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MOTIVES OF HYPER-KÄHLER VARIETIES

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INTRODUCTION

Hyper-Kähler manifolds are an important class of higher dimensional Kähler manifolds. Together with complex tori and Calabi–Yau manifolds, they constitute the building blocks of Kähler manifolds with trivial first Chern class, by the Beauville–Bogomolov decomposition theorem.

Hyper-Kähler manifolds are higher dimensional analogues of K3 surfaces; in dimension 2 any hyper-Kähler manifold is a K3 surface. The topological classification of hyper-Kähler manifolds is a very hard open problem. For the time being, it is known that there are two distinct deformation classes in any even dimension $2n \geq 4$, called the $\mathrm{K3}^{[n]}$ and Kum^n-types respectively first discovered by Beauville [**9**], and two more deformation classes OG10 and OG6 in dimension 10 and 6 respectively, found by O'Grady [**68**], [**69**].

Despite this difficulty, thanks to work of Huybrechts, Markman, O'Grady, Verbitsky and many others, we have a rich theory of hyper-Kähler manifolds which parallels in many ways that of K3 surfaces. For instance, the second cohomology $H^2(X, \mathbb{Z})$ is equipped with a non-degenerate symmetric bilinear form, the Beauville–Bogomolov form. One of the highlights in the theory is certainly the Torelli theorem due to Huybrechts and Verbitsky, [**42**] [**84**] [**44**]: the global deformations of a hyper-Kähler manifold X are controlled by the Hodge structure on the lattice $H^2(X, \mathbb{Z})$.

The main theme of this thesis is the study of motives of hyper-Kähler varieties. The similarity between the theory of higher dimensional hyper-Kähler varieties and that of K3 surfaces suggests that the motives of hyper-Kähler varieties should be controlled by smaller, "surface-like" motives.

Most of the known constructions of hyper-Kähler varieties involve taking a moduli space of sheaves on a K3 or abelian surfaces; in such cases a relation between the Chow motive of the moduli space and that of the surface is expected. Bülles [**15**], building on work of Markman [**53**], has proven that the rational Chow motive of the moduli space belongs to the tensor category generated by the motive of the surface. This gives many examples of hyper-Kähler varieties of $\mathrm{K3}^{[n]}$-type whose motive is controlled by the motive of a K3 surface.

However, the general projective deformation of such a hyper-Kähler variety is, a priori, no longer related to a surface. Nevertheless, we still expect to be able to control the motive of X via a motive of weight 2: the natural replacement for the motive of the surface would be the component of the motive of X in degree 2. To make this precise we need to leave the category of Chow motives, since in this setting we do not even know that the Künneth projectors are algebraic. To circumvent this issue we will work within the category of André motives [**4**], denoted by AM. The following conjecture summarizes our expectations on the motives of hyper-Kähler varieties.

0.1. Conjecture. — *Let $K \subset \mathbb{C}$ be an algebraically closed field. Let X be a hyper-Kähler variety over K, and let $\mathcal{H}^\bullet(X) = \bigoplus_i \mathcal{H}^i(X) \in \mathsf{AM}_K$ be its André motive. Then:*

- *the even part $\mathcal{H}^+(X) = \bigoplus_i \mathcal{H}^{2i}(X)$ of the motive of X belongs to the Tannakian category $\langle \mathcal{H}^2(X) \rangle \subset \mathsf{AM}_K$ generated by $\mathcal{H}^2(X)$;*
- *if X has non-trivial cohomology in some odd degree, then the motive $\mathcal{H}^\bullet(X)$ belongs to the Tannakian category $\langle \mathcal{H}^1(A) \rangle \subset \mathsf{AM}_K$, where A is the Kuga–Satake abelian variety obtained from the Hodge structure $H^2(X)$.*

In any case, the motive of X is abelian.

The abelian variety A is obtained from $H^2(X)$ via the Kuga–Satake construction ([**22**]). This abelian variety is not uniquely determined (not even up to isogeny), but we will show that the conjecture is independent of choices involved in the Kuga–Satake construction.

As a consequence of the work of Looijenga–Lunts [**51**] and Verbitsky [**83**], the conjecture holds at the level of Hodge structures. Soldatenkov proves in [**80**] that deformation equivalent hyper-Kähler varieties with Hodge-isometric second cohomology have isomorphic total Hodge structure.

The category of André motives provides also a natural framework to study the Mumford–Tate conjecture. Let $K \subset \mathbb{C}$ be a field which is finitely generated over $\mathbb{Q}$, let $\bar{K} \subset \mathbb{C}$ be an algebraic closure of K, and let ℓ be a fixed prime number. Given a smooth and projective variety X over K, we have on the one hand the rational Hodge structure $H^\bullet(X)$ on the singular cohomology of $X(\mathbb{C})$, and on the other hand the ℓ-adic Galois representation $H^\bullet_\ell(X)$ on the étale cohomology $H^\bullet_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$.

Roughly speaking, the Mumford–Tate conjecture predicts that $H^\bullet(X)$ and $H^\bullet_\ell(X)$ contain the same information. Due to the very different nature of these two objects, the Tannakian formalism is needed in order to formulate the comparison. The conjecture is a very difficult open problem; we refer to [**62**] for a survey of known cases. If the Mumford–Tate conjecture is true for X, then the conjectures of Hodge and Tate are equivalent for all powers of X.

Results. — Our main contributions to the study of motives of hyper-Kähler varieties are summarized below:

(i) we prove that the Chow motive of a (ten dimensional) O'Grady moduli space on a K3 or abelian surface belongs to the category generated by the Chow motive of the surface;

(ii) we introduce the notion of *defect group* of a hyper-Kähler variety, and use it to prove the Mumford–Tate conjecture and Conjecture 0.1 for all hyper-Kähler varieties of known deformation type;

(iii) we prove that deformation equivalent hyper-Kähler varieties with $b_2 > 6$ with Hodge-isometric H^2 have isomorphic André motives, modulo a technical assumption in presence of non-trivial cohomology in odd degree.

The first of these results is obtained via a refinement of Bülles method in [**15**], using the geometry of O'Grady moduli spaces [**68**]. This result is joint work with Lie Fu and Ziyu Zhang, published in [**29**].

In the same article with Fu and Zhang we introduced the defect group of a hyper-Kähler variety, and we used it to prove the Mumford–Tate conjecture and Conjecture 0.1 for all hyper-Kähler varieties of known deformation type. This improves previous work of the author [**27**], where the Mumford–Tate conjecture for varieties of deformation type $\mathrm{K3}^{[n]}$ and OG10 is proven.

The defect group attached to a hyper-Kähler variety is an algebraic group $P(X)$ which measures the failure of Conjecture 0.1: the conjecture holds for X if and only if $P(X)$ is trivial, which in turn implies the Mumford–Tate conjecture. Even if, for a yet to be discovered deformation class of hyper-Kähler varieties, it turns out that Conjecture 0.1 is false, the defect group still allows to control the motives of these varieties via their degree 2 component, in the sense of the third result listed above. We obtain a similar statement about Galois representations on the étale cohomology of hyper-Kähler varieties. These results appeared in the author's article [**28**].

The key inputs used to establish the properties of the defect group are: the action on the cohomology of X of the Lie algebra $\mathfrak{g}(X)$ introduced by Looijenga–Lunts [**51**] and Verbitsky [**83**], and the André theorem [**3**] saying that if X is a hyper-Kähler variety with $b_2(X) > 3$ then $\mathcal{H}^2(X)$ is an abelian motive for which the Mumford–Tate conjecture holds true.

We now review in more detail the contents of the thesis, and give precise statements of the main results.

Motives of moduli spaces

Let S be a K3 or abelian surface. Denote by $\widetilde{\mathrm{NS}}(S)$ the Mukai extension of the Néron–Severi lattice of S: we have $\widetilde{\mathrm{NS}}(X) = H^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S,\mathbb{Z})$, where

$$((a,b,c),(a',b',c')) = (b,b') - ac' - a'c.$$

Elements of $\widetilde{\mathrm{NS}}(S)$ are called Mukai vectors; to any coherent sheaf E on S is associated its Mukai vector $\mathrm{ch}(E)\cdot\sqrt{\mathrm{td}_S}$.

It is known ([**46**]) that, if $\mathrm{v} \in \widetilde{\mathrm{NS}}(S)$ and H is a v-generic polarization on S, there exist a non-singular quasi-projective moduli space $\mathcal{M}^{\mathrm{st}}$ of H-stable sheaves on S with Mukai vector v and a projective, but possibly singular, moduli space $\mathcal{M}$ of H-semistable sheaves on S with Mukai vector v. The moduli space $\mathcal{M}^{\mathrm{st}}$ is an open subvariety of $\mathcal{M}$; the singular locus of $\mathcal{M}$ consists of the strictly H-semistable sheaves. By work of Mukai ([**64**]), $\mathcal{M}^{\mathrm{st}}$ is a symplectic manifold, i.e. it carries a nowhere degenerate holomorphic closed 2-form σ. A crepant resolution $\widetilde{\mathcal{M}} \to \mathcal{M}$ is a projective birational morphism with $\widetilde{\mathcal{M}}$ non-singular such that the pull-back of the form σ on $\mathcal{M}^{\mathrm{st}}$ extends to a holomorphic symplectic form on $\widetilde{\mathcal{M}}$.

With S as above, assume that $\mathrm{v} \in \widetilde{\mathrm{NS}}(S)$ is a primitive Mukai vector and let H be a v-generic polarization on S. In this case there are no strictly H-semistable sheaves with Mukai vector v and therefore $\mathcal{M} = \mathcal{M}^{\mathrm{st}}$ is a smooth projective variety of dimension $\mathrm{v}^2 + 2$. By [**67**], when $\mathrm{v}^2 > 0$ the second cohomology of $\mathcal{M}$ is identified with $\mathrm{v}^{\perp} \subset \widetilde{H}^2(S, \mathbb{Z})$; building on work of Markman [**53**], Bülles has shown in [**15**] that the rational Chow motive $\mathfrak{h}(\mathcal{M})$ of the moduli space belongs to the pseudo-abelian tensor category of motives generated by the rational Chow motive $\mathfrak{h}(S)$ of the surface.

If the Mukai vector v is not primitive, the moduli space $\mathcal{M}$ of H-semistable sheaves on S with Mukai vector v will be singular. When $\mathrm{v} = 2\,\mathrm{v}_0$ with v_0 primitive such that $\mathrm{v}_0^2 = 2$ and H is a v_0-generic polarization, O'Grady has constructed in [**68**] a crepant resolution $\widetilde{\mathcal{M}} \to \mathcal{M}$. This is the only case in which $\mathcal{M}$ admits a crepant resolution: by [**47**], given an abelian or K3 surface S, a vector $\mathrm{v} \in \widetilde{\mathrm{NS}}(S)$ and a v-generic polarization H on S, if the moduli space $\mathcal{M}$ admits a crepant resolution then either v is primitive or $\mathrm{v} = 2\,\mathrm{v}_0$ with v_0 a primitive Mukai vector such that $\mathrm{v}_0^2 = 2$. We obtain the following generalization of Bülles' result.

