangle$ (recall that $S_{\tau,\alpha}$ and $S_{\tau,\alpha+\tau+\tau'}$ restrict to the same curve $R_{\tau,\tau',\bar{\alpha}}$). Switching the roles of τ and τ' , we find

$$\xi|_{V_{\tau,\tau'}} = \sum_{\beta \in A_{2,\tau}/\langle \tau+\tau' \rangle} r_{\tau,\tau',\beta} + \sum_{\gamma \in A_{2,\tau'}/\langle \tau+\tau' \rangle} r_{\tau,\tau',\gamma}.$$

Finally, combining this with Proposition 3.10.(i), the multiplicity k of $E \cap W_\tau$ along D_τ must be 2; otherwise, ξ could not restrict to the above class in $H^2(V_{\tau,\tau'}, \mathbb{Z})$. \square

Remark 3.12. — Let $\iota_\tau: W_\tau \hookrightarrow K^3(A)$ denote the inclusion. If $\nu: K^3(A) \rightarrow A_0^{(4)}$ is the Hilbert-Chow morphism, an explicit computation shows that $\nu(\iota_\tau(S_{\tau,\alpha})) \cap \nu(\iota_{\tau'}(S_{\tau',\alpha'})) = \emptyset$ and $\nu(\iota_\tau(\delta_\tau)) \cap \nu(\iota_{\tau'}(\delta_{\tau'})) = \emptyset$, for any $\tau \neq \tau' \in A_2$. Thus, for any $\tau \neq \tau'$, we have

$$\int_{K^3(A)} \iota_{\tau,*}(s_{\tau,\alpha}) \cdot \iota_{\tau',*}(s_{\tau',\alpha'}) = 0 \quad \text{and} \quad \int_{K^3(A)} \iota_{\tau,*}(\delta_\tau) \cdot \iota_{\tau',*}(\delta_{\tau'}) = 0.$$

Remark 3.13. — The bundles in the exact sequence

$$0 \rightarrow T_{W_\tau} \rightarrow T_{K^3(A)|_{W_\tau}} \rightarrow N_{W_\tau|K^3(A)} \rightarrow 0$$

have trivial first Chern class. Hence, $c_2(K^3(A))|_{W_\tau} = c_2(W_\tau) + c_2(N_{W_\tau|K^3(A)})$, and

$$(9) \quad c_4(K^3(A))|_{W_\tau} = c_4(W_\tau) + c_2(W_\tau) \cdot c_2(N_{W_\tau|K^3(A)}).$$

We also have an exact sequence

$$0 \rightarrow N_{V_{\tau,\tau'}|W_\tau} \rightarrow N_{V_{\tau,\tau'}|K^3(A)} \rightarrow N_{W_\tau|K^3(A)}|_{V_{\tau,\tau'}} \rightarrow 0$$

of bundles with trivial first Chern class. Since $V_{\tau,\tau'}$ is the transversal intersection of W_τ and $W_{\tau'}$, the last term is identified with $N_{V_{\tau,\tau'}|W_{\tau'}}$. Moreover, translation by $\epsilon \in A_{2,\tau+\tau'}$ maps $V_{\tau,\tau'}$ to itself and exchanges W_τ with $W_{\tau'}$, inducing an isomorphism of vector bundles $N_{V_{\tau,\tau'}|W_\tau} \cong N_{V_{\tau,\tau'}|W_{\tau'}}$. Via the normal bundle sequence for the embedding of $V_{\tau,\tau'}$ into $K^3(A)$, we find

$$c_2(K^3(A))|_{V_{\tau,\tau'}} = c_2(V_{\tau,\tau'}) + c_2(N_{V_{\tau,\tau'}|W_\tau}) + c_2(N_{V_{\tau,\tau'}|W_{\tau'}}).$$

Since $V_{\tau,\tau'}$ is a K3 surface, Proposition 3.11 implies that

$$(10) \quad \deg \left(c_2(K^3(A))|_{V_{\tau,\tau'}} \right) = 48.$$

4. Canonical Hodge classes of degree four

In this section we will show that canonical Hodge classes in $H^4(K, \mathbb{Q})$ are analytic, for any manifold K of Kum³-type. As in the previous section, let A be a complex torus of dimension 2 and let $K^3(A)$ be the generalized Kummer sixfold on A . Recall its submanifolds W_τ and $V_{\tau,\tau'}$ from Definition 3.3.

Definition 4.1. — We denote by $w_\tau \in H^4(K^3(A), \mathbb{Z})$ (resp. by $v_{\tau,\tau'} \in H^8(K^3(A), \mathbb{Z})$) the fundamental class of $W_\tau \subset K^3(A)$ (resp. of $V_{\tau,\tau'}$), for $\tau \neq \tau'$ in A_2 .

Note that, for any $\tau \neq \tau' \in A_2$, we have $w_\tau \cdot w_{\tau'} = v_{\tau,\tau'}$, because W_τ and $W_{\tau'}$ meet transversely in $V_{\tau,\tau'}$ by Proposition 3.4.(iii).

Lemma 4.2. — The classes w_τ and $v_{\tau,\tau'}$ are canonical Hodge classes, and moreover they remain analytic on any deformation of $K^3(A)$.

Proof. — The Lemma follows immediately from Remark 3.6. \square

The Proposition below will then imply that canonical Hodge classes of degree 4 are analytic on any deformation of $K^3(A)$.

Proposition 4.3. — *Any canonical Hodge class in $H^4(K^3(A), \mathbb{Q})$ is a linear combination of the 17 linearly independent classes c_2 and w_τ , for $\tau \in A_2$.*

In fact, since we know from §2.4 that the space of canonical Hodge classes has rank 17, it will be sufficient to check that the classes in the statement are linearly independent. In turn, this will be proven using the following intersection numbers.

Proposition 4.4. — *We have*

$$w_\tau^3 = 60, \quad w_\tau^2 \cdot w_{\tau'} = 12, \quad w_\tau \cdot w_{\tau'} \cdot w_{\tau''} = 4,$$

for any τ, τ', τ'' in A_2 pairwise distinct.

Part of the statement follows directly from Propositions 3.10 and 3.11.

