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Salvatore Floccari

# On the Mumford–Tate conjecture for hyperkähler varieties

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**Abstract.** We study the Mumford–Tate conjecture for hyperkähler varieties. We show that the full conjecture holds for all varieties deformation equivalent to either an Hilbert scheme of points on a K3 surface or to O'Grady's ten dimensional example, and all of their self-products. For an arbitrary hyperkähler variety whose second Betti number is not 3, we prove the Mumford–Tate conjecture in every codimension under the assumption that the Künneth components in even degree of its André motive are abelian. Our results extend a theorem of André.

**Key words.** Mumford–Tate conjecture · Hyperkähler varieties · Motives · Hodge theory

## 1. Introduction

Let $k \subset \mathbb{C}$ be a finitely generated field, with algebraic closure $\bar{k} \subset \mathbb{C}$, and let $\ell$ be a prime number. Given a smooth and projective variety $X$ over $k$, Artin's comparison theorem gives a canonical identification of $\mathbb{Q}_\ell$-vector spaces

$$H^i_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^i_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell)$$

between singular cohomology groups of $X(\mathbb{C})$ and $\ell$-adic cohomology groups of $X_{\bar{k}}$.

Both sides come with additional structure, namely, a Hodge structure on the left hand side and a Galois representation on the right hand side. These data are encoded in the corresponding tannakian fundamental groups. The Mumford–Tate conjecture predicts that Artin's comparison isomorphism identifies the two groups. We refer to this statement for $i = 2j$ as the Mumford–Tate conjecture in codimension $j$ for $X$.

The Mumford–Tate conjecture is a difficult open problem. It is known only in a very limited number of cases, see [16, Sects. 2.4, 3.3, 4.4] for a recent survey.

S. Floccari (✉): Leibniz University Hannover, Hannover, Germany
e-mail: floccari@math.uni-hannover.de

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### 1.1. Results

Our main result establishes the Mumford–Tate conjecture for hyperkähler varieties $X$ over $k$ that are of $\mathrm{K3}^{[m]}$ or OG10-type, *i.e.*, such that the complex manifold $X(\mathbb{C})$ is a deformation of the Hilbert scheme of zero-dimensional subschemes of length $m$ on a K3 surface [3] or of O'Grady's ten dimensional hyperkähler manifold [18]. The second Betti number of $X$ is 23 in the first case and 24 in the second.

**Theorem 1.1.** *Let $X$ be a hyperkähler variety of either* $\mathrm{K3}^{[m]}$ *or* OG10*-type. Then, the Mumford–Tate conjecture holds in any codimension for $X$ and for all self-products $X^j$.*

Our second result establishes the Mumford–Tate conjecture in any codimension for a hyperkähler variety with $b_2 > 3$ whose even André motive is abelian.

**Theorem 1.2.** *Let $X$ be a hyperkähler variety such that $b_2(X) > 3$. Assume that, for all $i \geq 0$, the component in degree $2i$ of the André motive of $X$ is an abelian motive. Then, the Mumford–Tate conjecture holds in any codimension for $X$. In particular, the Hodge and Tate conjecture for $X$ are equivalent.*

The work [13] suggests that any hyperkähler variety has abelian André motive, however, for the time being, this statement remains a conjecture (but see Sect. 1.3). By definition, the second Betti number of a hyperkähler variety is always at least 3; all known examples satisfy $b_2 > 3$ and it is believed that no hyperkähler variety with $b_2 = 3$ exists [4, Question 4].

### 1.2. Overview of the contents

We recall in Sect. 2 the statement of the Mumford–Tate conjecture and its motivic version; throughout, we will use the category of motives constructed by André in [2]. The following theorem is essentially proven in [1], and it has been generalized in [17].

**Theorem 1.3.** *(André). Let $X$ be a hyperkähler variety such that $b_2(X) > 3$. Then, the motivic Mumford–Tate conjecture in codimension 1 holds for $X$.*

The main tool used in the proof of this result is the Kuga-Satake construction in families, building on ideas due to Deligne [6]. The assumption that $b_2(X) > 3$ ensures the existence of non-trivial deformations of $X$, as otherwise the moduli space of hyperkähler varieties deformation equivalent to $X$ would be zero-dimensional.

With $X$ as above, we consider the even part $H^+_{\mathrm{B}}(X)$ of the singular cohomology with rational coefficients of $X(\mathbb{C})$, so $H^+_{\mathrm{B}}(X) = \bigoplus_i H^{2i}_{\mathrm{B}}(X)$. A crucial ingredient for us is the action of a $\mathbb{Q}$-Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ on $H^+_{\mathrm{B}}(X)$. This construction is due to Verbitsky [21] and Looijenga-Lunts [14]; we recall it in Sect 3. The even singular cohomology of $X$ is the Hodge realization of a motive $\mathcal{H}^+(X)$, whose motivic Galois group is denoted by $\mathrm{G}^+_{\mathrm{mot}}(X)$. We study the interplay between the actions of this group and of the Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ on $H^+_{\mathrm{B}}(X)$. This cohomology algebra carries a Hodge structure, whose Mumford–Tate group is denoted by $\mathrm{MT}^+(X)$; we show in Sect. 4 that $\mathrm{MT}^+(X)$ is a direct factor of the motivic Galois group $\mathrm{G}^+_{\mathrm{mot}}(X)$. Here, we need to assume that $b_2(X) > 3$ since we use André's Theorem 1.3.

In Sect. 5 we prove that if $\mathrm{MT}^+(X)$ has finite index in the motivic Galois group $\mathrm{G}^+_{\mathrm{mot}}(X)$, then the Mumford–Tate conjecture holds in arbitrary codimension for $X$, see Proposition 5.1. The proof of Theorem 1.2 is given in Sect. 5.2; this is in fact a direct consequence of the proposition and a general result on abelian motives due to André.