0.2. Theorem. — *Let S be an abelian or* K3 *surface. Let $\mathrm{v} = 2\,\mathrm{v}_0 \in \widetilde{\mathrm{NS}}(S)$, with v_0 a primitive Mukai vector such that $\mathrm{v}_0^2 = 2$, and let H be a v_0-generic polarization on S. Let $\widetilde{\mathcal{M}} \to \mathcal{M}$ be O'Grady's crepant resolution. Then the Chow motive of $\widetilde{\mathcal{M}}$ belongs to the pseudo-abelian tensor subcategory generated by $\mathfrak{h}(S)$.*

Theorem 0.2 is joint work with Lie Fu and Ziyu Zhang, published in [**29**]. In that article we offer generalizations in various directions, most notably to Bridgeland moduli spaces on Calabi–Yau categories; for the sake of simplicity, we decided to not include them here.

Bülles' result and our Theorem 0.2 give examples of hyper-Kähler varieties satisfying Conjecture 0.1. Indeed, when S is a K3 surface and $\mathrm{v} \in \widetilde{\mathrm{NS}}(S)$ is a primitive Mukai vector, the smooth and projective moduli space $\mathcal{M}$ of stable sheaves with Mukai vector v is a hyper-Kähler variety of $\mathrm{K3}^{[n]}$-type, while O'Grady's crepant resolutions $\widetilde{\mathcal{M}} \to \mathcal{M}$ are hyper-Kähler varieties of OG10-type. When S is instead an abelian surface, O'Grady resolution $\widetilde{\mathcal{M}}$ is not a hyper-Kähler variety; nevertheless it admits an isotrivial fibration over $S \times \hat{S}$, where $\hat{S}$ is the dual abelian surface, whose fibre $\widetilde{\mathcal{M}}_0$ is an hyper-Kähler variety of OG6-type. The analogue of Theorem 0.2 for $\widetilde{\mathcal{M}}_0$ is not known.

Defect groups of hyper-Kähler varieties

The word motive will now indicate an object of André category of motives. Via the Tannakian formalism, to any motive M is attached its motivic Galois group $\mathrm{G}_{\mathrm{mot}}(M)$, which is a reductive $\mathbb{Q}$-algebraic group whose category of representations is equivalent to the Tannakian category generated by M.

Let $K \subset \mathbb{C}$ be an algebraically closed field. To any hyper-Kähler variety X over K we associate its *defect group* $P(X)$, as follows. We denote by $\mathcal{H}^\bullet(X) = \bigoplus_i \mathcal{H}^i(X)$ the André motive of X.

If X has trivial cohomology in odd degree, we simply define $P(X)$ as the kernel of the surjective homomorphism of motivic Galois groups

$$P(X) \coloneqq \ker\Big(\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X))\Big),$$

coming from the fact that $\mathcal{H}^2(X)$ is a summand of $\mathcal{H}^\bullet(X)$.

In presence of non-trivial cohomology in odd degree, we first define the extended defect group $\widetilde{P}(X)$ as the kernel of $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X))$. We then show that there is a central element $\iota \in \widetilde{P}(X)$ of order 2, which acts on $\mathcal{H}^j(X)$ as multiplication by $(-1)^j$, and define the defect group $P(X)$ as the quotient $\widetilde{P}(X)/\langle\iota\rangle$.

The motivic Galois group $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$ always contains the Mumford–Tate group $\mathrm{MT}(H^\bullet(X))$ as a subgroup. The defect group is a complement for the Mumford–Tate group in the motivic Galois group.

0.3. Theorem. — *Let $K \subset \mathbb{C}$ be an algebraically closed field and let X be a hyper-Kähler variety over K. Assume that $b_2(X) > 3$. Then, if X has trivial cohomology in odd degrees, the defect group $P(X)$ is a direct complement of* $\mathrm{MT}(H^\bullet(X))$*:*

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) = \mathrm{MT}(H^\bullet(X)) \times P(X).$$

In presence of non-trivial cohomology in odd degree, the extended defect group $\widetilde{P}(X)$ is a complement of $\mathrm{MT}(H^\bullet(X))$ *inside* $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$*, in the sense that:*

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) = \mathrm{MT}(H^\bullet(X)) \cdot \widetilde{P}(X),$$

and the subgroups $\mathrm{MT}(H^\bullet(X))$ *and $\widetilde{P}(X)$ commute with each other and intersect in the central subgroup $\langle\iota\rangle \cong \mathbb{Z}/2\mathbb{Z}$.*

The assumption that $b_2(X) > 3$ comes from the work of André [**3**], who has shown that under this assumption the motive $\mathcal{H}^2(X)$ is abelian. Conjecturally, any hyper-Kähler variety satisfies this assumption.

The following result justifies the name defect group.

0.4. Corollary. — *Let $K \subset \mathbb{C}$ be an algebraically closed field, and let X be a hyper-Kähler variety over K such that $b_2(X) > 3$. Then the following are equivalent:*

(i) The defect group $P(X)$ is trivial.

(ii) Conjecture 0.1 holds for X.

(iii) The motive $\mathcal{H}^\bullet(X)$ is abelian.

(iv) On any power of X, all Hodge classes are motivated; equivalently,

$$\mathrm{MT}(H^\bullet(X)) = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)).$$

Another remarkable property of defect groups is their deformation invariance.

0.5. Theorem. — *Let $\mathfrak{X} \to B$ be a smooth and projective morphism (of schemes) with fibres hyper-Kähler varieties with $b_2 > 3$, where B is a non-singular and connected complex variety. For any two points $s, s' \in B$, the defect groups of the corresponding fibres are isomorphic: $P(\mathfrak{X}_s) \cong P(\mathfrak{X}_{s'})$.*

To prove the above theorem we apply the results on families of André motives formalized by Moonen in [**62**].

Defect groups and the Mumford–Tate conjecture. — Let now $K \subset \mathbb{C}$ be a field which is finitely generated over $\mathbb{Q}$, and let $\bar{K}$ be an algebraic closure of K in $\mathbb{C}$. We fix a prime number ℓ. The notion of defect group is useful in studying the Mumford–Tate conjecture for hyper-Kähler varieties: we have the following criterion.

0.6. Theorem. — *Let $K \subset \mathbb{C}$ be a finitely generated field over $\mathbb{Q}$, and let ℓ be a prime number. Let X be a hyper-Kähler variety over K, with $b_2(X) > 3$. Assume that the defect group $P(X_{\bar{K}})$ is finite. Then the Mumford–Tate conjecture holds for X.*

The above results on defect groups appeared in a joint paper with Lie Fu and Ziyu Zhang [**29**], and are based on previous work of the author [**27**].

We now discuss our applications. The key to these is that two deformation equivalent projective hyper-Kähler manifolds can be joined via a sequence of birational equivalences and polarized deformations over algebraic varieties, provided that $b_2 > 6$.

0.7. Theorem. — *Let X_1 and X_2 be deformation equivalent (in the complex analytic sense) complex hyper-Kähler varieties. Assume that $b_2(X_i) > 6$. Then there exist:*

- *finitely many connected and non-singular complex varieties S_i, for $i = 1, \dots, N$;*
- *for each $i = 1, \dots, N$, a smooth and projective morphism of schemes $\mathfrak{X}^i \to S_i$ with fibres hyper-Kähler varieties;*
- *for $i = 1, \dots, N$, points $a_i, b_i \in S_i$ together with birational maps*

$$X_1 \dashrightarrow \mathfrak{X}^1_{a_1}, \qquad \mathfrak{X}^i_{b_i} \dashrightarrow \mathfrak{X}^{i+1}_{a_{i+1}}, \text{ for } i = 1, \dots, N-1, \qquad \mathfrak{X}^N_{b_N} \dashrightarrow X_2.$$

This result uses the description of the ample cone of a hyper-Kähler variety given by Amerik–Verbitsky [**2**]. Soldatenkov states a similar result in [**79**] (which does not seem to require that $b_2 > 6$), but we have not been able to completely follow his argument.

As a consequence of Theorem 0.7, combining Theorem 0.5 with Riess' result in [**76**] that birational hyper-Kähler varieties have isomorphic Chow motives, we obtain that deformation equivalent hyper-Kähler varieties with $b_2 > 6$ have isomorphic defect groups.

0.8. Corollary. — *Let $K \subset \mathbb{C}$ be an algebraically closed field. Let X_1 and X_2 be hyper-Kähler varieties over K such that $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent (in the complex analytic sense), and assume that $b_2(X_i) > 6$. Then, the equivalent statements in Corollary 0.4 hold for X_1 if and only if they hold for X_2.*

Motives of known hyper-Kähler varieties. — Let $K \subset \mathbb{C}$ be a finitely generated field, with algebraic closure $\bar{K} \subset \mathbb{C}$, and let ℓ be a fixed prime number.

In what follows, a hyper-Kähler variety X over K is called *known* if $X_{\mathbb{C}}$ is deformation equivalent (in the complex analytic sense) to one of the known examples. Thus, $X_{\mathbb{C}}$ is of one of the deformation types $\mathrm{K3}^{[n]}$, Kum^n, OG10 or OG6.

0.9. Theorem. — *The defect group $P(X_{\bar{K}})$ of any known hyper-Kähler variety X over K is trivial. If $Y = X_1 \times X_2 \times \cdots \times X_k$ is any product of known hyper-Kähler varieties over K, we have:*

- *$Y_{\bar{K}}$ has abelian motive. Any Hodge class on $\mathcal{H}^\bullet(Y)$ is motivated.*
- *The Mumford–Tate conjecture holds for Y. In particular, the conjectures of Hodge and Tate are equivalent for Y.*

All known hyper-Kähler varieties have $b_2 > 6$. By Corollary 0.8, it suffices to find in each of the known deformation class a hyper-Kähler variety with trivial defect group or, equivalently, with abelian André motive; this is done analyzing a specific example in each deformation class. The statement about products is then deduced applying a result of Commelin [**20**]. That the motives of varieties of $\mathrm{K3}^{[n]}$-type are abelian had already been proven by Schlickewei [**77**] using work of Markman [**55**]. Theorem 0.9 appeared in our joint work [**29**].

The full motive is determined by $\mathcal{H}^2(X)$. — Even if we are as yet unable to prove in general that the defect group is trivial, we can show that motives of hyper-Kähler varieties are determined by their component in degree 2.

0.10. Theorem. — *Let $K \subset \mathbb{C}$ be an algebraically closed field. Let X_1 and X_2 be hyper-Kähler varieties over K with $b_2(X_i) > 6$ such that $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent. If the odd cohomology of X_i is not trivial, assume that the motive $\mathcal{H}^1(A_i)$ of the Kuga–Satake abelian variety A_i on $H^2(X_i)$ belongs to $\langle \mathcal{H}^\bullet(X_i) \rangle$. Let $f\colon H^2(X_1) \xrightarrow{\sim} H^2(X_2)$ be a Hodge isometry. Then, there exists an isomorphism of graded algebras $F\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ which is the realization of an isomorphism of motives $\mathcal{H}^\bullet(X_1) \xrightarrow{\sim} \mathcal{H}^\bullet(X_2)$ in AM_K.*

So far, the only known hyper-Kähler varieties with non-trivial odd cohomology are those of Kum^n-type, $n \geq 2$.

Galois representations attached to hyper-Kähler varieties. — With the Mumford–Tate conjecture being proven for all known hyper-Kähler varieties in Theorem 0.9, we can show that the Galois representations on their étale cohomology are determined by their component in degree 2, in the following strong sense.

We consider fields $K_1, K_2 \subset \mathbb{C}$ which are finitely generated over $\mathbb{Q}$, with algebraic closure $\bar{K}_1, \bar{K}_2 \subset \mathbb{C}$ respectively. We fix a prime number ℓ.