Lemma 4.5. — *The generalized Fujiki constants are*

$$C(w_\tau) = 12 \quad \text{and} \quad C(w_\tau \cdot w_{\tau'}) = 4,$$

for any $\tau \neq \tau' \in A_2$. For pairwise distinct elements τ, τ', τ'' of A_2 we have

$$c_2 \cdot w_\tau \cdot w_{\tau'} = 48; \quad w_\tau^2 \cdot w_{\tau'} = 12; \quad w_\tau \cdot w_{\tau'} \cdot w_{\tau''} = 4.$$

Proof. — Let $\xi \in H^2(K^3(A), \mathbb{Z})$ be half the class of the Hilbert-Chow divisor E ; recall that $q_{K^3(A)}(\xi, \xi) = -8$. The Fujiki constant of $K3^{[2]}$ -manifolds equals 3 (see [42]). Therefore, Fujiki relation (1), the projection formula, and Proposition 3.11 give

$$C(w_\tau) \cdot 64 = \int_{K^3(A)} w_\tau \cdot \xi^4 = \int_{W_\tau} \left(\xi|_{W_\tau} \right)^4 = 3 \cdot q_{W_\tau} \left(\xi|_{W_\tau}, \xi|_{W_\tau} \right)^2 = 3 \cdot 256,$$

from which we find $C(w_\tau) = 12$. Similarly, by Proposition 3.10.(i) we have

$$C(w_\tau \cdot w_{\tau'}) \cdot (-8) = \int_{K^3(A)} v_{\tau, \tau'} \cdot \xi^2 = \int_{V_{\tau, \tau'}} \left(\xi|_{V_{\tau, \tau'}} \right)^2 = -32,$$

which leads to $C(w_\tau \cdot w_{\tau'}) = 4$. By Proposition 3.4, three distinct $W_\tau, W_{\tau'}, W_{\tau''}$ meet transversely in 4 points, and hence $w_\tau \cdot w_{\tau'} \cdot w_{\tau''} = 4$. The self-intersection of $V_{\tau, \tau'}$ as a submanifold of W_τ equals the degree of $c_2(N_{V_{\tau, \tau'}|W_\tau})$, which is 12 by Proposition 3.10.(ii). The restriction of $w_{\tau'}$ to W_τ is the fundamental class of $V_{\tau, \tau'}$ in $H^4(W_\tau, \mathbb{Z})$, and therefore

$$w_{\tau'}^2 \cdot w_\tau = \int_{W_\tau} [V_{\tau, \tau'}]^2 = 12.$$

Finally, by (10), we have $c_2 \cdot v_{\tau, \tau'} = 48$. \square

4.1. The Γ -invariant classes. — In order to prove Proposition 4.4, we consider

$$w := \sum_{\tau \in A_2} w_\tau \in H^4(K^3(A), \mathbb{Z}) \quad \text{and} \quad v = \frac{1}{2} \sum_{\tau \neq \tau' \in A_2} v_{\tau, \tau'} \in H^8(K^3(A), \mathbb{Z}).$$

By Proposition 3.5, these canonical Hodge classes are invariant under the action of $\Gamma = A_4$. We shall express them in terms of the basis of Lemma 2.5.

Lemma 4.6. — *We have*

$$w = \frac{16}{11}\bar{q} - 3z \quad \text{and} \quad v = \frac{40}{33}\bar{q}^2 - \frac{45}{7}\bar{q} \cdot z.$$

Proof. — By the above Lemma 4.5, we have

$$C(v) = 120 \cdot 4, \quad C(w) = 16 \cdot 12, \quad c_2 \cdot v = 120 \cdot 48.$$

Write

$$v = \frac{C(v)}{C(\bar{q}^2)}\bar{q}^2 + \gamma\bar{q} \cdot z = \frac{40}{33}\bar{q}^2 + \gamma\bar{q} \cdot z,$$

for some coefficient $\gamma \in \mathbb{Q}$. Then, via Lemma 2.5, we find

$$\begin{aligned} 120 \cdot 48 &= c_2 \cdot v = \left(\frac{24}{11}\bar{q} + z\right) \cdot \left(\frac{40}{33}\bar{q}^2 + \gamma\bar{q} \cdot z\right) \\ &= \frac{24}{11} \cdot \frac{40}{33}\bar{q}^3 + \gamma\bar{q} \cdot z^2 \\ &= \frac{80640}{11} + \gamma\frac{2688}{11}, \end{aligned}$$

which gives $\gamma = -\frac{45}{7}$. We now use Lemma 4.5 to compute the intersection

$$\begin{aligned} w \cdot v &= \left(\sum_{\tau} w_\tau\right) \cdot \left(\frac{1}{2} \sum_{\tau \neq \tau'} w_\tau \cdot w_{\tau'}\right) \\ &= \sum_{\tau \neq \tau'} w_\tau^2 \cdot w_{\tau'} + \frac{1}{2} \sum_{\tau \neq \tau' \neq \tau'' \neq \tau} w_\tau \cdot w_{\tau'} \cdot w_{\tau''} \\ &= 16 \cdot 15 \cdot 12 + \frac{1}{2} \cdot 16 \cdot 15 \cdot 14 \cdot 4 \\ &= 9600. \end{aligned}$$

We can write

$$w = \frac{C(w)}{C(\bar{q})}\bar{q} + \lambda z = \frac{16}{11}\bar{q} + \lambda z$$

for some coefficient $\lambda \in \mathbb{Q}$, which is then determined by the equation

$$9600 = w \cdot v = \left(\frac{16}{11}\bar{q} + \lambda z\right) \cdot \left(\frac{40}{33}\bar{q}^2 - \frac{45}{7}\bar{q} \cdot z\right) = \frac{53760}{11} - \frac{17280}{11}\lambda;$$

we find $\lambda = -3$. □

Remark 4.7. — We obtain the integral linear relations

$$w = 8\bar{q} - 3c_2 \quad \text{and} \quad 3v = 7c_4 - c_2^2.$$

We can now complete the proof of the main results of this section.

Proof of Proposition 4.4. — By Lemma 4.5, it remains to prove that $w_\tau^3 = 60$. By Proposition 3.5, the action of Γ is transitive on the classes w_τ and hence the number w_τ^3 does not depend on $\tau \in A_2$. Via the expression for w found in Lemma 4.6 above and the constants determined in Lemma 2.5 we calculate $w^3 = 23040$. Expanding $w^3 = (\sum_\tau w_\tau)^3$ and using the intersection numbers already computed in Lemma 4.5, we find the equation

$$\begin{aligned} 23040 &= \sum_\tau w_\tau^3 + 3 \sum_{\tau \neq \tau'} w_\tau^2 \cdot w_{\tau'} + \sum_{\tau \neq \tau' \neq \tau'' \neq \tau} w_\tau \cdot w_{\tau'} \cdot w_{\tau''} \\ &= 16 \cdot w_\tau^3 + 3 \cdot 16 \cdot 15 \cdot 12 + 16 \cdot 15 \cdot 14 \cdot 4 \\ &= 16 \cdot w_\tau^3 + 22080, \end{aligned}$$

which gives $w_\tau^3 = 60$. \square

Proof of Proposition 4.3. — Assume given a linear combination $u = \sum_{\tau \in A_2} \beta_\tau w_\tau$ in the classes w_τ such that $u = 0$. By Proposition 4.4, we have

$$u \cdot (w_\tau^2 - w_{\tau'}^2) = (60 - 12)(\beta_\tau - \beta_{\tau'}) = 0,$$

and therefore $\beta_\tau = \beta_{\tau'}$ for any $\tau, \tau' \in A_2$. Thus $u = \beta w$, and hence $\beta = 0$, which means that the w_τ are linearly independent. Since the Γ -invariant classes in the subspace generated by the w_τ are the multiples of w , if c_2 was in this subspace we would obtain a non-trivial linear relation between the independent classes \bar{q} and c_2 , a contradiction. \square