In Sect. 6 we complete the proof of our main result Theorem 1.1. By Proposition 4.1, $\mathrm{MT}^+(X)$ is a direct product factor of $\mathrm{G}^+_{\mathrm{mot}}(X)$; moreover, the complement satisfies various constraints and in particular it commutes with the action of $\mathfrak{g}_{\mathrm{tot}}(X)$, see Lemma 4.3. For the $\mathrm{K3}^{[m]}$-type, we have a very effective understanding of this action thanks to work of Markman [15], and we deduce from his results that the Mumford–Tate group has finite index in the motivic Galois group. For the OG10-type, this finiteness follows instead from the complete description of the $\mathfrak{g}_{\mathrm{tot}}(X)$-representation on the cohomology given by Green-Kim-Laza-Robles [10]. In both cases, we apply Proposition 5.1 to conclude.

### *1.3. Related works*

The abelianity of the André motives of varieties of deformation type $\mathrm{K3}^{[m]}$, $\mathrm{Kum}_m$ and OG6 has been recently established by Soldatenkov in [19]; our Theorem 1.2 then implies the Mumford–Tate conjecture in arbitrary codimension for these varieties. Successively, together with Lie Fu and Ziyu Zhang we have shown in [8] the abelianity of the André motives of varieties of OG10-type, the fourth and last known deformation type of hyperkähler manifolds, and established the full statement of the Mumford–Tate conjecture for all products of hyperkähler varieties of known deformation type.

In each case, the proof requires a deformation to an explicit example in the given deformation type. We remark that the proof of Theorem 1.1 presented here is different and simpler: it uses neither deformation to a specific example, nor abelianity of the motives involved. We hope that a refinement of this method might lead to a proof of the Mumford–Tate conjecture for arbitrary hyperkähler varieties with $b_2 > 3$.

### *1.4. Notation and conventions*

Throughout the whole text, $k \subset \mathbb{C}$ will be a finitely generated field with algebraic closure $\bar{k} \subset \mathbb{C}$, and $\ell$ will be a fixed prime number. A hyperkähler variety over $k$ is a smooth projective variety over $k$ such that $X(\mathbb{C})$ is a hyperkähler manifold, as defined in Sect. 2. Given a complex variety $X$, we denote by $H^i(X)$ its rational singular cohomology groups. The word "motive" always indicates an object of André's category of motives (see Sect. 2.4).

## 2. The Mumford–Tate conjecture

We refer to [16] and the references therein. With notations and assumptions as in Sect. 1.4, we let $X$ be a smooth projective variety over the field $k$. We can extract information about $X$ by looking at various cohomology groups.

### *2.1. Betti cohomology*

We denote by $H^i_{\mathrm{B}}(X)$ the $i$-th singular cohomology group with rational coefficients of the complex manifold $X(\mathbb{C})$. It carries a pure polarizable $\mathbb{Q}$-Hodge structure of weight $i$. Associated to $H^i_{\mathrm{B}}(X)$ is its Mumford–Tate group $\mathrm{MT}\big(H^i_{\mathrm{B}}(X)\big)$; it is a reductive, connected algebraic subgroup of $\mathrm{GL}\big(H^i_{\mathrm{B}}(X)\big)$.

### *2.2. $\ell$-adic cohomology*

We write $H^i_\ell(X)$ for the $i$-th étale cohomology group of $X_{\bar{k}}$ with $\mathbb{Q}_\ell$-coefficients, which comes with a continuous representation $\sigma_\ell\colon \mathrm{Gal}(\bar{k}/k) \to \mathrm{GL}\big(H^i_\ell(X)\big)$; we denote by $\mathcal{G}\big(H^i_\ell(X)\big)$ the Zariski closure of the image of $\sigma_\ell$. It is an algebraic group over $\mathbb{Q}_\ell$. If $k'/k$ is a field extension, and if $X_{k'}$ denotes the base change of $X$ to $k'$, it may happen that $\mathcal{G}\big(H^i_\ell(X_{k'})\big)$ becomes smaller than $\mathcal{G}\big(H^i_\ell(X)\big)$; however, the connected component of the identity $\mathcal{G}\big(H^i_\ell(X)\big)^0$ is stable under finite field extensions, and there exists a finite field extension $k'/k$ such that $\mathcal{G}\big(H^i_\ell(X_{k'})\big)$ becomes connected.

### *2.3. The statement*

Artin's comparison theorem states that, for all $X$ and $i$ as above, there is a canonical isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$H^i_{\mathrm{B}}(X) \otimes \mathbb{Q}_\ell \cong H^i_\ell(X).$$

**Conjecture 2.1.** *(Mumford–Tate). Under the isomorphism of algebraic groups* $\mathrm{GL}\big(H^i_{\mathrm{B}}(X)\big) \otimes \mathbb{Q}_\ell \cong \mathrm{GL}\big(H^i_\ell(X)\big)$ *induced by Artin's isomorphism, we have*

$$\mathrm{MT}\big(H^i_{\mathrm{B}}(X)\big) \otimes \mathbb{Q}_\ell = \mathcal{G}\big(H^i_\ell(X)\big)^0.$$

*The Mumford–Tate conjecture in codimension $j$ for $X$ is this statement for $i = 2j$.*

### *2.4. Motives*

A third algebraic group is often useful in order to compare the two groups involved in the Mumford–Tate conjecture. Let $\mathrm{Mot}_k$ be the category of André motives over $k$ from [2]; it is a $\mathbb{Q}$-linear neutral tannakian semisimple category. We will denote motives by calligraphic letters. Let $\mathcal{M} \in \mathrm{Mot}_k$. For a field extension $k'/k$, we let $\mathcal{M}_{k'}$ be the motive over $k'$ obtained from $\mathcal{M}$ via base change.