0.11. Theorem. — *Let X_i be a hyper-Kähler variety over K_i of known deformation type, for $i = 1, 2$. Assume that $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent. Let Γ be a subgroup of $\mathrm{Gal}(\bar{K}_1/K_1)$, and let $\epsilon\colon \Gamma \to \mathrm{Gal}(\bar{K}_2/K_2)$ be a homomorphism. Assume that $f\colon H^2_\ell(X_1) \xrightarrow{\sim} H^2_\ell(X_2)$ is a Γ-equivariant isometry with respect to the Beauville–Bogomolov form. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index, and an isomorphism of graded algebras $F\colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$ which is Γ'-equivariant.*

We also obtain an analogue for hyper-Kähler varieties over finite fields. Despite the study of such varieties being still in an early stage, certain moduli spaces of sheaves on K3 surfaces in positive characteristic play a crucial role in Charles' proof of the Tate conjecture for K3 surfaces [**19**]. Other examples of hyper-Kähler varieties over finite fields are studied in [**30**], [**31**] and [**32**]; in each case, the varieties studied can be lifted to characteristic 0.

Let k be a finite field, and let Z_1 and Z_2 be smooth projective varieties over k. We assume that there exist hyper-Kähler varieties X_1 and X_2 over fields of characteristic 0 which lift Z_1 and Z_2. We let ℓ be a prime number invertible in k and consider $H^\bullet_\ell(Z_i) \coloneqq \bigoplus_j H^j_{\text{ét}}(Z_{i,\bar{k}}, \mathbb{Q}_\ell)$. By the smooth and proper base-change theorems we have an isomorphism of graded algebras $H^\bullet_\ell(X_i) \cong H^\bullet_\ell(Z_i)$; the Beauville–Bogomolov form induces a non-degenerate symmetric bilinear form on $H^2_\ell(Z_i)$.

0.12. Theorem. — *With notation and assumptions as above, assume that X_1 and X_2 are known hyper-Kähler varieties, and that $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent (in the complex analytic sense). Let $f\colon H^2_\ell(Z_1) \xrightarrow{\sim} H^2_\ell(Z_2)$ be a* $\mathrm{Gal}(\bar{k}/k)$*-equivariant isometry. Then, there exist a finite field extension k' of k and a* $\mathrm{Gal}(\bar{k}/k')$*-equivariant isomorphism of graded algebras $F\colon H^\bullet_\ell(Z_1) \xrightarrow{\sim} H^\bullet_\ell(Z_2)$.*

This result generalizes the work of Frei [**30**], which deals with the special case in which Z_1 and Z_2 are moduli spaces of stable sheaves on K3 surfaces.

Overview of the contents

In Chapter 1, we recall basic facts on hyper-Kähler varieties. Due to the prominent role which it plays in our arguments, we present in detail the LLV-Lie algebra $\mathfrak{g}(X)$ of a hyper-Kähler variety, introduced by Looijenga–Lunts [**51**] and Verbitsky [**83**]. We have included a complete proof of their theorem describing the Lie algebra $\mathfrak{g}(X)$.

In Chapter 2, we recall the statement of the Mumford–Tate conjecture and introduce various category of motives. We discuss in particular André motives, their motivic Galois groups and their behaviour under deformations, following [**62**].

In Chapter 3, we study the Chow motives of O'Grady moduli spaces.

In Chapter 4, we study the Hodge structure on the cohomology of a hyper-Kähler variety. We use the LLV-Lie algebra to show that the total Hodge structure is controlled by the second cohomology. Similar results appeared already in [**51**], [**83**], and, more recently, in [**36**] and [**80**]. We also give a Tannakian characterization of the Kuga–Satake construction, which appears to be new.

In Chapter 5, we define the defect group of hyper-Kähler varieties, and prove its main properties, Theorem 0.3, Corollary 0.4 and Theorems 0.5, 0.6.

In Chapter 6, we present our applications, and prove Corollary 0.8 and Theorems 0.9, 0.10, 0.11, 0.12.

In Chapter 7, we present a conjecture on the cohomology algebras of hyper-Kähler varieties, which we call the conjecture of cohomological rigidity. It suggests a different approach to show that defect groups are finite and hence to prove the Mumford–Tate conjecture, not relying on a deformation to a known example. We establish the conjecture for fourfolds of Kum^2-type, varieties of $\mathrm{K3}^{[n]}$-type for any n, and varieties of OG10-type. This approach was in fact used in our work [**27**] to establish the Mumford–Tate conjecture for varieties of $\mathrm{K3}^{[n]}$ or OG10-type.

In the Appendix, we prove Theorem 0.7.

Notation and conventions

- Reductive algebraic groups are not necessarily connected. Given an algebraic group G, we denote by G^0 its connected component containing the identity.
- We denote by $\mathbb{H}$ the real division algebra of Hamilton's quaternions. Any $h \in \mathbb{H}$ can be written as $h = a + bI + cJ + dK$ for real numbers a, b, c, d, where I, J, and K satisfy
$$I^2 = J^2 = K^2 = IJK = -1.$$
The norm and trace of h as above are $\mathrm{Nm}(h) = a^2 + b^2 + c^2 + d^2$ and $\mathrm{Tr}(h) = 2a$ respectively; quaternions of trace zero are called pure. We have $h^2 = -1$ if and only if $\mathrm{Tr}(h) = 0$ and $\mathrm{Nm}(h) = 1$, i.e. $h = \alpha I + \beta J + \gamma K$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$. Thus, the set of square roots of -1 in $\mathbb{H}$ is a 2-dimensional real sphere. Any $\mathbb{R}$-algebra automorphism $f\colon \mathbb{H} \to \mathbb{H}$ is given by $f(a) = hah^{-1}$ for some $h \in \mathbb{H}$.

– Let $X = (M, I)$ be a complex manifold; here M denotes the underlying differentiable manifold and I is the complex structure of X. A Kähler metric g on X is a Riemannian metric on M such that $g(I(u), I(v)) = g(u, v)$ for all vector fields u, v, and the 2-form defined by $\omega(u, v) = g(I \cdot u, v)$ for all vector fields u, v is closed, i.e. $d\omega = 0$. The differential form ω is called the Kähler form of g; any two elements of the set $\{I, g, \omega\}$ determine the third.

– If X is a complex manifold of dimension n, we let $\mathcal{O}_X$ and $\mathcal{T}_X$ denote the structure sheaf and the holomorphic tangent bundle of X respectively. The cotangent bundle of X is denoted by $\Omega^1_X = \mathcal{T}_X^\vee$; we let Ω^p_X denote the p-th exterior power of Ω^1_X. The canonical bundle of X is by definition $\mathcal{K}_X = \Omega^n_X$. The Betti numbers of X are $b_k = \dim_{\mathbb{Q}} H^k(X, \mathbb{Q})$, where $H^k(X, \mathbb{Q})$ is the k-th singular cohomology group of X. If X is compact Kähler we have the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p,q \geq 0,\ p+q=k} H^{p,q}(X),$$

which satisfies $\overline{H^{p,q}(X)} = H^{q,p}(X)$. The space $H^{p,q}(X)$ is isomorphic to $H^q(X, \Omega^p_X)$; the Hodge numbers of X are $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$.

– All cohomology groups and Chow groups are with rational coefficients, if not specified otherwise.

CHAPTER 1

HYPER-KÄHLER MANIFOLDS AND THE LLV-LIE ALGEBRA

1.1. Basics

This section contains some basic facts on hyper-Kähler manifolds. We follow mainly the references [**9**] and [**42**].

1.1.1. — Let X be a complex manifold. We will say that X is a *hyper-Kähler manifold* if:

- X admits a Kähler metric,
- X is compact and simply-connected,
- the cohomology group $H^0(X, \Omega_X^2)$ is one-dimensional, generated by the class of an everywhere non-degenerate holomorphic closed two-form σ.

It follows immediately from the definition that the dimension of a hyper-Kähler manifold X is even, and that $K_X \cong \mathcal{O}_X$. The form σ is called a holomorphic symplectic form; it induces an isomorphism $\mathcal{T}_X \cong \Omega_X^1$. A *holomorphic symplectic* manifold is a manifold X which carries a holomorphic symplectic form. In the literature, hyper-Kähler manifolds are often called *irreducible holomorphic symplectic* manifolds.

1.1.2. — Let (M, g) be a $4n$-dimensional compact and connected Riemannian manifold. We say that g is a *hyper-Kähler metric* if its holonomy group equals the compact symplectic group $\mathrm{Sp}(n)$. The group $\mathrm{Sp}(n)$ is the subgroup of $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n$ preserving the standard $\mathbb{H}$-Hermitian inner product. Hence, a hyper-Kähler metric g uniquely determines an isometric action of the Hamiltonian quaternions $\mathbb{H}$ on the tangent bundle TM of M, and there exist complex structures I, J and K

on M such that

$$IJ = K, \quad JK = I, \quad KI = J.$$

In this situation, the metric g is Kähler on the complex manifolds (M, I), (M, J) and (M, K). In fact, any $h \in \mathbb{H}$ such that $h^2 = -1$ gives a complex structure on M, and the metric g is Kähler on (M, h). We denote by ω_h the Kähler form of the Kähler metric g on (M, h); the corresponding cohomology class is denoted with the same symbol $\omega_h \in H^2(M, \mathbb{R})$. Since $h^2 = -1$ if and only if $h = \alpha I + \beta J + \gamma K$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$, the metric g is Kähler with respect to a 2-dimensional real sphere of complex structures on M, which we call the *complex structures induced* by the hyper-Kähler metric g.

1.1.3. Definition. — *The characteristic* 3*-space associated to the hyper-Kähler metric g is the* 3*-dimensional real subspace*

$$P_g \coloneqq \langle \omega_I, \omega_J, \omega_K \rangle \subset H^2(M, \mathbb{R}).$$

Equivalently, P_g is the span of the Kähler classes ω_h for all complex structures h induced by g. We will sometimes call (g, I, J, K) a *hyper-Kähler structure* on M. Any two hyper-Kähler structures (g, I, J, K) and (g, I', J', K') associated with the same hyper-Kähler metric are conjugate, in the sense that there exists a quaternion $h \in \mathbb{H}$ such that $I' = hIh^{-1}$, $J' = hJh^{-1}$, $K' = hKh^{-1}$.

1.1.4. — The existence of holomorphic symplectic forms on Kähler manifolds is closely related to hyper-Kähler metrics. Proofs of the following assertions can be found in [**9**, Proposition 4].

Let X be a compact complex manifold of dimension $2n$ and let g be a Kähler metric on X. Assume that g is a hyper-Kähler metric on the underlying real manifold M. This already implies that X is simply connected. Moreover, for a suitable hyper-Kähler structure (g, I, J, K) we have $X = (M, I)$ and $\sigma_I \coloneqq \omega_J + i\omega_K$ is a holomorphic symplectic form on X, unique up to scalar. Therefore, X is a hyper-Kähler manifold.