Remark 4.8. — For later use we compute that, for any $\tau \in A_2$, we have

$$C(w_\tau^2) = 12 \quad \text{and} \quad c_4 \cdot w_\tau = 408.$$

Indeed, from the expression for w found in Lemma 4.6 and thanks to Lemma 2.5 we find $C(w^2) = 1152$. Expanding the square of w we obtain

$$1152 = 16 \cdot C(w_\tau^2) + 16 \cdot 15 \cdot C(v_{\tau, \tau'}) = 16 \cdot C(w_\tau^2) + 16 \cdot 15 \cdot 4,$$

which yields $C(w_\tau^2) = 12$, as $C(v_{\tau, \tau'}) = 4$ by Lemma 4.5. We also easily calculate via Lemma 2.5

$$c_4 \cdot w_\tau = \frac{1}{16} c_4 \cdot w = \frac{1}{16} \cdot \left(\frac{40}{33} \bar{q}^2 - \frac{47}{21} \bar{q} \cdot z \right) \cdot \left(\frac{16}{11} \bar{q} - 3z \right) = 408.$$

Remark 4.9. — Cup-product with $\bar{q}_{K^3(A)}$ is injective on the space of canonical Hodge classes in $H^4(K^4(A), \mathbb{Q})$. This follows via an argument entirely analogous to that used in the proof of Proposition 4.3, using the intersection numbers

$$\bar{q} \cdot w_\tau^2 = 84 \quad \text{and} \quad \bar{q} \cdot w_\tau \cdot w_{\tau'} = 28,$$

which are calculated from Lemma 4.5 via [4, Proposition 2.4].

5. Restriction of canonical Hodge classes

The main result of this section is Proposition 5.8, which gives the second Chern class of the normal bundle of $\iota_\tau: W_\tau \hookrightarrow K^3(A)$ as an element of the cohomology of W_τ . In the next section we will use this to show that canonical Hodge classes of degree 6 are analytic.

5.1. Cohomology of $K3^{[2]}$ -manifolds. — Following [39], for any manifold X of $K3^{[2]}$ -type we have

$$H^4(X, \mathbb{Q}) = \text{Sym}^2(H^2(X, \mathbb{Q})).$$

The intersection product defines a non-degenerate symmetric bilinear form on the space $\text{Sym}^2(H^2(X, \mathbb{Q}))$, determined by

$$(11) \quad (a_1 a_2, a_3 a_4) = q(a_1, a_2) \cdot q(a_3, a_4) + q(a_1, a_3) \cdot q(a_2, a_4) + q(a_1, a_4) \cdot q(a_2, a_3)$$

where $q = q_X$ is the Beauville-Bogomolov form. The Euler characteristic of X is $\deg(c_4(X)) = 324$. The class $\bar{q}_X \in H^4(X, \mathbb{Q})$ given by the Beauville-Bogomolov form satisfies $c_2(X) = \frac{6}{5} \cdot \bar{q}_X$ and $\int_X \bar{q}_X^2 = 575$. The generalized Fujiki constant is $C(\bar{q}_X) = 25$; this means that, for every $\alpha, \beta \in H^2(X, \mathbb{Q})$, we have

$$(12) \quad \int_X \bar{q}_X \cdot \alpha \cdot \beta = 25 \cdot q_X(\alpha, \beta).$$

5.2. The cohomology of W_τ . — Let A be a 2-dimensional complex torus and let $K^3(A)$ be the generalized Kummer variety on it. For any $\tau \in A_2$, we consider the submanifold $\iota_\tau: W_\tau \hookrightarrow K^3(A)$ (Definition 3.3); by Proposition 3.4, $W_\tau = \text{Km}^\tau(A)^{[2]}$. Recall from Definition 3.8 that we defined divisor classes $s_{\tau, \alpha}$ and δ_τ on W_τ , which are pairwise orthogonal -2 classes in the lattice $H^2(W_\tau, \mathbb{Z})$.

Lemma 5.1. — *Consider the 19-dimensional subspace of $H^4(W_\tau, \mathbb{Q})$ generated by:*

$$\bar{q}_{W_\tau}; \quad \delta_\tau^2; \quad \sum_{\alpha \in A_{2, \tau}} s_{\tau, \alpha}^2; \quad \sum_{\alpha \in A_{2, \tau}} s_{\tau, \alpha} \cdot s_{\tau, \alpha + \theta} \text{ for } 0 \neq \theta \in A_2; \quad \delta \cdot \sum_{\alpha \in A_{2, \tau}} s_{\tau, \alpha}.$$

The intersection form on this subspace is given (in the above ordered basis) by the matrix:

$$\begin{pmatrix} 575 & -50 & -800 & 0 & \cdots & 0 & 0 \\ -50 & 12 & 64 & 0 & \cdots & 0 & 0 \\ -800 & 64 & 1152 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 128 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 128 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 64 \end{pmatrix}$$

Proof. — The calculation is elementary using the description (11) of the intersection product and the Fujiki relation (12). One computes

$$\bar{q}_{W_\tau} \cdot \delta_\tau^2 = -50; \quad \bar{q}_{W_\tau} \cdot \delta_\tau \cdot s_{\tau,\alpha} = 0; \quad \bar{q}_{W_\tau} \cdot s_{\tau,\alpha} \cdot s_{\tau,\alpha'} = \begin{cases} -50 & \text{if } \alpha = \alpha'; \\ 0 & \text{else.} \end{cases}$$

Next, we have

$$(s_{\tau,\alpha} \cdot s_{\tau,\beta}) \cdot (s_{\tau,\alpha'} \cdot s_{\tau,\beta'}) = \begin{cases} 12 & \text{if } \alpha = \alpha' = \beta = \beta'; \\ 4 & \text{if } \alpha = \alpha' \text{ and } \beta = \beta'; \\ 4 & \text{if } \alpha \neq \alpha' \text{ and } \{\beta, \beta'\} = \{\alpha, \alpha'\}; \\ 0 & \text{else;} \end{cases}$$

and

$$\delta_\tau^4 = 12; \quad \delta_\tau^3 \cdot s_{\tau,\alpha} = 0; \quad \delta_\tau^2 \cdot (s_{\tau,\alpha} \cdot s_{\tau,\beta}) = \begin{cases} 4 & \text{if } \alpha = \beta; \\ 0 & \text{else.} \end{cases}$$

By the bilinearity of the intersection product, the matrix in the statement is now obtained via a straightforward calculation. \square

By Remark 3.6, any deformation $\mathcal{K} \rightarrow B$ induces a deformation of the $\text{K3}^{[2]}$ -manifold W_τ . However, the classes $s_{\tau,\alpha}$ and δ_τ do not remain Hodge classes on all such induced deformations. We are thus led to introduce the following divisor classes.

Definition 5.2. — For any $\tau \in A_2$ and $\alpha \in A_{2,\tau}$ we define the Hodge class

$$s'_{\tau,\alpha} := (4s_{\tau,\alpha} - \delta_\tau) \in H^2(W_\tau, \mathbb{Z}).$$

The point of the definition is explained by the next lemma.