### 2.5. *Realization I*

The inclusion $k \subset \mathbb{C}$ determines a realization functor from $\mathrm{Mot}_k$ to the category of polarizable $\mathbb{Q}$-Hodge structures, and we write $\mathcal{M}_{\mathrm{B}}$ for the Hodge realization of the motive $\mathcal{M}$. The composition with the forgetful functor to $\mathbb{Q}$-vector spaces is a fibre functor on $\mathrm{Mot}_k$; the tannakian formalism then yields a reductive $\mathbb{Q}$-algebraic group $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$, which is a subgroup of $\mathrm{GL}(\mathcal{M}_{\mathrm{B}})$. The tannakian subcategory $\langle\mathcal{M}\rangle^{\otimes}$ of $\mathrm{Mot}_k$ generated by $\mathcal{M}$ is equivalent to the category of finite dimensional representations of $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$. We call this group the motivic Galois group of $\mathcal{M}$.

### 2.6. *Realization II*

The prime $\ell$ determines another realization functor to the category of $\ell$-adic Galois representations; we write $\mathcal{M}_\ell$ for the Galois representation attached to the motive $\mathcal{M}$. We obtain an algebraic group $\mathrm{G}_{\mathrm{mot},\ell}(\mathcal{M}) \subset \mathrm{GL}(\mathcal{M}_\ell)$ over $\mathbb{Q}_\ell$ such that the category of its finite dimensional representations is equivalent to $\langle\mathcal{M}\rangle^{\otimes} \otimes \mathbb{Q}_\ell$. Artin's comparison theorem yields an isomorphism $\mathcal{M}_{\mathrm{B}} \otimes \mathbb{Q}_\ell \cong \mathcal{M}_\ell$, inducing an identification $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}) \otimes \mathbb{Q}_\ell = \mathrm{G}_{\mathrm{mot},\ell}(\mathcal{M})$ of subgroups of $\mathrm{GL}(\mathcal{M}_{\mathrm{B}}) \otimes \mathbb{Q}_\ell \cong \mathrm{GL}(\mathcal{M}_\ell)$.

### 2.7. *The motivic Mumford–Tate conjecture*

We refer to [16, Sect. 3.1] for an enlightening discussion of the behaviour of $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$ under extensions of the base field. It suffices to say that there exists a finite field extension $k^\diamond/k$ such that $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{k'}) \cong \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{k^\diamond})$ for all field extensions $k'/k^\diamond$.

**Conjecture 2.2.** *(Motivic Mumford–Tate). For any motive $\mathcal{M} \in \mathrm{Mot}_k$, we have*

$$\mathrm{MT}\big(\mathcal{M}_{\mathrm{B}}\big) = \mathrm{G}_{\mathrm{mot}}\big(\mathcal{M}_{\bar{k}}\big), \text{ and } \mathcal{G}\big(\mathcal{M}_\ell\big)^0 = \mathrm{G}_{\mathrm{mot},\ell}\big(\mathcal{M}_{\bar{k}}\big).$$

The conjecture is the conjunction of the motivic Hodge and Tate conjectures: the first says that Hodge classes are motivated, hence $\mathrm{MT}\big(\mathcal{M}_{\mathrm{B}}\big) = \mathrm{G}_{\mathrm{mot}}\big(\mathcal{M}_{\bar{k}}\big)$, and the second says that Tate classes are motivated, and hence $\mathcal{G}\big(\mathcal{M}_\ell\big)^0 = \mathrm{G}_{\mathrm{mot},\ell}\big(\mathcal{M}_{\bar{k}}\big)$. These statements are weak versions of the usual Hodge and Tate conjectures respectively.

We summarize a few known facts about these groups.

- There are natural inclusions

$$\mathrm{MT}(\mathcal{M}_{\mathrm{B}}) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{k}}) \text{ and } \mathcal{G}(\mathcal{M}_\ell)^0 \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{k}}) \otimes \mathbb{Q}_\ell.$$

- The algebraic group $\mathrm{MT}(\mathcal{M}_{\mathrm{B}})$ is connected and reductive. On the other hand, $\mathcal{G}(\mathcal{M}_\ell)^0$ is not known to be reductive, while $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{k}})$ is reductive, but not known to be connected in general.

There are contravariant functors $\mathcal{H}^i$ from the category of smooth projective varieties over $k$ to $\mathrm{Mot}_k$, such that, for any smooth projective variety $X$ over $k$, we have

$$\mathcal{H}^i(X)_{\mathrm{B}} = H^i_{\mathrm{B}}(X) \text{ and } \mathcal{H}^i(X)_\ell = H^i_\ell(X).$$

Therefore, Conjecture 2.2 implies Conjecture 2.1. We refer to Conjecture 2.2 for the motive $\mathcal{H}^{2j}(X)$ as the motivic Mumford–Tate conjecture for $X$ in codimension $j$.

### *2.8. Abelian motives*

A motive $\mathcal{M} \in \mathrm{Mot}_k$ is abelian if it belongs to the tannakian subcategory generated by the motives of all abelian varieties over $k$. We will need the following theorem due to André [2], which improves Deligne's result on absolute Hodge classes on abelian varieties from [7].

**Theorem 2.3.** *Let $\mathcal{M} \in \mathrm{Mot}_k$ be an abelian motive. Then we have*

$$\mathrm{MT}(\mathcal{M}_{\mathrm{B}}) = \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{k}}).$$

## 3. Hyperkähler varieties

In this section we work over the complex numbers. A hyperkähler manifold $X$ is a connected, simply connected, compact Kähler manifold admitting a nowhere degenerate holomorphic 2-form which spans $H^{0,2}(X)$. At times, we use the expression "hyperkähler variety" instead of writing "projective hyperkähler manifold". The dimension of such a manifold is always even; hyperkähler surfaces are K3 surfaces. The second cohomology group of a hyperkähler manifold $X$ carries a canonical symmetric bilinear form, the Beauville–Bogomolov form, which is non-degenerate and deformation invariant, and yields a morphism of Hodge structures $H^2(X)(1)\otimes H^2(X)(1) \to \mathbb{Q}$. We refer to [3] and [12] for a proper introduction to the subject.