Conversely, let X be a hyper-Kähler manifold as in §1.1.1. Let $\alpha \in H^2(X, \mathbb{R})$ be a Kähler class. Then, by Yau's solution to Calabi conjecture [**91**], there exists a unique Ricci-flat metric g on X with Kähler class α, and Beauville deduces from the existence and uniqueness of the holomorphic symplectic form that g is a hyper-Kähler metric. If $\sigma \in H^2(X, \mathbb{C})$ denotes the cohomology class of the holomorphic symplectic form, then the characteristic 3-space P_g associated to the hyper-Kähler metric g is

$$P_g = \langle \alpha, \mathfrak{Re}(\sigma), \mathfrak{Im}(\sigma) \rangle \ \subset \ H^2(X, \mathbb{R}).$$

Hence the notions introduced in §1.1.1 and §1.1.2 are essentially equivalent: any Kähler class α on the hyper-Kähler manifold $X = (M, I)$ represents a hyper-Kähler metric g_α on M, while if (M, g) is a compact Riemannian manifold with g a hyper-Kähler metric, then M admits a 2-sphere of complex structures h for which the complex manifold $X_h = (M, h)$ is hyper-Kähler.

The holonomy principle allows to calculate the holomorphic differential forms on a hyper-Kähler manifold, see [**9**, Proposition 3].

1.1.5. Theorem. — *Let X be a hyper-Kähler manifold of dimension $2n$. Let $\sigma \in H^0(X, \Omega^2)$ be a generator. Then*

$$H^0(X, \Omega^k) = \begin{cases} \mathbb{C} \cdot \sigma^j, & \text{if } k = 2j \leq 2n; \\ 0, & \text{otherwise.} \end{cases}$$

1.1.6. — Since a hyper-Kähler manifold X is simply connected by definition, it satisfies $H^1(X, \mathbb{Z}) = 0$. The second cohomology group $H^2(X, \mathbb{Z})$ is torsion free and resembles very much that of a K3 surface. The Hodge decomposition gives $H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$, with $H^{2,0}(X) = \overline{H^{0,2}(X)}$ of dimension 1. Moreover, there exists a symmetric bilinear form

$$q\colon H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z},$$

the *Beauville–Bogomolov form*, which enjoys the following remarkable properties.

1.1.7. Theorem. — *(i) The form q is non-degenerate, of signature $(3, b_2 - 3)$. It does not depend on the complex structure of X, but only on its topology. The Hodge decomposition is orthogonal for the $\mathbb{C}$-linear extension of q.*

(ii) Let $\alpha \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ be a Kähler class, and let g be the corresponding hyper-Kähler metric. Then the form q is positive definite on the characteristic 3-space $P_g \subset H^2(X, \mathbb{R})$ associated to g.

(iii) The quadratic form q satsfies Fujiki's relation: there exists a constant $c > 0$ such that

$$\int_X \alpha^{2n} = c \cdot q(\alpha, \alpha)^n.$$

Parts (*i*) and (*ii*) of the Theorem are [**9**, Theorem 5], while (*iii*) is [**34**, Theorem 4.7].

1.1.8. — Let X be a hyper-Kähler manifold. A *deformation* of a compact Kähler manifold X is a proper and smooth morphism $\mathfrak{X} \to S$ of connected complex spaces such that for a distinguished point $0 \in S$ we have $\mathfrak{X}_0 = X$. All fibres $\mathfrak{X}_s$ for $s \in S$ sufficiently close to 0 are then compact Kähler manifolds. By [**9**, Remarque 1], the fibre $\mathfrak{X}_s$ is a hyper-Kähler manifold whenever it is Kähler. Via deformation theory one can show that there exists a universal local deformation space $\mathrm{Def}(X)$ parametrizing local deformations of X, see [**42**] and the references therein. The local deformation space should be thought as the germ of a complex space at a distinguished point $0 \in \mathrm{Def}(X)$. There exists a universal family $f\colon \mathfrak{X} \to \mathrm{Def}(X)$ such that $\mathfrak{X}_0 = X$: for any deformation $Y \to S$ of X such that $Y_s = X$ for the point $s \in S$, we obtain a classifying morphism $U \to \mathrm{Def}(X)$ in a neighborhood $U \subset S$ of s, such that $Y|_U$ is the pull-back of the universal family along $U \to \mathrm{Def}(X)$.

It is a result of Bogomolov [**11**] that $\mathrm{Def}(X)$ is smooth of dimension $b_2(X) - 2$ for any hyper-Kähler manifold X. We may then assume that $\mathrm{Def}(X)$ is a complex ball; in particular $\mathrm{Def}(X)$ is simply connected and the local system $R^2 f_*\mathbb{Z}$ is constant on $\mathrm{Def}(X)$. Let Λ denote the lattice $H^2(X, \mathbb{Z})$ equipped with the Beauville–Bogomolov form q, and let $\Lambda_{\mathrm{Def}(X)}$ be the constant local system on $\mathrm{Def}(X)$ with fibre Λ. The identification $H^2(\mathfrak{X}_0, \mathbb{Z}) = \Lambda$ determines an isomorphism of local systems $\phi\colon R^2 f_*\mathbb{Z} \xrightarrow{\sim} \Lambda_{\mathrm{Def}(X)}$. The *local period map* is the map

$$\mathcal{P}\colon \mathrm{Def}(X) \to \mathbb{P}(\Lambda \otimes \mathbb{C}), \quad t \mapsto \phi_t(H^{2,0}(\mathfrak{X}_t)).$$

It is holomorphic. The next result is [**9**, Theorem 5].

***1.1.9. Theorem* (Local Torelli theorem).** — *Let $D \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ be the non-singular quadric defined by $q(x, x) = 0$. The image of $\mathcal{P}$ lies in the open real analytic subset*

$$D^o = \{x \in D \mid q(x, \bar{x}) > 0\} \;\subset\; D.$$

Moreover $\mathcal{P}\colon \mathrm{Def}(X) \to D^o$ is a local isomorphism.

The existence of the Beauville–Bogomolov form q is closely related with the local Torelli theorem: one can prove the existence of q by showing that the image of the local period map is a local isomorphism onto a non-singular quadric.

1.1.10. — Let X and Λ be as above. A Λ-*marked* hyper-Kähler manifold is a pair (Y, τ) where Y is a hyper-Kähler manifold and $\tau\colon H^2(Y, \mathbb{Z}) \to \Lambda$ is an isometry. It is known ([**42**]) that there exists a coarse moduli space $\mathfrak{M}$ of Λ-marked hyper-Kähler manifolds, which, roughly speaking, is constructed by gluing together the local deformation spaces. Its points correspond bijectively to Λ-marked hyper-Kähler manifolds up to isomorphism, where (Y, τ) and (Y', τ') are isomorphic if there exists an isomorphism $f\colon Y \xrightarrow{\sim} Y'$ such that $f^*\colon H^2(Y', \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ equals $\tau^{-1} \circ \tau'$. For any connected component $\mathfrak{M}_0$ of $\mathfrak{M}$, we obtain the *global period map*

$$\mathcal{P}\colon \mathfrak{M}_0 \to D^o.$$

It is holomorphic and is a local isomorphism by the local Torelli theorem.

A global Torelli theorem for hyper-Kähler manifolds can be stated in terms of $\mathcal{P}$. The space $\mathfrak{M}_0$ is a complex manifold, but it is not Hausdorff. Nevertheless, it admits a universal Hausdorff quotient $\mathfrak{M}_0 \to \mathfrak{M}_0^\dagger$, obtained identifying the inseparable points, where $x, y \in \mathfrak{M}_0$ are inseparable if any two open neighborhoods $x \in U$ and $y \in V$ have non-trivial intersection. Any continuous map from $\mathfrak{M}_0$ to a Hausdorff space factors through $\mathfrak{M}_0^\dagger$. In [**42**], Huybrechts proved the surjectivity of the global period map and that inseparable points of $\mathfrak{M}_0$ correspond to bimeromorphic manifolds. The theorem below was then proved by Verbitsky in [**84**], see also [**44**] and [**56**].

***1.1.11. Theorem* (Global Torelli theorem).** — *For any connected component $\mathfrak{M}_0$ of $\mathfrak{M}$ the period map $\mathcal{P}\colon \mathfrak{M}_0 \to D^o$ induces an isomorphism $\mathfrak{M}_0^\dagger \cong D^o$. If $x, y \in \mathfrak{M}_0$ are inseparable points then the corresponding hyper-ähler manifolds are bimeromorphic.*

1.1.12. — Let X and Y be hyper-Kähler manifolds. We say that X and Y are *deformation equivalent* if there exists a proper and smooth morphism $\mathfrak{X} \to S$ of connected complex spaces and two points $s, s' \in S$ together with isomorphisms such that $\mathfrak{X}_s \cong X$ and $\mathfrak{X}_{s'} \cong Y$. By the smoothness of the local deformation spaces, we may assume that S is a complex manifold; in this situation, it is known ([**89**, Chapter 9]) that any two fibres of $\mathfrak{X} \to S$ are diffeomorphic. The known examples of hyper-Kähler manifolds are therefore divided into deformation classes.

The only hyper-Kähler surfaces are K3 surfaces. In higher dimension the classification is an open problem; the first higher dimensional examples were constructed by

Beauville [**9**]. For the time being, any known hyper-Kähler manifold is deformation equivalent to one of the following examples.

1.1.13. — Let S be a K3 surface. For any integer $n \geq 1$, the Douady space $S^{[n]}$ of zero-dimensional subspaces $(Z, \mathcal{O}_Z)$ of S with $\dim_{\mathbb{C}}(\mathcal{O}_Z) = n$ on S is a hyper-Kähler manifold of dimension $2n$. If S is projective then $S^{[n]}$ is the Hilbert scheme parametrizing zero-dimensional subschemes of lenght n of S; for $n = 1$, this is just S. The hyper-Kähler manifolds $S^{[n]}$ are all deformation equivalent to each other, and we say that a hyper-Kähler manifold is of $\mathrm{K3}^{[n]}$-type if it is a deformation of some $S^{[n]}$. These varieties have $b_2 = 23$.

1.1.14. — Let T be a complex torus. Let $n \geq 1$ be an integer and consider the Douady space $T^{[n+1]}$ of zero-dimensional subspaces $(Z, \mathcal{O}_Z)$ with $\dim_{\mathbb{C}}(\mathcal{O}_Z) = n + 1$; if T is projective, then $T^{[n+1]}$ is the Hilbert scheme of zero-dimensional subschemes of lenght $n + 1$ in T. It is a compact Kähler manifold and admits a holomorphic symplectic form, but it is not simply connected and $H^{2,0}(T^{[n+1]})$ is not 1-dimensional.

Let $T^{(n+1)}$ denote the $(n+1)$-th symmetric power of T. The sum operation of T yields a morphism $s \colon T^{(n+1)} \to T$; composing it with the Hilbert-Chow morphism $T^{[n+1]} \to T^{(n+1)}$ we obtain a holomorphic map $\Sigma \colon T^{[n+1]} \to T$. This morphism is smooth and proper, and all of its fibres are isomorphic. The generalized Kummer manifold K_T^n is by definition the fibre $\Sigma^{-1}(0)$: it is a hyper-Kähler manifold of dimension $2n$, and if T' is another complex torus then K_T^n and $K_{T'}^n$ are deformation equivalent. We say that a hyper-Kähler manifold is of $\mathrm{Kum}^{[n]}$-type if it is a deformation of a generalized Kummer manifold K_T^n; they have $b_2 = 7$.

1.1.15. — For any $n \geq 2$ the manifolds of $\mathrm{K3}^{[n]}$ and $\mathrm{Kum}^{[n]}$-type are not deformation equivalent. Later, two more deformation classes were discovered by O'Grady [**68**], [**69**], one in dimension 10 and the other in dimension 6. We refer to these deformation classes as the OG10 and OG6-types respectively. Their second Betti numbers are 24 and 8 respectively, as calculated by Rapagnetta [**74**], [**75**].