Lemma 5.3. — Let $\mathcal{W}_\tau \rightarrow B$ be a deformation of W_τ induced by a deformation $\mathcal{K} \rightarrow B$ of $K^3(A)$. Then the classes $s'_{\tau,\alpha}$ remain Hodge on $\mathcal{W}_{\tau,b}$, for any $b \in B$.

Proof. — Each $s'_{\tau,\alpha}$ is orthogonal to $\iota_\tau^*(H^2(K^3(A), \mathbb{Z}))$. To prove this, we may assume that A is a general complex torus such that $H^2(A, \mathbb{Z})$ is an irreducible Hodge structure. Then $H^2(K^3(A), \mathbb{Z}) = H^2(A, \mathbb{Z}) \oplus \mathbb{Z} \cdot \xi$, and the first summand equals the transcendental part $H_{\text{tr}}^2(K^3(A), \mathbb{Z})$. Hence, $s'_{\tau,\alpha} \in \text{NS}(W_\tau)$ is orthogonal to $\iota_\tau^*(H^2(A, \mathbb{Q})) = H_{\text{tr}}^2(W_\tau, \mathbb{Q})$. By Proposition 3.11, we have $q_{W_\tau}(\iota_\tau^*(\xi), s'_{\tau,\alpha}) = 0$, and we conclude that $s'_{\tau,\alpha}$ is orthogonal to $\iota_\tau^*(H^2(K^3(A), \mathbb{Z}))$.

Consider now a deformation as in the statement and let $s'_b \in H^2(\mathcal{W}_{\tau,b}, \mathbb{Z})$ be a class obtained via parallel transport of some $s'_{\tau,\alpha}$, for some $b \in B$. By the above, s'_b is orthogonal to the image of $\iota_{\tau,b}^*: H^2(\mathcal{K}_b, \mathbb{Z}) \rightarrow H^2(\mathcal{W}_\tau, \mathbb{Z})$ of the pull-back; but this image contains a symplectic form, and hence s'_b is a Hodge class. \square

5.3. Restriction of canonical Hodge classes. — We will now explicitly describe the restriction to $\iota_\tau: W_\tau \hookrightarrow K^3(A)$ of the canonical Hodge classes in $H^4(K^3(A), \mathbb{Q})$ from Proposition 4.3. First, we have the following.

Lemma 5.4. — *Let $\omega \in H^4(K^3(A), \mathbb{Q})$ be a canonical Hodge class. Then its pull-back $\iota_\tau^*(\omega) \in H^4(W_\tau, \mathbb{Q})$ is a linear combination of the following 17 classes:*

$$\iota_\tau^*(\bar{q}_{K^3(A)}); \quad \sum_{\alpha \in A_{2,\tau}} s'_{\tau,\alpha} \cdot s'_{\tau,\alpha+\theta}, \text{ for } \theta \in A_2.$$

Proof. — By Proposition 4.3, every canonical Hodge class in $H^4(K^3(A), \mathbb{Q})$ is invariant under the action of $G = A_2 \times \langle -1 \rangle$. The group G induces an action of A_2 on W_τ , such that $\theta \in A_2$ maps $s'_{\tau,\alpha}$ to $s'_{\tau,\alpha+\theta}$ (see Remark 3.9).

Recall from Remark 3.6 that any deformation of $K^3(A)$ induces a deformation of W_τ . Considering the orthogonal decomposition

$$H^2(W_\tau, \mathbb{Q}) = \iota_\tau^*(H^2(K^3(A), \mathbb{Q})) \oplus \langle s'_{\tau,\alpha} \rangle_{\alpha \in A_{2,\tau}}$$

and the induced splitting of $H^4(W_\tau, \mathbb{Q})$, we see that the classes in $H^4(W_\tau, \mathbb{Q})$ staying Hodge on every such deformation are the multiples of the pull-back of $\bar{q}_{K^3(A)}$ and those contained in the subspace $\text{Sym}^2(\langle s'_{\tau,\alpha} \rangle_{\alpha \in A_{2,\tau}})$. The A_2 -invariants in this subspace are spanned by the 16 vectors $\sum_{\alpha \in A_{2,\tau}} s'_{\tau,\alpha} \cdot s'_{\tau,\alpha+\theta}$, for θ varying in A_2 . \square

Remark 5.5. — For any $\theta \in A_2$, we have

$$\sum_{\alpha \in A_{2,\tau}} s'_{\tau,\alpha} \cdot s'_{\tau,\alpha+\theta} = 16 \cdot \delta_\tau^2 + 16 \cdot \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\theta} - 8 \cdot \delta_\tau \cdot \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha}.$$

In particular, the pull-back $\iota_\tau^*(\omega)$ of a canonical Hodge class $\omega \in H^4(K^3(A), \mathbb{Q})$ is a linear combination of the vectors of Lemma 5.1.

Lemma 5.6. — *For any $\tau \in A_2$, we have in $H^4(W_\tau, \mathbb{Q})$:*

$$\iota_\tau^*(\bar{q}_{K^3(A)}) = 2\bar{q}_{W_\tau} + \frac{1}{2}\delta_\tau^2 + \frac{31}{32} \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha}^2 - \frac{1}{32} \sum_{0 \neq \theta \in A_2} \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\theta} - \frac{1}{4}\delta_\tau \cdot \sum_{\alpha} s_{\tau,\alpha}.$$

Proof. — We note that, for any $\gamma \in H^2(K^3(A), \mathbb{Q})$, we have

$$q_{W_\tau}(\iota_\tau^*(\gamma), \iota_\tau^*(\gamma)) = 2q_{K^3(A)}(\gamma, \gamma).$$

Indeed, as the Fujiki constant of W_τ is 3, we find

$$q_{W_\tau}(\iota_\tau^*(\gamma), \iota_\tau^*(\gamma))^2 = \frac{1}{3} \int_{W_\tau} \iota_\tau^*(\gamma)^4 = \frac{1}{3} \int_{K^3(A)} w_\tau \cdot \iota_\tau^*(\gamma)^4 = \frac{C(w_\tau)}{3} q_{K^3(A)}(\gamma, \gamma)^2,$$

which equals $4q_{K^3(A)}(\gamma, \gamma)^2$ because $C(w_\tau) = 12$, by Lemma 4.5. As Kähler classes have positive Beauville-Bogomolov square, we find that $q_{W_\tau}(\iota_\tau^*(\gamma), \iota_\tau^*(\gamma)) = 2q_{K^3(A)}(\gamma, \gamma)$.