Let $X$ be a complex hyperkähler variety of dimension $2n$. The rational cohomology $H^*(X)$ of $X$ is a graded algebra via cup product. Verbitsky and Looijenga-Lunts studied in [21] and [14] a Lie algebra action on $H^*(X)$, which we describe below.

### *3.1. $\mathfrak{sl}_2$-triples*

Let $\theta \in \mathrm{End}\big(H^*(X)\big)$ be the degree 0 endomorphism whose action on $H^j(X)$ is multiplication by $j - 2n$, for all $j$. Given $x \in H^2(X)$, we denote by $L_x$ the endomorphism of $H^*(X)$ which maps a cohomology class $\alpha$ to the product $x \wedge \alpha$. We say that a class $x \in H^2(X)$ has the Lefschetz property if, for all positive integers $j$, the map $L_x^j \colon H^{2n-j}(X) \to H^{2n+j}(X)$ is an isomorphism. The Lefschetz property for $x \in H^2(X)$ is equivalent to the existence of $\Lambda_x \in \mathrm{End}\big(H^*(X)\big)$ such that $L_x, \theta$, and $\Lambda_x$ form an $\mathfrak{sl}_2$-triple, *i.e.*, we have

$$[\theta, L_x] = 2L_x, \quad [\theta, \Lambda_x] = -2\Lambda_x \quad \text{and} \quad [L_x, \Lambda_x] = \theta.$$

Once it exists, the endomorphism $\Lambda_x$ is uniquely determined, see for instance Proposition 1.4.6 in [9, Exposé X].

### *3.2. The total Lie algebra*

We define $\mathfrak{g}_{\mathrm{tot}}(X)$ as the smallest Lie subalgebra of $\mathfrak{gl}\big(H^*(X)\big)$ containing $L_x$, for all $x \in H^2(X)$, and $\Lambda_x$, for all $x \in H^2(X)$ with the Lefschetz property. The first Chern class of an ample divisor on $X$ has the Lefschetz property by the Hard Lefschetz theorem. It is shown in [14, Sect. (1.9)] that $\mathfrak{g}_{\mathrm{tot}}(X)$ is a semisimple $\mathbb{Q}$-Lie algebra, which is evenly graded by the adjoint action of $\theta$, so that $\mathfrak{g}_{\mathrm{tot}}(X) = \bigoplus_i \mathfrak{g}_{2i}(X)$. The action of $\mathfrak{g}_{\mathrm{tot}}(X)$ on the cohomology of $X$ preserves the even and odd cohomology, and the Lie subalgebra $\mathfrak{g}_0(X)$ consists of the endomorphisms contained in $\mathfrak{g}_{\mathrm{tot}}(X)$ which preserve the grading of $H^*(X)$. The construction does not depend on the complex structure of $X$; therefore, $\mathfrak{g}_{\mathrm{tot}}(X)$ is deformation invariant.

### *3.3. A theorem of Looijenga-Lunts and Verbitsky*

Let now $H$ denote the space $H^2(X)$ equipped with the Beauville-Bogomolov form. Let $\tilde{H}$ denote the orthogonal direct sum of $H$ with $U = \langle v, w\rangle$ equipped with the form $-2vw$. We summarize the main properties of the Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$.

**Theorem 3.1.**

(a) *There is an isomorphism $\mathfrak{g}_{\mathrm{tot}}(X) \cong \mathfrak{so}(\tilde{H})$ of $\mathbb{Q}$-Lie algebras, which maps the element $\theta \in \mathfrak{g}_{\mathrm{tot}}(X)$ to the element of $\mathfrak{so}(\tilde{H})$ which acts as multiplication by $-2$ on $v$, by $2$ on $w$, and by $0$ on $H$.*

(b) *We have*

$$\mathfrak{g}_{\mathrm{tot}}(X) = \mathfrak{g}_{-2}(X) \oplus \mathfrak{g}_0(X) \oplus \mathfrak{g}_2(X).$$

*Moreover, $\mathfrak{g}_0(X) \cong \mathfrak{so}(H) \oplus \mathbb{Q}\cdot\theta$, and $\theta$ is central in $\mathfrak{g}_0(X)$. The abelian subalgebra $\mathfrak{g}_2(X)$ is the linear span of the endomorphisms $L_x$, and $\mathfrak{g}_{-2}(X)$ is the span of the $\Lambda_x$, for $x \in H^2(X)$ with the Lefschetz property.*

(c) *The Lie subalgebra $\mathfrak{g}_0(X)$ acts via derivations on the graded algebra $H^*(X)$. The induced action of $\mathfrak{so}(H) \subset \mathfrak{g}_0(X)$ on $H^2(X) = H$ is the standard representation.*

The above theorem is proven in [21], and in [14, Proposition 4.5]. A proof can also be found in the appendix of [13]. These proofs are carried out with real coefficients, but immediately imply the result with rational coefficients: since $\mathfrak{g}_{\mathrm{tot}}(X)$ is defined over $\mathbb{Q}$, the equality $\mathfrak{g}_{\mathrm{tot}}(X) \otimes \mathbb{R} = \mathfrak{so}(\tilde{H}) \otimes \mathbb{R}$ of Lie subalgebras of $\mathfrak{gl}(\tilde{H}) \otimes \mathbb{R}$ shows that the same equality already holds with rational coefficients.

### 3.4. The integrated representation

We know from Theorem 3.1 that the semisimple part of $\mathfrak{g}_0(X)$ is isomorphic to $\mathfrak{so}(H)$. We denote by

$$\rho\colon \mathfrak{so}(H) \to \prod_j \mathfrak{gl}\big(H^j(X)\big)$$

the restriction of the representation $\mathfrak{g}_0(X) \to \prod_j \mathfrak{gl}\big(H^j(X)\big)$ to the Lie subalgebra $\mathfrak{so}(H)$. We also let $\rho^+\colon \mathfrak{so}(H) \to \prod_i \mathfrak{gl}\big(H^{2i}(X)\big)$ denote the representation induced by $\rho$ on the even cohomology of $X$.