1.2. The LLV-construction

In this section we construct a $\mathbb{Q}$-Lie algebra $\mathfrak{g}(X)$ acting on the cohomology of a hyper-Kähler manifold, following Looijenga–Lunts [**51**] and Verbitsky [**83**].

1.2.1. — In this text, a $\mathbb{Q}$-*Frobenius algebra of level* m is a graded associative $\mathbb{Q}$-algebra $V^\bullet = \bigoplus_{i=0}^{2m} V^i$ with unity and which is graded-commutative, with an isomorphism $\int \colon V^{2m} \to \mathbb{Q}$ such that $\alpha, \beta \mapsto \int \alpha \cdot \beta$ defines a non-degenerate pairing on $V^\bullet$. Any class $x \in V^2$ defines a nilpotent endomorphism of $V^\bullet$

$$L_x \colon V^\bullet \to V^{\bullet+2}, \qquad L_x(\alpha) \coloneqq x \cdot \alpha.$$

We say that x has the *Lefschetz property* if $L_x^k \colon V^{m-k} \to V^{m+k}$ is an isomorphism for all $k > 0$. Let $\theta \colon V^\bullet \to V^\bullet$ be multiplication by $j - m$ on V^j. Then x has the Lefschetz property if and only if there exists a $\mathbb{Q}$-linear map $\Lambda_x \colon V^\bullet \to V^{\bullet-2}$ such that (L_x, θ, Λ_x) is an $\mathfrak{sl}_2$-triple; explicitly this means that we have the relations

$$[L_x, \Lambda_x] = \theta, \qquad [\theta, L_x] = 2L_x, \qquad [\theta, \Lambda_x] = -2\Lambda_x.$$

The set of $x \in V^2$ with the Lefschetz property is Zariski open in V^2.

1.2.2. — Let us briefly recall how to construct Λ_x, if $x \in V^2$ has the Lefschetz property. For $0 \le j \le m$, we define the primitive (with respect to x) part $P_x^j \subset V^j$ as

$$P_x^j \coloneqq \ker(L_x^{m-j+1} \colon V^j \to V^{2m-j+2}).$$

We define $P_x^j = 0$ for $j \notin \{0, 1, \dots, m\}$. Then any $y \in V^k$ can be written uniquely as a sum $y = \sum_{i \ge 0} L_x^i(y_{k-2i})$ with $y_{k-2i} \in P_x^{k-2i}$. For $i \ge 0$, we obtain projectors $\pi_{k,i} \colon V^k \to P_x^{k-2i}$ such that $y = \sum_{i \ge 0} L_x^i(\pi_{k,i}\, y)$; clearly, if $k - 2i \notin \{0, 1, \dots, m\}$ then $\pi_{k,i} = 0$. Explicitly, if $k \le m$, the projectors $\pi_{k,i}$ are inductively defined by

$$\pi_{k,i} \coloneqq (L_x^{m-k+2i}|_{V^{k-2i}})^{-1} L_x^{m-k+i}(y - \sum_{i' > i} L_x^{i'}(\pi_{k,i'}\, y)).$$

If $k > m$ and $y \in V^k$, then $y = L_x^{k-m}(z)$ for a unique $z \in V^{2m-k}$. We therefore obtain $y = \sum_i L^{k-m+i}(\pi_{2m-k,i} z)$. In other words, for any $i \ge 0$, we have

$$\pi_{k,i} = \pi_{2m-k,i-k+m} \circ (L_x^{k-m}|_{V^{2m-k}})^{-1}.$$

For $y \in V^k$, we define

$$\Lambda_x(y) = \Lambda_x\Big(\sum_{i \ge 0} L_x^i(\pi_{k,i}\, y)\Big) = \sum_{i \ge 1} i(m - k + i + 1) \cdot L_x^{i-1}(\pi_{k,i}\, y);$$

a straightforward computation shows that (L_x, θ, Λ_x) is an $\mathfrak{sl}_2$-triple.

By standard representation theory of $\mathfrak{sl}_2$, for any $\mathfrak{sl}_2$-triple (L_x, θ, Λ_x), the primitive cohomology P_x^j coincides with $\ker(\Lambda_x) \cap V^j$. This implies that Λ_x is uniquely determined by L_x and θ, since if $(L_x, \theta, \Lambda'_x)$ is also an $\mathfrak{sl}_2$-triple then $\Lambda_x - \Lambda'_x$ commutes with L_x and hence $(\Lambda_x - \Lambda'_x)(\sum_{i\geq 0} L_x^i(\pi_{k,i}y)) = 0$ for all $y \in V^k$.

1.2.3. Definition. — *Let $V^\bullet$ be a Frobenius algebra. Assume that there exists some $x \in V^2$ with the Lefschetz property. The LLV-Lie algebra $\mathfrak{g}(V^\bullet)$ of $V^\bullet$ is the Lie subalgebra of $\mathfrak{gl}(V^\bullet)$ generated by the $\mathfrak{sl}_2$-triples (L_x, θ, Λ_x) for $x \in V^2$ with the Lefschetz property.*

We define a non-degenerate bilinear form ϕ on $V^\bullet$ by $\phi(\alpha, \beta) = (-1)^k \int \alpha \cdot \beta$ for α of degree $2k+m$ or $2k+m+1$. The sign correction ensures that ϕ is $\mathfrak{g}(V^\bullet)$-invariant, in the sense that $\mathfrak{g}(V^\bullet) \subset \mathfrak{so}(V^\bullet, \phi)$; we call ϕ the *canonical blinear form* on the Frobenius algebra $V^\bullet$.

The adjoint action of θ on $\mathfrak{g}(X)$ has even integers as eigenvalues, and we write $\mathfrak{g}(V^\bullet) = \oplus_i \mathfrak{g}_{2i}(V^\bullet)$ accordingly. We collect here two easy observations.

1.2.4. Lemma. — *(i) Let $V_1^\bullet$, $V_2^\bullet$ be two Frobenius algebras and assume that there exists $x \in V_1^2$ with the Lefschetz property. Let $F\colon V_1^\bullet \xrightarrow{\sim} V_2^\bullet$ be an isomorphism of graded algebras; we denote by $F_*\colon \mathfrak{gl}(V_1^\bullet) \xrightarrow{\sim} \mathfrak{gl}(V_2^\bullet)$ the isomorphism $A \mapsto FAF^{-1}$. Then F_* restricts to an isomorphism of graded Lie algebras $F_*\colon \mathfrak{g}(V_1^\bullet) \xrightarrow{\sim} \mathfrak{g}(V_2^\bullet)$.*

(ii) Let $V^\bullet$ be a Frobenius algebra, and assume that there exists $x \in V^2$ with the Lefschetz property. Let $G \subset \prod_j \mathrm{GL}(V^j)$ be a group which acts on $V^\bullet$ by graded algebra automorphisms. Denote by K the kernel of the restriction $G \to \mathrm{GL}(V^2)$. Then the action of K on V commutes with the action of $\mathfrak{g}(X)$.

Proof. — *(i)*. For any $x \in V_1^2$ we have $F_*(L_x) = L_{F(x)}$ because F is an algebra homomorphism. We clearly also have $F_*(\theta_1) = \theta_2$. Moreover, if $x \in V_1^2$ has the Lefschetz property then $F(x)$ has it as well, and $(F_*(L_x), F_*(\theta_1), F_*(\Lambda_x))$ is an $\mathfrak{sl}_2$-triple, so $F_*(\Lambda_x) = \Lambda_{F(x)}$.

(ii). For $k \in K$, we have $k(L_x)k^{-1} = L_{k(x)} = L_x$, and $k\theta k^{-1} = \theta$. By the above argument we also have $k\Lambda_x k^{-1} = \Lambda_x$ whenever $x \in V^2$ has the Lefschetz property. ☐

1.2.5. — The following example will be relevant for us. Let H be a $\mathbb{Q}$-vector space equipped with a non-degenerate symmetric bilinear form q. Let $\tilde{H} = \mathbb{Q}\cdot v \oplus H \oplus \mathbb{Q}\cdot w$ be the orthogonal direct sum of (H, q) with $U = \mathbb{Q}\cdot v \oplus \mathbb{Q}\cdot w$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. We denote by $\tilde{q}$ the resulting bilinear form on $\tilde{H}$.

We put on $\tilde{H}$ the structure of a Frobenius algebra of level 2 as follows: we declare v, V, w to have degree 0, 2, 4 respectively; the algebra structure is given by

$$\begin{aligned}
&v \cdot \alpha = \alpha && \text{for } \alpha \in \tilde{H}, \\
&x \cdot y = q(x, y) \cdot w && \text{for } x, y \in H, \\
&z \cdot w = 0 && \text{for } z \in H \oplus \mathbb{Q} \cdot w.
\end{aligned}$$

It is readily seen that any $x \in H$ with $q(x, x) \neq 0$ has the Lefschetz property, since $L_x^2\colon \mathbb{Q} \cdot v \to \mathbb{Q} \cdot w$ maps $a \cdot v$ to $aq(x, x) \cdot w$. The Lie algebra $\mathfrak{g}(\tilde{H})$ is thus defined; the semisimple element θ acts as multiplication by -2 (resp. 2) on $\mathbb{Q} \cdot v$ (resp. on $\mathbb{Q} \cdot w$), and it is zero on H.

The canonical bilinear form on the Frobenius algebra $\tilde{H}$ coincides with $\tilde{q}$. Therefore, we have $\mathfrak{g}(\tilde{H}) \subset \mathfrak{so}(\tilde{H}, \tilde{q})$.

1.2.6. Proposition. — *The Lie algebra $\mathfrak{g}(\tilde{H})$ equals $\mathfrak{so}(\tilde{H}, \tilde{q})$. The adjoint action of θ on $\mathfrak{g}(X)$ has eigenvalues $-2, 0, 2$ only. We have*

$$\mathfrak{g}(\tilde{H}) = \mathfrak{g}_{-2}(\tilde{H}) \oplus \mathfrak{g}_0(\tilde{H}) \oplus \mathfrak{g}_2(\tilde{H}),$$

where $\mathfrak{g}_{-2}(\tilde{H})$ and $\mathfrak{g}_2(\tilde{H})$ are abelian Lie subalgebras isomorphic to H and

$$\mathfrak{g}_0(\tilde{H}) = \mathbb{Q} \cdot \theta \oplus \mathfrak{so}(H).$$

Proof. — Let $n = \dim V$. We fix a basis $v, e_1, \ldots, e_n, w$ of $\tilde{H}$. Let Q be the matrix of the bilinear form q. The matrix corresponding to $\tilde{q}$ is then given by

$$\tilde{Q} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & Q & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We write $A \in \mathfrak{gl}(\tilde{H})$ as

$$A = \begin{pmatrix} a & b^T & c \\ d & A' & e \\ f & g^T & h \end{pmatrix},$$

in which $a, c, f, h \in \mathbb{Q}$ are scalars, b, d, e, g are vectors in H and $A' \in \mathfrak{gl}(H)$. The matrix $A \in \mathfrak{so}(\tilde{H}, \tilde{Q})$ if and only if $A^T\tilde{Q} + \tilde{Q}A = 0$; computing this expression we obtain the equations

$$\begin{aligned} a+h&=0, & A'&\in\mathfrak{so}(H,Q),\\ Qd-g&=0, & Qe-b&=0,\\ 2c&=0, & 2f&=0. \end{aligned}$$

Given a vector $x \in H$ with $x^T Q\, x \neq 0$, the corresponding $\mathfrak{sl}_2$-triple is given by

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x^T Q & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \Lambda_x = \frac{2}{x^T Q\, x} \cdot \begin{pmatrix} 0 & x^T Q & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

Extending the coefficients to the complex numbers we may assume that $\{e_1, \ldots, e_n\}$ is an orthonormal basis of H; then it is easy to see that the $\mathbb{C}$-Lie algebra generated by the commutators $[L_x, \Lambda_y]$ is $\mathbb{C}\cdot\theta \oplus \mathfrak{so}(H_{\mathbb{C}})$. It follows that $\mathfrak{g}(\tilde{H})$ coincides with $\mathfrak{so}(\tilde{H}, \tilde{q})$. The other assertions are now clear. □

1.2.7. — Let X be a non-singular and projective complex algebraic variety of dimension n. By Poincaré duality, the cohomology algebra $H^\bullet(X, \mathbb{Q})$ is a Frobenius algebra of level n over $\mathbb{Q}$. By the Hard Lefschetz theorem, the first Chern class of an ample line bundle on X has the Lefschetz property.