We let $\tilde{\lambda}_i^+, \tilde{\lambda}_i^-$, $i = 1, 2, 3$, be an orthogonal basis of $H^2(A, \mathbb{Q})$ such that $\tilde{\lambda}_i^\pm$ has self-intersection ± 2 . The $\tilde{\lambda}_i^\pm$ and ξ give an orthogonal basis of $H^2(K^3(A), \mathbb{Q})$, and

$$\bar{q}_{K^3(A)} = \frac{1}{2} \left(\sum_{i=1}^3 (\tilde{\lambda}_i^+)^2 - \sum_{i=1}^3 (\tilde{\lambda}_i^-)^2 \right) - \frac{1}{8} \xi^2.$$

Set $\lambda_i^\pm := \iota_\tau^*(\tilde{\lambda}_i^\pm)$. Then the classes λ_i^+, λ_i^- , $i = 1, 2, 3$; $s_{\tau, \alpha}$, $\alpha \in A_{2, \tau}$; δ_τ , are an orthogonal basis of $H^2(W_\tau, \mathbb{Q})$, with $q_{W_\tau}(\lambda_i^\pm, \lambda_i^\pm) = \pm 4$. Indeed, if $\pi^\tau: A \dashrightarrow \text{Km}^\tau(A)$ denotes the natural rational map, we have the orthogonal decomposition

$$H^2(W_\tau, \mathbb{Q}) = \pi_*^\tau(H^2(A, \mathbb{Q})) \oplus \bigoplus_{\alpha \in A_{2, \tau}} \mathbb{Q} \cdot s_{\tau, \alpha} \oplus \mathbb{Q} \cdot \delta_\tau.$$

The restriction $\iota_\tau^*: H^2(K^3(A), \mathbb{Q}) \rightarrow H^2(W_\tau, \mathbb{Q})$ induces an isomorphism between the first summand in $H^2(K^3(A), \mathbb{Q}) = H^2(A, \mathbb{Q}) \oplus \mathbb{Q} \cdot \xi$ and $\pi_*^\tau(H^2(A, \mathbb{Q}))$. To see this, we may assume that A is a general complex torus. Then the summand $H^2(A, \mathbb{Q}) \subset H^2(K^3(A), \mathbb{Q})$ (resp. $\pi_*^\tau(H^2(A, \mathbb{Q})) \subset H^2(W_\tau, \mathbb{Q})$) is the transcendental part of the cohomology of $K^3(A)$ (resp. of W_τ). Since ι_τ^* is a morphism of Hodge structures, we have our claim.

By Remark 2.2, we have

$$\bar{q}_{W_\tau} = \frac{1}{4} \left(\sum_{i=1}^3 (\lambda_i^+)^2 - \sum_{i=1}^3 (\lambda_i^-)^2 \right) - \frac{1}{2} \sum_{\alpha \in A_{2, \tau}} s_{\tau, \alpha}^2 - \frac{1}{2} \delta_\tau^2.$$

From Proposition 3.11 we know that $\iota_\tau^*(\xi) = 2\delta_\tau + \frac{1}{2} \sum_{\alpha} s_{\tau, \alpha}$. Expanding the square of this class leads to the claimed expression for $\iota_\tau^*(\bar{q}_{K^3(A)})$. \square

Lemma 5.7. — *Let $\tau \neq \tau' \in A_2$. In $H^4(W_\tau, \mathbb{Q})$, we have:*

$$\iota_\tau^*(W_{\tau'}) = \frac{2}{5} \bar{q}_{W_\tau} + \frac{1}{4} \sum_{\alpha \in A_{2, \tau}} s_{\tau, \alpha}^2 - \frac{1}{4} \sum_{\alpha \in A_{2, \tau}} s_{\tau, \alpha} \cdot s_{\tau, \alpha + \tau + \tau'}.$$

Proof. — From Proposition 3.10 we obtain the following intersection numbers:

$$\begin{aligned} \iota_\tau^*(w_{\tau'}) \cdot \delta_\tau^2 &= \int_{V_{\tau, \tau'}} \delta_\tau^2|_{V_{\tau, \tau'}} = -4; \\ \iota_\tau^*(w_{\tau'}) \cdot \delta_\tau \cdot s_{\tau, \alpha} &= 0 \quad \text{for any } \alpha \in A_{2, \tau}; \\ \iota_\tau^*(w_{\tau'}) \cdot s_{\tau, \alpha} \cdot s_{\tau, \alpha + \theta} &= \begin{cases} -2 & \text{if } \theta \in \{0, \tau + \tau'\}; \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Moreover, $\iota_\tau^*(w_{\tau'}) \cdot \bar{q}_{W_\tau} = 30$. Indeed, since $\iota_\tau^*(w_{\tau'})$ is the fundamental class of $V_{\tau, \tau'} \subset W_\tau$ and $\bar{q}_{W_\tau} = \frac{5}{6} c_2(W_\tau)$, the normal bundle sequence and Proposition 3.10.(ii) yield

$$\iota_\tau^*(w_{\tau'}) \cdot \bar{q}_{W_\tau} = \frac{5}{6} \iota_\tau^*(w_{\tau'}) \cdot c_2(W_\tau) = \frac{5}{6} \left(\deg c_2(V_{\tau, \tau'}) + \deg c_2(N_{V_{\tau, \tau'}|W_\tau}) \right) = 30.$$

By Remark 5.5, we can write $\iota_\tau^*(w_{\tau'})$ as linear combination of the classes of Lemma 5.1:

$$\iota_\tau^*(w_{\tau'}) = \eta_1 \bar{q}_{W_\tau} + \eta_2 \delta_\tau^2 + \eta_3 \delta_\tau \cdot \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} + \sum_{\theta \in A_2} \beta_\theta \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\theta},$$

for suitable rational numbers η_j and β_θ , which we are going to determine using the intersection numbers computed above and the intersection matrix from Lemma 5.1.

First, by the above, $\iota_\tau^*(w_{\tau'})$ is orthogonal to $\delta_\tau \cdot \sum_\alpha s_{\tau,\alpha}$, and hence $\eta_3 = 0$. Moreover, $\iota_\tau^*(w_{\tau'})$ is orthogonal to $\sum_\alpha s_{\tau,\alpha} \cdot s_{\tau,\alpha+\theta}$ unless $\theta = 0$ or $\theta = \tau + \tau'$, and hence $\beta_\theta = 0$ for $\theta \notin \{0, \tau + \tau'\}$. For $\theta = \tau + \tau'$, we find

$$\iota_\tau^*(w_{\tau'}) \cdot \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\tau+\tau'} = -32,$$

which gives $128\beta_{\tau+\tau'} = -32$, i.e., $\beta_{\tau+\tau'} = -\frac{1}{4}$. Thus,

$$\iota_\tau^*(w_{\tau'}) = \eta_1 \bar{q}_{W_\tau} + \eta_2 \delta_\tau^2 + \beta_0 \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha}^2 - \frac{1}{4} \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\tau+\tau'}.$$

Computing now the intersection of $\iota_\tau^*(w_{\tau'})$ with the three classes \bar{q}_{W_τ} , δ_τ^2 and $\sum_\alpha s_{\tau,\alpha}^2$ via Lemma 5.1 we obtain the equations

$$\begin{cases} 575\eta_1 - 50\eta_2 - 800\beta_0 = 30; \\ -50\eta_1 + 12\eta_2 + 64\beta_0 = -4; \\ -800\eta_1 + 64\eta_2 + 1152\beta_0 = -32. \end{cases}$$

Solving this linear system we find $\eta_1 = \frac{2}{5}$, $\eta_2 = 0$, $\beta_0 = \frac{1}{4}$, as claimed. \square

We finally calculate $\iota_\tau^*(w_\tau)$. The strategy is similar to that used in the above proof.