**Proposition 3.2.** *The representation $\rho^+\colon \mathfrak{so}(H) \to \prod_i \mathfrak{gl}\big(H^{2i}(X)\big)$ integrates to a faithful representation*

$$\tilde{\rho}^+\colon\ \mathrm{SO}(H) \to \prod_i \mathrm{GL}\big(H^{2i}(X)\big),$$

*such that $\pi_2 \circ \tilde{\rho}^+\colon\ \mathrm{SO}(H) \to \mathrm{GL}\big(H^2(X)\big) = \mathrm{GL}(H)$ is the standard representation, where $\pi_2$ is the obvious projection $\prod_i \mathrm{GL}\big(H^{2i}(X)\big) \to \mathrm{GL}\big(H^2(X)\big)$.*

We refer to [20, Sect. 8] for a proof. Note that under the representation $\tilde{\rho}^+$, the group $\mathrm{SO}(H)$ acts via graded algebra automorphisms on the even cohomology of $X$, by part (c) of Theorem 3.1.

### 3.5. The Weil operator

We need to recall one more result. Let $W_{\mathbb{C}} \in \mathrm{End}\big(H^*(X, \mathbb{C})\big)$ be the endomorphism which acts on each $H^{p,q}(X)$ as multiplication by $i(p-q)$. It is known that $W_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of an endomorphism $W \in \mathrm{End}\big(H^*(X, \mathbb{R})\big)$, which is called the Weil operator.

**Theorem 3.3.** *The Weil operator $W$ is an element of $\rho\big(\mathfrak{so}(H)\big) \otimes \mathbb{R}$.*

This is proven in [21]; see also the appendix to the paper [13].

### 3.6. Hodge theory of hyperkähler varieties

Let $H^+(X)$ denote the weight 0 Hodge structure $\bigoplus_i H^{2i}(X)(i)$, and let $\mathrm{MT}^+(X)$ denote its Mumford–Tate group. Let $\pi_2\colon\ \mathrm{MT}^+(X) \twoheadrightarrow \mathrm{MT}\big(H^2(X)(1)\big)$ be the projection induced by the inclusion of $H^2(X)(1)$ into $H^+(X)$. We will deduce the following result from Theorem 3.3.

**Corollary 3.4.** *The map $\pi_2$ is an isomorphism*

$$\mathrm{MT}^+(X) \cong \mathrm{MT}\big(H^2(X)(1)\big).$$

*In particular, the weight 0 Hodge structure $H^+(X)$ belongs to the tensor subcategory of polarizable $\mathbb{Q}$-Hodge structures generated by $H^2(X)(1)$.*

We first prove a lemma.

**Lemma 3.5.** *We have*

$$\mathrm{MT}^+(X) \subset \tilde{\rho}^+\big(\mathrm{SO}(H)\big).$$

*Proof.* We identify $\mathrm{SO}(H)$ with its image under the representation $\tilde{\rho}^+$ from Proposition 3.2. Let $T$ be a tensor construction on $H^+(X)$, by which we mean that $T$ is a finite sum

$$T = \bigoplus_i \big(H^+(X)\big)^{\otimes m_i} \otimes \big(H^+(X)\big)^{\vee, \otimes n_i}.$$

for some integers $m_i$ and $n_i$. Both $\mathrm{MT}^+(X)$ and $\mathrm{SO}(H)$ act on the space $T$, as they are both subgroups of $\mathrm{GL}\big(H^+(X)\big)$. In order to show that $\mathrm{MT}^+(X)$ is contained into $\mathrm{SO}(H)$, it suffices to check that, for all tensor constructions $T$ as above, every element $\alpha$ of $T$ fixed by the latter is also fixed by $\mathrm{MT}^+(X)$. Indeed, both groups are reductive, and we can then apply [7, Proposition 3.1] to conclude. Let $\alpha \in T$ be invariant for the $\mathrm{SO}(H)$-action. Then, the image of $\alpha$ in $T \otimes \mathbb{C}$ is in the kernel of every element of $\mathfrak{so}(H) \otimes \mathbb{C}$. By Theorem 3.3, this implies that $\alpha$ is of type $(0,0)$; hence $\alpha$ is a Hodge class and it is therefore fixed by the Mumford–Tate group. □

*Proof of Corollary 3.4.* It suffices to show that the restriction of the projection $\pi_2$ to $\mathrm{MT}^+(X)$ is injective. The composition $\pi_2 \circ \tilde{\rho}^+ \colon \mathrm{SO}(H) \to \mathrm{GL}\big(H^2(X)\big)$ is injective thanks to Proposition 3.2. As $\mathrm{MT}^+(X) \subset \tilde{\rho}^+\big(\mathrm{SO}(H)\big)$ by Lemma 3.5, it follows that the restriction of $\pi_2$ to $\mathrm{MT}^+(X)$ is injective, too. □

*Remark 3.6.* The conclusion of Corollary 3.4 is true even without the projectivity assumption on $X$, with the only difference that the Hodge structures involved are not necessarily polarizable.

## 4. A splitting of the motivic Galois group

In this section, $X$ is a complex hyperkähler variety; we further assume that $b_2(X) > 3$. We consider the weight 0 motive

$$\mathcal{H}^+(X) := \bigoplus_i \mathcal{H}^{2i}(X)(i) \in \mathrm{Mot}_{\mathbb{C}},$$

and we denote by $\mathrm{G}^+_{\mathrm{mot}}(X) \subset \prod_i \mathrm{GL}\big(H^{2i}(X)(i)\big)$ its motivic Galois group. We let $\bar{\pi}_2$ be the projection $\mathrm{G}^+_{\mathrm{mot}}(X) \twoheadrightarrow \mathrm{G}_{\mathrm{mot}}\big(\mathcal{H}^2(X)(1)\big)$ induced by the inclusion of $\mathcal{H}^2(X)(1)$ into $\mathcal{H}^+(X)$, and we define

$$P(X) := \ker(\bar{\pi}_2) \subset \mathrm{G}^+_{\mathrm{mot}}(X).$$

**Proposition 4.1.** *We have*

$$\mathrm{G}^+_{\mathrm{mot}}(X) = P(X) \times \mathrm{MT}^+(X).$$

We will first establish some preliminary results.