1.2.8. Definition. — *The LLV-Lie algebra $\mathfrak{g}(X)$ of X is by definition the LLV-Lie algebra attached to the Frobenius algebra $H^\bullet(X, \mathbb{Q})$.*

Recall (Definition 1.2.3) that this means that $\mathfrak{g}(X)$ is generated by all $\mathfrak{sl}_2$-triples (L_x, θ, Λ_x) for $x \in H^2(X, \mathbb{Q})$ with the Lefschetz property, where L_x is given by cup-product with x and θ is multiplication by $j - n$ on $H^j(X, \mathbb{Q})$. It is clear that the Lie algebra $\mathfrak{g}(X)$ does not depend on the complex structure on X but only on its topology, since this is the case for the algebra $H^\bullet(X, \mathbb{Q})$. We let ϕ be the canonical bilinear form on the Frobenius algebra $H^\bullet(X, \mathbb{Q})$; it is given by $\alpha, \beta \mapsto (-1)^k \int \alpha \cdot \beta$ for α of degree $2k + n$ or $2k + n + 1$, where $\int \colon H^\bullet(X, \mathbb{Q}) \to H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}$ denotes the projection. The form ϕ is $\mathfrak{g}(X)$-invariant, i.e. $\mathfrak{g}(X) \subset \mathfrak{so}(H^\bullet(X, \mathbb{Q}), \phi)$.

1.2.9. Proposition **([51, Proposition 1.6]).** — *For any non-singular and projective complex variety X, the $\mathbb{Q}$-Lie algebra $\mathfrak{g}(X)$ is semisimple. Moreover ϕ restricts to a non-degenerate pairing on any $\mathfrak{g}(X)$-submodule of $H^\bullet(X, \mathbb{Q})$.*

1.2.10. Corollary. — *Let $A^\bullet \subset H^\bullet(X, \mathbb{Q})$ be a subalgebra. Assume that $A^\bullet$ is stable under $\mathfrak{g}(X)$. Then $A^\bullet$ contains the subalgebra generated by $H^2(X, \mathbb{Q})$, and $A^\bullet$ is a Frobenius algebra of level n.*

Proof. — Indeed, by Proposition 1.2.9, the restriction of ϕ to $A^\bullet$ is non-degenerate. This implies that $H^{2n}(X, \mathbb{Q})$ is contained in $A^\bullet$, and that the latter is a Frobenius algebra. But the $\mathfrak{g}(X)$-submodule containing $H^{2n}(X, \mathbb{Q})$ clearly also contains $H^0(X, \mathbb{Q})$ and any product $x_1 \cdot x_2 \cdot \ldots \cdot x_k$ of classes $x_1, x_2, \ldots, x_k \in H^2(X, \mathbb{Q})$. □

1.2.11. — Let now X be a hyper-Kähler manifold of dimension $2n$. By [**42**, Theorem 3.5], X deforms to a projective hyper-Kähler variety X'. Since X and X' are diffeomorphic, their cohomology algebras are isomorphic; hence, the $\mathbb{Q}$-Lie algebra $\mathfrak{g}(X)$ is defined. The following theorem due to Verbitsky [**83**] and Looijenga–Lunts [**51**, Proposition 4.5] describes $\mathfrak{g}(X)$. Let H denote the space $H^2(X, \mathbb{Q})$ equipped with the Beauville–Bogomolov form q, and let $\tilde{H}$ denote the orthogonal direct sum of H with $U = \mathbb{Q} \cdot v \oplus \mathbb{Q} \cdot w$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$.

1.2.12. Theorem. — *(i) There exists a unique isomorphism of $\mathbb{Q}$-Lie algebras*

$$\varphi \colon \mathfrak{g}(X) \xrightarrow{\sim} \mathfrak{so}(\tilde{H})$$

such that $\varphi(\theta)$ acts as multiplication by -2 (resp. by 2) on $\mathbb{Q} \cdot v$ (resp. on $\mathbb{Q} \cdot w$) and it is zero on H, and, for any $x, y \in H$, we have

$$\varphi(L_x)(v) = x, \qquad \varphi(L_x)(y) = q(x, y) \cdot w, \qquad \varphi(L_x)(w) = 0.$$

(ii) We have $\mathfrak{g}(X) = \mathfrak{g}_{-2}(X) \oplus \mathfrak{g}_0(X) \oplus \mathfrak{g}_2(X)$, where $\mathfrak{g}_{-2}(X)$ and $\mathfrak{g}_2(X)$ are abelian subalgebras isomorphic to H. The isomorphism φ restricts to

$$\mathfrak{g}_0(X) \xrightarrow{\sim} \mathbb{Q} \cdot \varphi(\theta) \oplus \mathfrak{so}(H).$$

(iii) The induced action of $\mathfrak{so}(H)$ on $H^\bullet(X, \mathbb{Q})$ is by derivations, and its action on $H^2(X, \mathbb{Q})$ is the standard representation.

Note that the isomorphism φ needs to map $\mathfrak{sl}_2$-triples to $\mathfrak{sl}_2$-triples; this immediately shows its uniqueness. Also note that (ii) follows from (i) via Proposition 1.2.6.

1.2.13. — In the next two sections we will give the proof of Theorem 1.2.12. Let M be the real manifold underlying X. The starting observation is that any hyper-Kähler metric g on M determines an action of a real Lie algebra $\mathfrak{h}_g$ on the cohomology $H^\bullet(M,\mathbb{R})$. Let (g,I,J,K) be a hyper-Kähler structure associated with g. Since g is Kähler with respect to the three complex structures I,J,K, the Kähler classes $\omega_I,\omega_J,\omega_K \in H^2(X,\mathbb{R})$ have the Lefschetz property.

1.2.14. Definition. — *The Lie algebra* $\mathfrak{h}_g \subset \mathfrak{gl}(H^\bullet(X,\mathbb{R}))$ *is the real Lie subalgebra generated by*

$$L_{\omega_I},\quad L_{\omega_J},\quad L_{\omega_K},\quad \Lambda_{\omega_I},\quad \Lambda_{\omega_J},\quad \Lambda_{\omega_K}.$$

We will explicitly compute the Lie algebra $\mathfrak{h}_g$ in Theorem 1.3.8; it turns out that it does not depend on the chosen hyper-Kähler metric, being always isomorphic to $\mathfrak{so}(4,1)$, but its action on the cohomology of M does. Thus, each hyper-Kähler metric g on M yields a different embedding $\mathfrak{h}_g \hookrightarrow \mathfrak{g}(X)\otimes\mathbb{R}$.

The key fact in order to compute $\mathfrak{g}(X)$ is then that the image of the different embeddings $\mathfrak{h}_g \hookrightarrow \mathfrak{g}(X)\otimes\mathbb{R}$ for all hyper-Kähler metrics on M generate $\mathfrak{g}(X)\otimes\mathbb{R}$. This is a consequence of the local Torelli theorem, see Proposition 1.4.2, and it has strong implications for the structure of $\mathfrak{g}(X)$, see Proposition 1.4.4. As a consequence, the subalgebra $A_2^\bullet \subset H^\bullet(X,\mathbb{Q})$ generated by $H^2(X,\mathbb{Q})$ is a faithful and irreducible $\mathfrak{g}(X)$-module. Calculating the algebra $A_2^\bullet$, see Theorem 1.4.7, leads to the conclusion.

1.3. The Lie algebra of a hyper-Kähler metric

In this section we compute the Lie algebra $\mathfrak{h}_g$ attached to a hyper-Kähler metric g on M (Definition 1.2.14). The main idea is to use the Kähler identities to reduce this calculation to that of an analogous Lie algebra acting on the differential forms on M, which, in turn, is reduced to a linear algebraic computation.

1.3.1. — Let T be a finitely generated left $\mathbb{H}$-module equipped with a positive definite $\mathbb{H}$-invariant $\mathbb{R}$-bilinear symmetric pairing $\langle -,-\rangle$. Then T is a $4m$-dimensional real vector space, and it is in fact isomorphic to an orthogonal direct sum $T\cong\mathbb{H}^{\oplus m}$, were $\mathbb{H}$ is equipped with the standard inner product. To see this, choose any $e\in T$ with $\langle e,e\rangle=1$. Then $\{e, I\cdot e, J\cdot e, K\cdot e\}$ is an orthonormal basis of $\mathbb{H}\cdot e$, since

$$\langle e, I\cdot e\rangle = \langle I\cdot e, I^2\cdot e\rangle = -\langle e, I\cdot e\rangle = 0.$$

Thus we may write $T \cong \mathbb{H} \oplus \mathbb{H}^\perp$; by induction we conclude that $T \cong \mathbb{H}^{\oplus m}$ for some m.

Any pure quaternion h of norm 1 gives a complex structure on T. The induced orientations of T are all compatible, thus T is canonically oriented. Any such complex structure h determines a decomposition

$$T \otimes_{\mathbb{R}} \mathbb{C} = T_h^{1,0} \oplus T_h^{0,1}$$

such that the action of h on $T^{1,0}$ (resp. on $T^{0,1}$) is multiplication by i (resp. by $-i$).

1.3.2. — The inner product induces an isomorphism $T \cong T^\vee$. We obtain a right action of $\mathbb{H}$ on $T^\vee$ and on the exterior algebra $\bigwedge^\bullet T^\vee$. The inner product on T extends to an inner product $\langle -, - \rangle$ on $\bigwedge^\bullet T^\vee$: if $\{e_1, \ldots, e_{4m}\}$ is an orthonormal basis of T, then

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k},\ 1 \leq i_1 < i_2 < \ldots < i_k \leq 4m\},$$

is an orthonormal basis of $\bigwedge^\bullet T^\vee$; the resulting product does not depend on the chosen basis. We let $\mathrm{vol} \coloneqq e_1 \wedge e_2 \wedge \ldots \wedge e_{4m}$; it is independent on the chosen basis of T as long as it has the canonical orientation. The Hodge star operator $\star\colon \bigwedge^\bullet T^\vee \to \bigwedge^{4m-\bullet} T^\vee$ is defined by declaring that $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}$ for all $\alpha, \beta \in \bigwedge^\bullet T^\vee$.