Proposition 5.8. — *In $H^4(W_\tau, \mathbb{Q})$, we have*

$$\iota_\tau^*(w_\tau) = \frac{8}{5} \bar{q}_{W_\tau} + \delta_\tau^2 + \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha}^2 - \frac{1}{2} \delta_\tau \cdot \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha}.$$

Proof. — We start by calculating the following intersection numbers:

$$\begin{aligned} \iota_\tau^*(w_\tau) \cdot \bar{q}_{W_\tau} &= 70; \\ \iota_\tau^*(w_\tau) \cdot \iota_{\tau'}^*(w_{\tau'}) &= 12 \text{ for any } \tau \neq \tau' \in A_2; \\ \iota_\tau^*(w_\tau) \cdot \iota_\tau^*(\bar{q}_{K^3(A)}) &= 84. \end{aligned}$$

Recall that $\iota_\tau^*(w_\tau)$ equals the top Chern class $c_2(N_{W_\tau|K^3(A)})$ of the normal bundle. By §5.1, we have $\bar{q}_{W_\tau} = \frac{5}{6} c_2(W_\tau)$ and $\deg(c_4(W_\tau)) = 324$. Via (9) we then find

$$\bar{q}_{W_\tau} \cdot \iota_\tau^*(w_\tau) = \frac{5}{6} c_2(W_\tau) \cdot c_2(N_{W_\tau|K^3(A)}) = \frac{5}{6} \cdot (c_4(K^3(A)) \cdot w_\tau - \deg(c_4(W_\tau))) = 70,$$

as $c_4(K^3(A)) \cdot w_\tau = 408$ by Remark 4.8. For any $\tau \neq \tau'$ we have

$$\iota_\tau^*(w_\tau) \cdot \iota_\tau^*(w_{\tau'}) = \int_{K^3(A)} w_\tau^2 \cdot w_{\tau'} = 12$$

by Proposition 4.4. Finally,

$$\iota_\tau^*(w_\tau) \cdot \iota_\tau^*(\bar{q}_{K^3(A)}) = \int_{K^3(A)} w_\tau^2 \cdot \bar{q}_{K^3(A)} = C(\bar{q}_{K^3(A)} \cdot w_\tau^2) = 84,$$

where we used that $C(w_\tau^2) = 12$ (by Remark 4.8) and [4, Proposition 2.4] to calculate the generalized Fujiki constant $C(\bar{q}_{K^3(A)} \cdot w_\tau^2)$.

Now, according to Lemma 5.4 we can write

$$\iota_\tau^*(w_\tau) = \eta \cdot \iota_\tau^*(\bar{q}_{K^3(A)}) + \sum_{\theta \in A_2} \beta_\theta \cdot \sum_{\alpha \in A_{2,\tau}} s'_{\tau,\alpha} \cdot s'_{\tau,\alpha+\theta},$$

for some coefficients η and β_θ , $\theta \in A_2$. We claim that $\beta_\theta = \beta_{\theta'}$ whenever $\theta \neq 0, \theta' \neq 0$. Indeed, $\iota_\tau^*(w_\tau) \cdot (\iota_\tau^*(w_{\tau+\theta}) - \iota_\tau^*(w_{\tau+\theta'})) = 0$, and, by Lemma 5.7, we have

$$\iota_\tau^*(w_{\tau+\theta}) - \iota_\tau^*(w_{\tau+\theta'}) = -\frac{1}{4} \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\theta} + \frac{1}{4} \sum_{\alpha \in A_{2,\tau}} s_{\tau,\alpha} \cdot s_{\tau,\alpha+\theta'}.$$

Using Remark 5.5 and Lemma 5.1, we obtain the equation

$$-16\beta_\theta \cdot \frac{128}{4} + 16\beta_{\theta'} \cdot \frac{128}{4} = 0,$$

which proves the claim. We therefore have

$$\iota_\tau^*(w_\tau) = \eta \cdot \iota_\tau^*(\bar{q}_{K^3(A)}) + \beta \sum_{\alpha \in A_{2,\tau}} (s'_{\tau,\alpha})^2 + \gamma \sum_{0 \neq \theta \in A_2} \sum_{\alpha \in A_{2,\tau}} s'_{\tau,\alpha} \cdot s'_{\tau,\alpha+\theta},$$

for three coefficients η, β, γ . Via Remark 5.5 and Lemma 5.6, we write $\iota^*(w_\tau)$ in terms of the vectors of Lemma 5.1 and we calculate the intersection of $\iota_\tau^*(w_\tau)$ with the three classes \bar{q}_{W_τ} , $\sum_{\tau' \neq \tau} \iota^*(w_{\tau'})$ and $\iota_\tau^*(\bar{q}_{K^3(A)})$ in terms of the coefficients η, β, γ . The computation is easily done (with the help of a computer) via Lemma 5.1, using the explicit formulae for $\iota^*(w'_\tau)$ and $\iota^*(\bar{q}_{K^3(A)})$ found in Lemma 5.7 and Lemma 5.6 respectively. We obtain the linear system

$$\begin{cases} 350\eta - 13600\beta - 12000\gamma &= 70; \\ 420\eta - 8640\beta - 22080\gamma &= 180; \\ 252\eta - 7616\beta - 6720\gamma &= 84, \end{cases}$$

which gives the solution $\eta = \frac{4}{5}$, $\beta = \frac{9}{640}$, $\gamma = \frac{1}{640}$. Writing $\iota_\tau^*(w_\tau)$ in terms of the vectors of Lemma 5.1 leads to the claimed expression. \square

6. Canonical Hodge classes of degree six

Our next goal is to show that canonical Hodge classes in the middle cohomology of a Kum³-manifold K are analytic. We shall first determine the subalgebra of the cohomology which is fixed by the group G (§2.5). We keep using the notation from the previous sections.

Proposition 6.1. — *In terms of the LLV-decomposition, we have*

$$H^\bullet(K, \mathbb{Q})^G = V_{(3)} \oplus V_{(1,1)} \oplus 16V.$$

Proof. — We may and will assume that $K = K^3(A)$ is the generalized Kummer sixfold on an abelian surface A . Recall that the LLV-decomposition (3) gives

$$H^\bullet(K, \mathbb{Q}) = V_{(3)} \oplus V_{(1,1)} \oplus 16V \oplus 240\mathbb{Q} \oplus V_{\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}.$$

The group G acts trivially on the summand $V_{(3)}$ and it has no invariants in the odd cohomology. Moreover, G acts trivially on the summand $16V$, because, by Proposition 4.3, this is generated as LLV-module by the G -invariant classes c_2 and w_τ for $\tau \in A_2$.