**Lemma 4.2.** *There exists a section $s$ of the map $\bar{\pi}_2$,*

$$s\colon \mathrm{G}_{\mathrm{mot}}\big(\mathcal{H}^2(X)(1)\big) \hookrightarrow \mathrm{G}^+_{\mathrm{mot}}(X),$$

*whose image coincides with* $\mathrm{MT}^+(X) \subset \mathrm{G}^+_{\mathrm{mot}}(X)$.

*Proof.* We have a commutative diagram

$$\begin{tikzcd}
\mathrm{G}^+_{\mathrm{mot}}(X) \arrow[r, two heads, "\bar{\pi}_2"] & \mathrm{G}_{\mathrm{mot}}\big(\mathcal{H}^2(X)(1)\big) \\
\mathrm{MT}^+(X) \arrow[u, hook, "\iota_+"] \arrow[r, "\pi_2"] & \mathrm{MT}\big(H^2(X)(1)\big) \arrow[u, hook, "\iota_2"]
\end{tikzcd}$$

Here, $\iota_+$ and $\iota_2$ denote the natural inclusions; $\pi_2$ and $\iota_2$ are isomorphisms due to Corollary 3.4 and Theorem 1.3 respectively. We can now take $s = \iota_+ \circ (\iota_2 \circ \bar{\pi}_2)^{-1}$. □

**Lemma 4.3.** *The adjoint action of the group $P(X) \subset \mathrm{GL}\big(H^+(X)\big)$ on $\mathfrak{gl}\big(H^+(X)\big)$ restricts to the identity on the Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$.*

*Proof.* Note that $P(X)$ acts on $H^+(X)$ via algebra automorphisms since the cup-product is induced by an algebraic cycle, namely, the small diagonal $\delta \subset X^3$; moreover, by definition, its action preserves the grading and is trivial on $H^2(X)$. Hence, if $p \in P(X)$, then $p$ commutes with $\theta$ and $L_x$, for $x \in H^2(X)$. Further, if $x$ has the Lefschetz property, then $p$ commutes with $\Lambda_x$ as well: indeed, $L_x$, $\theta$ and $p\Lambda_x p^{-1}$ form an $\mathfrak{sl}_2$-triple, and this forces $p\Lambda_x p^{-1} = \Lambda_x$, see Sect. 3.1. As the various operators $L_x$ and $\Lambda_x$, for $x \in H^2(X)$, generate the Lie subalgebra $\mathfrak{g}_{\mathrm{tot}}(X) \subset \mathfrak{gl}\big(H^+(X)\big)$, we conclude that $P(X)$ commutes with the whole of $\mathfrak{g}_{\mathrm{tot}}(X)$. □

*Proof of Proposition 4.1.* By Lemma 4.2, $P(X) \cdot \mathrm{MT}^+(X) = \mathrm{G}^+_{\mathrm{mot}}(X)$, and the two subgroups have trivial intersection. By Lemma 3.5 and the above Lemma 4.3, $P(X)$ and $\mathrm{MT}(X)^+$ commute. It follows that $\mathrm{G}^+_{\mathrm{mot}}(X)$ is the direct product of these two subgroups. □

## 5. A sufficient condition

With notations and assumptions as in Sect. 1.4, let $X$ be a hyperkähler variety over $k$, and assume that $b_2(X) > 3$. Consider the weight 0 motive

$$\mathcal{H}^+(X) = \bigoplus_i \mathcal{H}^{2i}(X)(i) \in \mathrm{Mot}_k,$$

and write $\mathrm{G}^+_{\mathrm{mot}}(X)$ for its motivic Galois group. Let $H^+_{\mathrm{B}}(X)$ and $H^+_\ell(X)$ denote respectively the Hodge and $\ell$-adic realization of $\mathcal{H}^+(X)$. We write $\mathrm{MT}^+(X)$ for $\mathrm{MT}\big(H^+_{\mathrm{B}}(X)\big)$ and $\mathcal{G}^+_\ell(X)$ for $\mathcal{G}\big(H^+_\ell(X)\big)$. We identify $H^+_{\mathrm{B}}(X) \otimes \mathbb{Q}_\ell$ with $H^+_\ell(X)$ via Artin's comparison isomorphism. Then both $\mathrm{MT}^+(X) \otimes \mathbb{Q}_\ell$ and $\mathcal{G}^+_\ell(X)$ are identified with subgroups of $\mathrm{GL}\big(H^+_\ell(X)\big)$.

### 5.1. *The criterion*

Conjecturally, under Artin's isomorphism we have $\mathrm{MT}^+(X) \otimes \mathbb{Q}_\ell \cong \mathcal{G}_\ell^+(X)^0$. We refer to this statement as the Mumford–Tate conjecture for $\mathcal{H}^+(X)$; it implies the Mumford–Tate conjecture in codimension $j$ for $X$ and for all integers $j$, and, if $X$ has trivial odd cohomology, it also implies the Mumford–Tate conjecture in any codimension for any self-power $X^k$. Recall from Sect. 2.7 that $\mathcal{G}_\ell^+(X)^0$ is a subgroup of $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}}) \otimes \mathbb{Q}_\ell \cong \mathrm{G}^+_{\mathrm{mot}}(X_{\mathbb{C}}) \otimes \mathbb{Q}_\ell$, and that, by Proposition 4.1, we have an equality $\mathrm{G}^+_{\mathrm{mot}}(X_{\mathbb{C}}) = P(X) \times \mathrm{MT}^+(X)$ of subgroups of $\mathrm{GL}\big(H^+_{\mathrm{B}}(X)\big)$.