For any pure quaternion h of norm 1, the decomposition $T \otimes_{\mathbb{R}} \mathbb{C} = T_h^{1,0} \oplus T_h^{0,1}$ induces a p,q-decomposition $\bigwedge^\bullet T^\vee \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} \bigwedge_h^{p,q} T^\vee$, where

$$\bigwedge\nolimits_h^{p,q} T^\vee \coloneqq \bigwedge^p (T_h^{1,0})^\vee \otimes \bigwedge^q (T_h^{0,1})^\vee.$$

The complex conjugate of $\bigwedge_h^{p,q} T^\vee$ is $\bigwedge_h^{q,p} T^\vee$. Let $W_h\colon \bigwedge^\bullet T^\vee \otimes_{\mathbb{R}} \mathbb{C} \to T^\vee \otimes_{\mathbb{R}} \mathbb{C}$ be multiplication by $i(p-q)$ on $\bigwedge_h^{p,q} T^\vee$. For any $x \in \bigwedge^\bullet T^\vee \otimes_{\mathbb{R}} \mathbb{C}$ we have $\overline{W_h(x)} = W_h(\overline{x})$; hence W_h is the complexification of a real endomorphism W_h of $\bigwedge^\bullet T^\vee$, called the *Weil operator*. In fact, W_h extends the action of h on $T^\vee$ to a derivation of $\bigwedge^\bullet T^\vee$.

1.3.3. — Let $\theta\colon \bigwedge^\bullet T^\vee \to \bigwedge^\bullet T^\vee$ be multiplication by $k - 2m$ on $\bigwedge^k T^\vee$. For any pure quaternion h of norm 1, we define $\omega_h \in \bigwedge^2 T^\vee$ by $\omega_h(\alpha, \beta) \coloneqq \langle h(\alpha), \beta \rangle$, and we introduce the endomorphism $L_{\omega_h}\colon \bigwedge^\bullet T^\vee \to \bigwedge^{\bullet+2} T^\vee$ given by $L_{\omega_h}(\alpha) \coloneqq \omega_h \wedge \alpha$.

We also let $\Lambda_{\omega_h} \coloneqq \star^{-1} L_{\omega_h} \star$. Then $(L_{\omega_h}, \theta, \Lambda_{\omega_h})$ is an $\mathfrak{sl}_2$-triple, see [**43**, Proposition 1.2.26]. In particular ω_h has the Lefschetz property.

1.3.4. Definition. — *The Lie algebra $\mathfrak{h}(T) \subset \mathfrak{gl}(\bigwedge^\bullet T^\vee)$ is the Lie subalgebra generated by*

$$L_{\omega_I},\quad L_{\omega_J},\quad L_{\omega_K},\quad \Lambda_{\omega_I},\quad \Lambda_{\omega_J},\quad \Lambda_{\omega_K}.$$

The next result due to Verbitsky gives generators and relations for the Lie algebra $\mathfrak{h}(T)$. Consider the 3-dimensional subspace $P \coloneqq \langle \omega_I, \omega_J, \omega_K \rangle \subset \bigwedge^2 T^\vee$, and let $\tilde{P}$ be the orthogonal direct sum of P with $U = \mathbb{R} \cdot u \oplus \mathbb{R} \cdot v$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. Consider the Weil operators W_I, W_J, W_K (see §1.3.2).

1.3.5. Proposition **([82]).** — *(i) We have*

$$[L_{\omega_I}, \Lambda_{\omega_J}] = W_K, \quad [L_{\omega_J}, \Lambda_{\omega_K}] = W_I, \quad [L_{\omega_K}, \Lambda_{\omega_I}] = W_J.$$

(ii) The Lie algebra $\mathfrak{h}(T)$ is 10-dimensional; a basis is given by

$$L_{\omega_I},\ L_{\omega_J}, L_{\omega_K},\ \theta,\ W_I,\ W_J,\ W_K,\ \Lambda_{\omega_I},\ \Lambda_{\omega_J},\ \Lambda_{\omega_K}.$$

Introducing the notation $K_{i,j} \coloneqq [L_{\omega_i}, \Lambda_{\omega_j}]$, for $i, j \in \{I, J, K\}$, the following is a full set of relations among the generators of $\mathfrak{h}(T)$:

$$\begin{aligned}
[\theta, L_{\omega_i}] &= 2L_{\omega_i}, & [\theta, \Lambda_{\omega_i}] &= -2\Lambda_{\omega_i},\\
[L_{\omega_i}, \Lambda_{\omega_i}] &= \theta, & [\theta, K_{i,j}] &= 0,\\
[L_{\omega_i}, L_{\omega_j}] &= 0, & [\Lambda_{\omega_i}, \Lambda_{\omega_j}] &= 0,\\
[K_{i,j}, K_{j,k}] &= 2K_{i,k}, & K_{i,j} &= -K_{j,i},\\
[K_{i,j}, L_{\omega_j}] &= 2L_{\omega_i}, & [K_{i,j}, \Lambda_{\omega_j}] &= 2\Lambda_{\omega_i},\\
[K_{i,j}, L_{\omega_k}] &= [K_{i,j}, \Lambda_{\omega_k}] = 0, & &\textit{for } k \neq i, j.
\end{aligned}$$

(iii) There exists a unique isomorphism $\varphi\colon \mathfrak{h}(T) \xrightarrow{\sim} \mathfrak{so}(\tilde{P}) \cong \mathfrak{so}(4,1)$ such that $\varphi(\theta)$ acts as multiplication by -2 (resp. 2) on $\mathbb{R} \cdot v$ (resp. on $\mathbb{R} \cdot w$) and it is zero on P, and, for $t = I, J, K$ and any $y \in P$, we have

$$\varphi(L_{\omega_t})(v) = \omega_t, \qquad \varphi(L_{\omega_t})(y) = q(\omega_t, y) \cdot w, \qquad \varphi(L_{\omega_t})(w) = 0.$$

This isomorphism restricts to $\varphi\colon \langle W_I, W_J, W_K \rangle \xrightarrow{\sim} \mathfrak{so}(P)$.

Proof. — It is clear that the operators $L_{\omega_I}, L_{\omega_J}, L_{\omega_K}$ commute. Then we have

$$[\Lambda_{\omega_I}, \Lambda_{\omega_J}] = \star^{-1}[L_{\omega_I}, L_{\omega_J}]\star = 0,$$

and hence $\Lambda_{\omega_I}, \Lambda_{\omega_J}, \Lambda_{\omega_K}$ commute as well.

The module T is an orthogonal direct sum of copies of $\mathbb{H}$ with the standard inner product, $T = \mathbb{H}^{\oplus m}$. It follows that $\bigwedge^\bullet T^\vee = (\bigwedge^\bullet \mathbb{H}^\vee)^{\otimes m}$. The classes $\omega_I, \omega_J, \omega_K$ belong to $(\bigwedge^2 \mathbb{H}^\vee)^{\oplus m}$, and we conclude that $\mathfrak{h}(T)$ equals $\mathfrak{h}(\mathbb{H})$ which acts on $\bigwedge^\bullet T^\vee$ via the m-fold product of its action on $\bigwedge^\bullet \mathbb{H}^\vee$. Thus, we may assume $m = 1$.

In this case we explicitly compute all the necessary relations. We have $T = \mathbb{H}$ with its standard inner product. We let e_1, e_I, e_J, e_K be the basis of $\mathbb{H}^\vee$ dual to the orthonormal basis $1, I, J, K$ of $\mathbb{H}$. We then calculate

$$\omega_I = e_1 \wedge e_I + e_J \wedge e_K,$$

$$\omega_J = e_1 \wedge e_J - e_I \wedge e_K,$$

$$\omega_K = e_1 \wedge e_K + e_I \wedge e_J.$$

Let $P \subset \bigwedge^2 \mathbb{H}^\vee$ be the subspace generated by $\omega_I, \omega_J, \omega_K$. It is readily seen that this is an orthogonal basis of P. We have $\bigwedge^2 \mathbb{H}^\vee = P \oplus P^\perp$; the Hodge star operator is the identity on P and it is -1 on $P^\perp$.

We now compute the action of $[L_{\omega_J}, \Lambda_{\omega_K}]$: on $\mathbb{H}^\vee$ this is given by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

on $\bigwedge^3 \mathbb{H}^\vee$, with respect to the basis $\hat{e}_1, \hat{e}_I, \hat{e}_J, \hat{e}_K$, it is given by the transpose matrix. Here, $\hat{e}_1 = e_I \wedge e_J \wedge e_K$, $\hat{e}_I = e_1 \wedge e_J \wedge e_K$, etc. Next, the action of $[L_{\omega_J}, \Lambda_{\omega_K}]$ is trivial on $\bigwedge^0 \mathbb{H}^\vee$, on $P^\perp \subset \bigwedge^2 \mathbb{H}^\vee$ and on $\bigwedge^4 \mathbb{H}^\vee$. On $P = \langle \omega_I, \omega_J, \omega_K \rangle$ this action is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}.$$

The action of W_I on $\mathbb{H}^\vee$ coincides with that of $[L_{\omega_J}, \Lambda_{\omega_K}]$. Using that W_I acts by derivations on the algebra $\bigwedge^\bullet T^\vee$ we easily verify that $[L_{\omega_J}, \Lambda_{\omega_K}] = W_I$.

By similar computations we check that the Lie algebra $\mathfrak{h}(\mathbb{H})$ is 10-dimensional with the basis given in (*ii*) and calculate the complete set of relations among these generators. This establishes (*i*) and (*ii*).

To prove (*iii*), we consider the non-degenerate form ϕ on $\bigwedge^\bullet \mathbb{H}^\vee$ given as follows. Let $\int\colon \bigwedge^\bullet \mathbb{H}^\vee \to \bigwedge^4 \mathbb{H}^\vee \cong \mathbb{R}$ be the projection. Then ϕ is given by

$$\alpha, \beta \mapsto (-1)^n \int \alpha \wedge \beta$$

for α of degree $2n+2$ or $2n+3$; we have $\mathfrak{h}(\mathbb{H}) \subset \mathfrak{so}(\bigwedge^\bullet \mathbb{H}^\vee, \phi)$.

The restriction to $P \subset \bigwedge^2 \mathbb{H}^\vee$ of the form ϕ coincides with the given inner product. Hence, the subspace $\bigwedge^0 \mathbb{H}^\vee \oplus P \oplus \bigwedge^4 \mathbb{H}^\vee \subset \bigwedge^\bullet \mathbb{H}^\vee$ equipped with ϕ is naturally identified with $\tilde{P}$. Moreover, this subspace is a faithful representation of $\mathfrak{h}(\mathbb{H})$. We thus obtain an injective homomorphism $\mathfrak{h}(\mathbb{H}) \hookrightarrow \mathfrak{so}(\tilde{P})$, which is surjective by dimension reasons, and, therefore, an isomorphism; this is the desired isomorphism φ. By Proposition 1.2.6, φ restricts to an isomorphism $\langle W_I, W_J, W_K \rangle \xrightarrow{\sim} \mathfrak{so}(P)$. $\square$

1.3.6. Remark. — By [**51**, Proof of Lemma 4.2], the induced representation of $\mathfrak{so}(\tilde{P})$ on $\bigwedge^{\mathrm{odd}} \mathbb{T}^\vee$ is a spin representation. This action integrates to a representation of the spin group $\mathrm{Spin}(\tilde{P})$ on $\bigwedge^\bullet T^\vee$; for each k, the element $-1 \in \mathrm{Spin}(\tilde{P})$ acts as multiplication by $(-1)^k$ on $\bigwedge^k T^\vee$.