We prove that $V_{(1,1)} \subset H^\bullet(K, \mathbb{Q})^G$. Let Y_K be the manifold of K3^[3]-type given by Theorem 3.1. By Remark 3.2, we have $Y_K = \tilde{K}/G$, where \tilde{K} is the successive blow-up of K along 16 smooth fourfolds \bar{W}_τ . Each \bar{W}_τ is birational to the K3^[2]-variety W_τ , and hence $h^{2,0}(\bar{W}_\tau) = 1$. Since $h^{3,1}(K) = 6$, the blow-up formula for cohomology yields the Hodge numbers $h^{4,0}(\tilde{K}) = 1$ and $h^{3,1}(\tilde{K}) = 6 + 16 = 22$. On the other hand, we have $h^{4,0}(Y_K) = 1$ and $h^{3,1}(Y_K) = 22$, as for any manifold of K3^[3]-type ([22]). By construction, we have an isomorphism $H^\bullet(Y_K, \mathbb{Q}) = H^\bullet(\tilde{K}, \mathbb{Q})^G$ and a G -equivariant embedding of $H^\bullet(K, \mathbb{Q})$ into $H^\bullet(\tilde{K}, \mathbb{Q})$. The Hodge numbers just calculated thus imply that the action of G on $H^4(K, \mathbb{C})$ is trivial on $H^{4,0}(K)$ and $H^{3,1}(K)$. As Hodge structure, $V_{(1,1)} \cap H^4(K, \mathbb{Q})$ is a Tate twist of $H^2(K, \mathbb{Q})$; it must therefore consists of G -invariants, because G commutes with the LLV-Lie algebra. We conclude that $V_{(1,1)}$ is contained into $H^\bullet(K, \mathbb{Q})^G$.

We next show that G has no invariants in the LLV-summand $240\mathbb{Q}$. To this end, we compute the traces of $g \in G = A_2 \times \langle -1 \rangle$ acting on this summand. Let

$$\chi(g) := \sum_i (-1)^i \operatorname{tr} \left(g_{|_{H^i(K^3(A), \mathbb{Q})}}^* \right).$$

By the Lefschetz trace formula, $\chi(g)$ equals the Euler characteristic of the fixed locus K^g , which consists of 8 K3 surfaces for non-trivial $g = (\tau, 1)$ and of the union of the K3^[2]-manifold W_τ and 140 isolated points for $g = (\tau, -1)$ (see [15, Proposition 2.10]). Hence,

$$\chi(g) = \begin{cases} 448 & \text{if } g = 1; \\ 464 & \text{if } g = (\tau, -1) \text{ for } \tau \in A_2; \\ 192 & \text{if } g = (\tau, 1) \text{ for } 0 \neq \tau \in A_2. \end{cases}$$

An automorphism $(\tau, \pm 1) \in G$ acts on the odd cohomology of K as multiplication by ± 1 . From the above and table (4), we deduce that

$$\mathrm{tr}(g_{|240\mathbb{Q}}^*) = \begin{cases} 240 & \text{if } g = 1; \\ 0 & \text{if } g = (\tau, -1) \text{ for } \tau \in A_2; \\ -16 & \text{if } g = (\tau, 1) \text{ for } 0 \neq \tau \in A_2. \end{cases}$$

We conclude that

$$\dim(240\mathbb{Q})^G = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(g_{|240\mathbb{Q}}^*) = \frac{1}{32}(240 - 16 \cdot 15) = 0.$$

□

Remark 6.2. — Automorphisms in $\mathrm{Aut}_0(K) \setminus G$ acts non-trivially on $H^4(K, \mathbb{Q})$, by Propositions 3.5 and 4.3. Therefore, the above proposition characterizes G as the group of automorphisms of K which act trivially on the cohomology in degrees 2 and 4.

6.1. Canonical Hodge classes are analytic. — As in the previous sections, we consider the generalized Kummer manifold $K^3(A)$ on a 2-dimensional complex torus A , and the submanifolds $\iota_\tau: W_\tau \hookrightarrow K^3(A)$, for $\tau \in A_2$. Recall that we introduced 16 divisor classes $s'_{\tau, \alpha}$ on each W_τ in Definition 5.2.

Definition 6.3. — For any $\tau \in A_2$ and $\alpha \in A_{2, \tau}$ we define

$$d_{\tau, \alpha} := \iota_{\tau, *}(s'_{\tau, \alpha}) \in H^6(K^3(A), \mathbb{Z}).$$

Lemma 6.4. — The classes $d_{\tau, \alpha}$ are canonical Hodge classes, and they remain analytic on any deformation of $K^3(A)$.

Proof. — Let $\mathcal{K} \rightarrow B$ be a deformation of $K^3(A)$, and consider the induced deformation $\mathcal{W}_\tau \rightarrow B$ of W_τ as in Remark 3.6. A class $d_b \in H^6(\mathcal{K}_b, \mathbb{Q})$ obtained as parallel transport of some $d_{\tau, \alpha}$ is of the form $\iota_{\tau, b, *}(s'_b)$, where $s'_b \in H^2(\mathcal{W}_{\tau, b}, \mathbb{Q})$ is the parallel transport of $s'_{\tau, \alpha}$; here, $\iota_{\tau, b}: \mathcal{W}_{\tau, b} \hookrightarrow \mathcal{K}_b$ is the inclusion. By Lemma 5.3, s'_b is a Hodge class. By Lefschetz (1,1) theorem we have $s'_b = c_1(L_b)$ for some line bundle L_b on $\mathcal{W}_{\tau, b}$. A Grothendieck-Riemann-Roch computation (using that hyper-Kähler manifolds have trivial first Chern class) shows that $d_b = c_3(\iota_{\tau, *}(L_b))$, and hence d_b is an analytic class on \mathcal{K}_b . □

The next statement thus implies that canonical Hodge classes of degree 6 are analytic, for any manifold of Kum^3 -type.

Proposition 6.5. — The classes $d_{\tau, \alpha}$ generate the subspace of canonical Hodge classes in $H^6(K^3(A), \mathbb{Q})$.

We will use the following intersection numbers to prove the above proposition.

Proposition 6.6. — Fix $\tau \in A_2$. For $\alpha, \alpha' \in A_{2,\tau}$, we have

$$d_{\tau,\alpha} \cdot d_{\tau,\alpha'} = \begin{cases} -52 & \text{if } \alpha = \alpha'; \\ 12 & \text{if } \alpha \neq \alpha'. \end{cases}$$

Proof. — Since $c_2(N_{W_\tau|K^3(A)}) = \iota_\tau^*(w_\tau)$, by [20, Corollary 6.3] we have

$$d_{\tau,\alpha} \cdot d_{\tau,\alpha'} = \int_{W_\tau} \iota_\tau^*(w_\tau) \cdot s'_{\tau,\alpha} \cdot s'_{\tau,\alpha'},$$

for any $\alpha \in A_{2,\tau}$. It is straightforward to compute these numbers with the intersection form (11), using the expression for $\iota_\tau^*(w_\tau)$ of Proposition 5.8, the definition of $s'_{\tau,\alpha}$ (Definition 5.2) and the intersections determined in the proof of Lemma 5.1. \square

Proof of Proposition 6.5. — Fix $\tau \in A_2$. With the intersection numbers of Proposition 6.6, an argument entirely analogous to that used in the proof of Proposition 4.3 shows that the 16 classes $d_{\tau,\alpha}$, for $\alpha \in A_{2,\tau}$, are linearly independent. The G -invariants in $\langle d_{\tau,\alpha} \rangle_{\alpha \in A_{2,\tau}}$ are the multiples of $\sum_{\alpha} d_{\tau,\alpha}$ (cf. Remark 3.9). This class is non-zero since $(\sum_{\alpha} d_{\tau,\alpha}) \cdot (\sum_{\alpha} d_{\tau,\alpha}) = 2048$ by Proposition 6.6. But, by Propositions 2.4 and 6.1, any G -invariant canonical Hodge class in $H^6(K^3(A), \mathbb{Q})$ is actually invariant under the action of the entire group $\text{Aut}_0(K^3(A))$. Since translation by $\epsilon \in A_4$ maps $d_{\tau,\alpha}$ to $d_{\tau+2\epsilon,\alpha+\epsilon}$, we must have $\sum_{\alpha \in A_{2,\tau}} d_{\tau,\alpha} = \sum_{\alpha' \in A_{2,\tau'}} d_{\tau',\alpha'}$, for any $\tau, \tau' \in A_2$.