**Proposition 5.1.** *Assume that $P(X)$ is finite (resp. trivial). Then the Mumford–Tate conjecture (resp. the motivic Mumford–Tate conjecture) holds for $\mathcal{H}^+(X)$.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{MT}^+(X) \otimes \mathbb{Q}_\ell & \hookrightarrow & \mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}}) \otimes \mathbb{Q}_\ell & \hookleftarrow & \mathcal{G}_\ell^+(X)^0 \\
\downarrow \wr & & \downarrow & & \downarrow \\
\mathrm{MT}\big(H^2_{\mathrm{B}}(X)(1)\big) \otimes \mathbb{Q}_\ell & \xrightarrow{\sim} & \mathrm{G}_{\mathrm{mot}}\big(\mathcal{H}^2(X_{\bar{k}})(1)\big) \otimes \mathbb{Q}_\ell & \xleftarrow{\sim} & \mathcal{G}\big(H^2_\ell(X)(1)\big)^0
\end{array}$$

The horizontal arrows on the bottom are isomorphisms due to Theorem 1.3, and the vertical map on the left is an isomorphism thanks to Corollary 3.4. By Proposition 4.1 we have $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}}) = P(X) \times \mathrm{MT}^+(X)$; if $P(X)$ is finite, it follows that we have $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}})^0 = \mathrm{MT}^+(X)$. Hence, replacing in the above diagram $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}})$ with its connected component of the identity, also the leftmost arrow on the top row becomes an isomorphism. Thus all arrows in the diagram become isomorphisms, and we obtain

$$\mathrm{MT}^+(X) \otimes \mathbb{Q}_\ell = \mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}})^0 \otimes \mathbb{Q}_\ell = \mathcal{G}_\ell^+(X)^0.$$

Moreover, if $P(X)$ is trivial then $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}})$ is connected and equal to $\mathrm{MT}^+(X)$, and therefore the motivic Mumford–Tate conjecture holds in this case. □

### 5.2. *Proof of Theorem 1.2*

By the above Proposition 5.1, it suffices to show that the assumption of abelianity of all even Künneth components $\mathcal{H}^{2i}(X_{\bar{k}})$ of the motive of $X$ implies that $P(X)$ is trivial. Note that this assumption is equivalent to the abelianity of $\mathcal{H}^+(X_{\bar{k}})$. But then the desired conclusion follows immediately from Proposition 4.1 and André's theorem 2.3: indeed, the first result implies that $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}}) = P(X) \times \mathrm{MT}^+(X)$ and the second that $\mathrm{G}^+_{\mathrm{mot}}(X_{\bar{k}}) = \mathrm{MT}^+(X)$.

## 6. Proof of Theorem 1.1

In this section we prove Theorem 1.1. To this end, we will establish the finiteness of the group $P(X)$ from Sect. 4 when $X$ is a hyperkähler variety of $\mathrm{K3}^{[m]}$ or OG10-type; Theorem 1.1 then follows via Proposition 5.1. We can, and will, work over the complex numbers, since $\mathrm{G}^{+}_{\mathrm{mot}}(X_{\bar{k}}) \cong \mathrm{G}^{+}_{\mathrm{mot}}(X_{\mathbb{C}})$.

Recall that a complex hyperkähler variety $X$ is of $\mathrm{K3}^{[m]}$-type if it is a deformation of a Hilbert scheme of 0-dimensional subschemes of length $m$ on some K3 surface. If $m = 1$, then $X$ is the original K3 surface; we will assume $m \geq 2$. In this case $\dim X = 2m$, the odd cohomology of $X$ vanishes, and the second Betti number equals 23, See [11]. We say that $X$ is of OG10-type if it is deformation equivalent to O'Grady's ten dimensional hyperkähler variety constructed in [18]. In this case the odd Betti numbers of $X$ vanish as well, and we have $b_2(X) = 24$, see [5].

### 6.1. Reduction

Let $\mathrm{Aut}\big(H^{+}(X)\big)$ be the group of automorphisms of the graded $\mathbb{Q}$-algebra $H^{+}(X) = \bigoplus_i H^{2i}(X)$. Let $K(X) \subset \mathrm{Aut}\big(H^{+}(X)\big)$ be the kernel of the natural restriction map $\mathrm{Aut}\big(H^{+}(X)\big) \to \mathrm{GL}\big(H^{2}(X)\big)$. The group $P(X)$ acts via algebra automorphisms, and, by construction, its action is trivial in degree 2. Hence, we have

$$P(X) \subset K(X).$$

To conclude the proof of Theorem 1.1 it therefore suffices to establish the following.

**Proposition 6.1.** *Assume $X$ is a hyperkähler variety of either* $\mathrm{K3}^{[m]}$ *or* OG10*-type. Then $K(X)$ is a finite group.*

As we are going to explain, this is a consequence of results due to Markman [15] in the first case and due to Green-Kim-Laza-Robles [10] in the second case.

### 6.2. The invariant pairing

We start by recalling from [14] some additional facts on the representation of $\mathfrak{g}_{\mathrm{tot}}(X)$ on the cohomology. Let $\int_X$ denote the projection $H^{+}(X) \to H^{4n}(X) \cong \mathbb{Q}$, where $\dim(X) = 2n$. Consider the Poincaré pairing $\phi\colon H^{+}(X) \otimes H^{+}(X) \to \mathbb{Q}$, which is defined via

$$\phi(\alpha, \beta) = (-1)^q \int_X \alpha \wedge \beta,$$

for $\alpha$ of degree $2n + 2q$. It is shown in [14, Proposition 1.6 and its proof], that the Lie algebra $\mathfrak{g}_{\mathrm{tot}}(X)$ preserves infinitesimally the Poincaré pairing, and that $\phi$ restricts to a non-degenerate pairing on every $\mathfrak{g}_{\mathrm{tot}}(X)$-submodule of $H^{+}(X)$.

### *6.3. The* OG10*-type*

The group $K(X)$ acts on $H^+(X)$ via graded algebra automorphisms and it acts trivially in degree 2; it follows that $K(X)$ preserves the pairing $\phi$. Moreover, the argument used to prove Lemma 4.3 shows that this group commutes with $\mathfrak{g}_{\mathrm{tot}}(X)$.

*Proof of Proposition* 6.1 *for the* OG10*-type.* Assume $X$ is of OG10-type. The representation of $\mathfrak{g}_{\mathrm{tot}}(X)$ on the cohomology has been fully described in [10, Theorem 1.1-(iv)]. We have

$$H^+(X) = V_1 \oplus V_2,$$

where $V_1$ is the subalgebra generated by $H^2(X)$ and $V_2$ is an absolutely irreducible $\mathfrak{g}_{\mathrm{tot}}(X)$-representation. We deduce that $K(X)$ is a subgroup of

$$\left(\mathrm{End}(V_2)^{\mathfrak{g}_{\mathrm{tot}}(X)}\right)^\times \cong \mathbb{Q}^\times,$$

by Schur's lemma. Further, we know that the pairing $\phi$ restricts to a non-degenerate invariant form on $V_2$, and we deduce that $K(X) \subset \{1, -1\}$. □

### *6.4. The* K3$^{[m]}$*-type*

As apparent from the proof, for the OG10-type the decomposition of the cohomology into $\mathfrak{g}_{\mathrm{tot}}(X)$-isotypical components already imposes the desired finiteness result. The analogous decomposition for the K3$^{[m]}$-type becomes more and more complicated as the dimension increases, see [10]. Nevertheless, it becomes more manageable once the algebra structure is taken into account, thanks to the following result of Markman. From now on, we assume that $X$ is a variety of K3$^{[m]}$-type.

For $l \geq 0$, we let $A_{2l} \subset H^+(X)$ be the subalgebra generated by $\bigoplus_{j \leq l} H^{2j}(X)$. Note that $A_{2l} = H^+(X)$ for $l \geq m$. Recall from Corollary 3.2 that we have a representation $\tilde{\rho}^+$ of $\mathrm{SO}(H)$ on $H^+(X)$.

**Theorem 6.2.** *(Markman). For all $i \geq 1$, there exists a subspace $C^{2i} \subset H^{2i}(X)$ with the following properties.*

(a) *We have a $\mathfrak{g}_0(X)$-invariant decomposition*

$$H^{2i}(X) = \left(A_{2i-2} \cap H^{2i}(X)\right) \oplus C^{2i}.$$

*Note that this implies $C^2 = H^2(X)$ and $C^{2i} = 0$ for $i > m$. Each $C^{2i}$ is in particular a subrepresentation for $\mathfrak{g}_0(X)$ and, hence, for* $\mathrm{SO}(H)$*. Moreover, the $\mathfrak{g}_{\mathrm{tot}}(X)$-module generated by $C^{2i}$ is orthogonal to $A_{2i-2}$ with respect to $\phi$.*

(b) *The sum $\bigoplus_{i \geq 1} C^{2i}$ generates the algebra $H^+(X)$.*

(c) *The* $\mathrm{SO}(H)$*-module $C^{2i}$ is a subrepresentation of the sum of a copy of the standard representation with a one dimensional trivial representation, for all $i \geq 2$.*

Parts (a) and (b) are proven in [15, Corollary 4.6], while (c) is [*loc. cit.*, Lemma 4.8]. We can now conclude the proof of our main result.

*Proof of Proposition 6.1 for the* $\mathrm{K3}^{[m]}$*-type.* We claim first of all that each subspace $C^{2i}$ is stable under the action of $K(X)$. In fact, since $K(X)$ acts via graded algebra automorphisms, the subalgebras $A_{2l}$ are $K(X)$-stable for all $l$. Since $\phi$ is $K(X)$-invariant, it follows that the orthogonal complement to each $A_{2l}$ is preserved as well; as $K(X)$ acts compatibly with the grading, it indeed stabilizes $C^{2i}$, for all $i$.

The subspaces $C^{2i}$ generate the cohomology by Theorem 6.2.(b), and $K(X)$ commutes with the representation $\tilde{\rho}$. Hence, we have

$$K(X) \subset \prod_{i\geq 2} \big(\mathrm{End}(C^{2i})^{\mathrm{SO}(H)}\big)^{\times}.$$

Let $V \subset C^{2i}$ be an irreducible-$\mathrm{SO}(H)$ representation. By Theorem 6.2.(c), the representation $V$ is absolutely irreducible and it appears in $C^{2i}$ with multiplicity one; it follows that $V$ is stable under $K(X)$ as well. By Schur's lemma, each element of $K(X)$ acts on $\mathfrak{g}_{\mathrm{tot}}(X) \cdot V$ via multiplication by some rational number. On the other hand $K(X)$ preserves the form $\phi$, whose restriction to the $\mathfrak{g}_{\mathrm{tot}}(X)$-module generated by $V$ is non-degenerate, and therefore the action of $K(X)$ on $\mathfrak{g}_{\mathrm{tot}}(X) \cdot V$ factors through $\{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$. Using again Theorem 6.2.(c), we conclude that, for all $i$, $\big(\mathrm{End}(C^{2i})^{\mathrm{SO}(H)}\big)^{\times}$ is a subgroup $\mathbb{Z}/2\mathbb{Z}^2$, and hence we have

$$K(X) \subset \prod_{i=2}^{m} \mathbb{Z}/2\mathbb{Z}^2.$$

□

Theorem 1.1 is proved.

*Remark 6.3.* The conclusion of Proposition 6.1 does not hold for the remaining deformation types $\mathrm{Kum}_m$ and OG6. This can be checked using the description of the $\mathfrak{g}_{\mathrm{tot}}(X)$-representation of the cohomology given in [10]: in fact, for these deformation types, there are $\mathfrak{g}_{\mathrm{tot}}(X)$-representations which appear in the cohomology with higher multiplicities, which cannot be explained only by taking into account the algebra structure on the cohomology.

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