Next, we observe that for $\tau \neq \tau'$ in A_2 the value $d_{\tau,\alpha} \cdot d_{\tau',\alpha'}$ is independent of α and α' . Indeed, by Definition 6.3 and Remark 3.12, the number in question equals

$$-8(\iota_{\tau,*}(s_{\tau,\alpha}) \cdot \iota_{\tau',*}(\delta_{\tau'}) + \iota_{\tau,*}(\delta_{\tau}) \cdot \iota_{\tau',*}(s_{\tau',\alpha'})),$$

and using the symmetries in G we see that the numbers $\iota_{\tau,*}(s_{\tau,\alpha}) \cdot \iota_{\tau',*}(\delta_{\tau'})$ do not depend on $\alpha \in A_{2,\tau}$. Since we have just shown that $\sum_{\alpha \in A_{2,\tau}} d_{\tau,\alpha} = \sum_{\alpha' \in A_{2,\tau'}} d_{\tau',\alpha'}$, Proposition 6.6 gives

$$\left(\sum_{\alpha \in A_{2,\tau}} d_{\tau,\alpha} \right) \cdot d_{\tau',\alpha'} = 128,$$

Therefore $d_{\tau,\alpha} \cdot d_{\tau',\alpha'} = 8$ whenever $\tau \neq \tau'$.

Assume now that a linear combination $u = \sum_{\tau \in A_2} \sum_{\alpha \in A_{2,\tau}} \beta_{\tau,\alpha} \cdot d_{\tau,\alpha}$ is zero. Taking the intersection product of u with $(d_{\tau,\alpha} - d_{\tau,\alpha'})$ we obtain that $\beta_{\tau,\alpha} = \beta_{\tau,\alpha'}$ for all $\alpha, \alpha' \in A_{2,\tau}$. Thus we can write $u = \sum_{\tau \in A_2} \gamma_{\tau} \sum_{\alpha \in A_{2,\tau}} d_{\tau,\alpha}$, and hence we must have $\sum_{\tau} \gamma_{\tau} = 0$. This shows that the \mathbb{Q} -vector space of linear relations between the 256 classes $d_{\tau,\alpha}$ is generated by $\sum_{\alpha} d_{\tau,\alpha} - \sum_{\alpha'} d_{\tau',\alpha'}$, for τ, τ' in A_2 . This space has dimension 15, and hence the 256 classes $d_{\tau,\alpha}$ span a subspace of $H^6(K^3(A), \mathbb{Q})$ of dimension 241, which therefore must be the whole 241-dimensional space of canonical Hodge classes in $H^6(K^3(A), \mathbb{Q})$. \square

7. Algebraic cycles on Kum³-varieties

The work done so far proves the following.

Theorem 7.1. — *Let K be any manifold of Kum³-type. All canonical Hodge classes on K are analytic.*

Proof. — The assertion for canonical Hodge classes in $H^4(K, \mathbb{Q})$ (resp. in $H^6(K, \mathbb{Q})$) follows from Lemma 4.2 and Proposition 4.3 (resp. from Lemma 6.4 and Proposition 6.5). By Remark 4.9, the cup-product with \bar{q}_K induces an isomorphism between canonical Hodge classes in $H^4(K, \mathbb{Q})$ and those in $H^8(K, \mathbb{Q})$, which therefore are analytic as well. \square

We can now complete the proof of our main result.

Proof of Theorem 1.1. — Let K be any projective manifold of Kum³-type, and consider the action of the group $G \subset \text{Aut}_0(K)$ on the cohomology. By Proposition 6.1, we can write:

$$H^\bullet(K, \mathbb{Q}) = (H^\bullet(K, \mathbb{Q}))^G \oplus H^{\text{odd}}(K, \mathbb{Q}) \oplus 240\mathbb{Q}.$$

The odd cohomology obviously does not contain Hodge classes. The last summand consists of canonical Hodge classes, which are all algebraic by Theorem 7.1. It remains to show that the Hodge classes in $(H^\bullet(K, \mathbb{Q}))^G$ are algebraic.

By Theorem 3.1, we have a diagram

$$\begin{array}{ccc} & \tilde{K} & \\ b \swarrow & & \searrow q \\ K & \xrightarrow{\quad r \quad} & Y_K, \end{array}$$

where $b: \tilde{K} \rightarrow K$ is the successive blow-up of 16 smooth fourfolds, the action of G extends to an action on \tilde{K} , and the quotient $q: \tilde{K} \rightarrow Y_K$ gives a variety of K3^[3]-type. Moreover, Y_K is a smooth and projective moduli space of stable sheaves on the K3 surface S_K . The Hodge conjecture holds for any power of S_K by [15, Corollary 5.8]; by a result of Bülles [8], this implies that the Hodge conjecture holds for Y_K as well.

The pull-back along q identifies the cohomology of Y_K with $H^\bullet(\tilde{K}, \mathbb{Q})^G$, while the pull-back along b embeds G -equivariantly the rational cohomology of K into that of \tilde{K} . The left-inverse of b^* is the push-forward b_* . Therefore, $r^* := b_* q^*$ induces a surjection

$$r^*: H^\bullet(Y_K, \mathbb{Q}) \rightarrow (H^\bullet(K, \mathbb{Q}))^G.$$

Note that r^* is induced by an algebraic correspondence. Since the Hodge conjecture holds for Y_K , any Hodge class in $(H^\bullet(K, \mathbb{Q}))^G$ is algebraic. \square

Proof of Theorem 1.2. — By the works [14], [44] and [17], the Mumford-Tate conjecture holds for any hyper-Kähler variety of known deformation type; the final result may be found in [17, Theorem 1.18]. See [34, §2.1] for the statement of this conjecture. If X/k is a smooth and projective variety over the finitely generated field $k \subset \mathbb{C}$ for which the Mumford-Tate conjecture holds, then the Galois representation on the ℓ -adic cohomology of X is semisimple and the Hodge conjecture for $X_{\mathbb{C}}$ is equivalent to the Tate conjecture for X/k . Therefore, Theorem 1.2 follows from Theorem 1.1. \square

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