1.3.7. — We now go back to the geometric situation and carry out the first part of the program outlined in §1.2.13. Let X be a hyper-Kähler manifold of dimension $2n$ and denote by H the vector space $H^2(X, \mathbb{Q})$ equipped with the Beauville–Bogomolov form q. Let M be the differentiable manifold underlying X, and assume given a hyper-Kähler metric g on M. Let $P_g \subset H^2(X, \mathbb{R})$ be the characteristic 3-space associated to g, and let $\tilde{P}_g$ be the orthogonal direct sum of P_g with $U = \mathbb{R} \cdot v \oplus \mathbb{R} \cdot w$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. The following result is again due to Verbitsky.

1.3.8. Theorem **([82]).** — *The Lie algebra $\mathfrak{h}_g$ (Definition 1.2.14) is isomorphic to $\mathfrak{h}(\mathbb{H})$. We have an isomorphism $\varphi \colon \mathfrak{h}_g \xrightarrow{\sim} \mathfrak{so}(\tilde{P}_g) \cong \mathfrak{so}(4,1)$.*

Hence, Proposition 1.3.5 gives generators and relations for the Lie algebra $\mathfrak{h}_g$.

Proof. — The hyper-Kähler metric g determines a left $\mathbb{H}$-action on the tangent bundle TM of M, and hence a right action of $\mathbb{H}$ on the graded vector bundle $\bigwedge^\bullet (TM)^\vee$. The metric g extends to a $\mathbb{H}$-invariant metric on $\bigwedge^\bullet (TM)^\vee$. Applying Proposition 1.3.5 to the $\mathbb{H}$-module TM we obtain an action of the Lie algebra $\mathfrak{h}(\mathbb{H})$ on $\bigwedge^\bullet (TM)^\vee$. Let $\mathcal{A}^\bullet(M)$ denote the algebra of differential forms on M, that is, the global sections of the bundle $\bigwedge^\bullet (TM)^\vee$. Recall ([**43**]) that the metric g determines a Laplacian operator Δ on $\mathcal{A}^\bullet(M)$, and that, by the Hodge theorem, the cohomology $H^\bullet(M, \mathbb{R})$ is canonically identified with the subspace $\ker(\Delta)$ of Δ-harmonic forms.

By the above, the Lie algebra $\mathfrak{h}(\mathbb{H})$ acts on $\mathcal{A}^\bullet(M)$. By the Kähler identities [**43**, Proposition 3.1.12], this action commutes with Δ, and hence $\mathfrak{h}(\mathbb{H})$ acts on the real cohomology of M; by construction, the action of $\mathfrak{h}(\mathbb{H})$ on $\mathcal{A}^\bullet(M)$ extends that of $\mathfrak{h}_g$

on $H^\bullet(M,\mathbb{R})$. Moreover, the action of $\mathfrak{h}(\mathbb{H})$ on $H^\bullet(M,\mathbb{R})$ is faithful, as it can be easily checked by looking at the basis of $\mathfrak{h}(\mathbb{H})$ given by Proposition 1.3.5. Thus $\mathfrak{h}_g$ is isomorphic to $\mathfrak{h}(\mathbb{H}) \cong \mathfrak{so}(4,1)$.

The Kähler classes $\omega_I, \omega_J, \omega_K$ span the 3-space $P_g \subset H \otimes_{\mathbb{Q}} \mathbb{R}$. By Theorem 1.1.7.(*ii*) the Beauville–Bogomolov form is positive definite on P_g, and Proposition 1.3.5.(*iii*) yields the isomorphism $\varphi\colon \mathfrak{h}_g \xrightarrow{\sim} \mathfrak{so}(\tilde{P}_g)$. □

1.3.9. Corollary. — *The sub-Lie algebra* $\mathfrak{so}(P_g) \cong \langle W_I, W_J, W_K\rangle$ *of* $\mathfrak{h}_g$ *acts by derivations on the algebra* $H^\bullet(X,\mathbb{R})$.

Proof. — As in 1.3.2, the action of any pure quaternion h of norm 1 gives a p,q-decomposition $\bigwedge^\bullet(TM)^\vee \otimes \mathbb{C} = \bigoplus_{p,q} \bigwedge_h^{p,q}(TM)^\vee$. Consequently, we have the decomposition $\mathcal{A}^\bullet(M) \otimes \mathbb{C} = \bigoplus_{p,q} \mathcal{A}_h^{p,q}(M)$ into p,q-differential forms, which induces the Hodge decomposition $H^\bullet(M,\mathbb{C}) = \bigoplus_{p,q} H_h^{p,q}(M)$ of the cohomology of the Kähler manifold (M,h). For any h as above, we have $H_h^{p,q}(M)\cdot H_h^{p',q'}(M) \subset H_h^{p+p',q+q'}(M)$. By definition, the Weil operator $W_h \otimes \mathbb{C}$ acts on $H_h^{p,q}(M)$ as multiplication by $i(p-q)$; hence for all cohomology classes $\alpha \in H_h^{p,q}(X)$, $\beta \in H_h^{p',q'}(X)$, we have

$$W_h(\alpha\cdot\beta) = i(p+p'-q-q')\alpha\cdot\beta = W_h(\alpha)\cdot\beta + \alpha\cdot W_h(\beta).$$

Therefore W_h is a derivation of the algebra $H^\bullet(M,\mathbb{R})$. □

1.4. The LLV-Lie algebra of a hyper-Kähler manifold

In this section, we complete the proof of Theorem 1.2.12. As before, X is a $2n$-dimensional hyper-Kähler manifold, M is the underlying real manifold, and H denotes the vector space $H^2(X,\mathbb{Q})$ equipped with the Beauville–Bogomolov form q. To ease notation, we write $H_{\mathbb{R}}$, $H_{\mathbb{C}}$, instead of $H\otimes\mathbb{R}$, $H\otimes\mathbb{C}$.

1.4.1. — We will say that a 3-dimensional real vector space $P \subset H_{\mathbb{R}}$ is a *characteristic* 3*-space* if it is the positive 3-space P_g associated to a hyper-Kähler metric g on M. By Theorem 1.3.8, any characteristic space P_g determines a Lie algebra embedding

$$\mathfrak{so}(\tilde{P}_g) \cong \mathfrak{h}_g \subset \mathfrak{g}(X)\otimes\mathbb{R}.$$

Thanks to the following result, these embeddings generate $\mathfrak{g}(X)\otimes\mathbb{R}$.

1.4.2. Proposition. — *Let* $\mathrm{Gr}(3, H_{\mathbb{R}})$ *be the Grassmannian parametrizing* 3*-dimensional spaces in* $H_{\mathbb{R}}$*. Then the subset*

$$\{P \in \mathrm{Gr}(3, H_{\mathbb{R}}) \textit{ such that } P \textit{ is a characteristic space}\}$$

is open (for the Euclidean topology) in $\mathrm{Gr}(3, H_{\mathbb{R}})$*.*

Proof. — We consider the partial flag variety

$$Z = \{(F, P) \mid F \subset P\} \subset \mathrm{Gr}(2, H_{\mathbb{R}}) \times \mathrm{Gr}(3, H_{\mathbb{R}}).$$

We let $Z^c \subset Z$ consists of those (F, P) such that P is a characteristic 3-space. We will show that Z^c is open in Z; this yields the desired conclusion.

We denote by $\mathrm{Gr}_+(k, H_{\mathbb{R}})$ the Grassmannian of k-dimensional subspaces of $H_{\mathbb{R}}$ which are positive with respect to the Beauville–Bogomolov form. Then $\mathrm{Gr}_+(k, H_{\mathbb{R}})$ is an open subset of the full Grassmaniann $\mathrm{Gr}(k, H_{\mathbb{R}})$ and hence

$$Z_+ \coloneqq Z \cap (\mathrm{Gr}_+(2, H_{\mathbb{R}}) \times \mathrm{Gr}_+(3, H_{\mathbb{R}}))$$

is open in Z. Since any characteristic 3-space is positive, we have $Z^c \subset Z_+$.

By [**42**, Lemma 8.2], the period domain

$$D^o = \{x \in \mathbb{P}(H_{\mathbb{C}}) \mid q(x, x) = 0,\ q(x, \bar{x}) > 0\}$$

is diffeomorphic to the Grassmannian $\mathrm{Gr}_+^{\mathrm{or}}(2, H_{\mathbb{R}})$ parametrizing oriented positive planes in $H_{\mathbb{R}}$, by mapping $x \in D^o$ to the real oriented plane $\vec{F}_x = \langle \mathfrak{Re}(x), \mathfrak{Im}(x)\rangle$. We have a natural map $\mathrm{Gr}_+^{\mathrm{or}}(2, H_{\mathbb{R}}) \to \mathrm{Gr}_+(2, H_{\mathbb{R}})$, obtained forgetting the orientation of a plane. This map is an étale double cover; we let $\tilde{Z}_+$ (resp. $\tilde{Z}^c$) be the preimage of Z_+ (resp. Z^c) along the induced double cover

$$\mathrm{Gr}_+^{\mathrm{or}}(2, H_{\mathbb{R}}) \times \mathrm{Gr}_+(3, H_{\mathbb{R}}) \to \mathrm{Gr}_+(2, H_{\mathbb{R}}) \times \mathrm{Gr}_+(3, H_{\mathbb{R}}).$$

It is enough to show that $\tilde{Z}^c$ is open in $\tilde{Z}_+$. The manifold $\tilde{Z}_+$ is the total space of the projectivization of the vector bundle $\tilde{Q} \subset \mathrm{Gr}_+^{\mathrm{or}}(2, H_{\mathbb{R}}) \times H_{\mathbb{R}}$ defined by

$$\tilde{Q} = \{(\vec{F}, v) \mid v \in F^{\perp}\};$$

the vector bundle $\tilde{Q}$ is identified with the universal quotient bundle on $\mathrm{Gr}_+^{\mathrm{or}}(2, H_{\mathbb{R}})$.

Let now $P_0 \subset H_{\mathbb{R}}$ be a characteristic 3-space. Choose an oriented plane $\vec{F}_0 \subset P_0$. Then $\vec{F}_0$ represents the period of a hyper-Kähler manifold X_0 and $P_0 = \langle \vec{F}_0, v_0\rangle$ for some Kähler class v_0 on X_0. By the local Torelli theorem, we can identify an open neighborhood $\tilde{U} \subset \mathrm{Gr}_+^{\mathrm{or}}(2, H_{\mathbb{R}})$ of $\vec{F}_0$ with the universal local deformation space $\mathrm{Def}(X_0)$. Thus any $\vec{F} \in \tilde{U}$ is the period of a hyper-Kähler manifold $X_{\vec{F}}$, and the fibre $\tilde{Q}_{\vec{F}}$ of $\tilde{Q}$ at $\vec{F}$ is identified with $H^{1,1}(X_{\vec{F}}) \cap H^2(X_{\vec{F}}, \mathbb{R})$. Consider

$$\tilde{V} = \{(\vec{F}, v) \mid \vec{F} \in \tilde{U},\ v \in H_{\mathbb{R}} \text{ is a Kähler class on } X_{\vec{F}}\}.$$

Since a Kähler class is of Hodge type $(1,1)$, we have $\tilde{V} \subset \tilde{Q}_{|_{\tilde{U}}}$. Moreover, by the openness of the Kähler cone in $H^{1,1}(X_{\vec{F}}) \cap H^2(X_{\vec{F}}, \mathbb{R})$ (see [**89**, Chapter 9]), the subset $\tilde{V}$ is open in $\tilde{Q}$. For any $(\vec{F}, v) \in \tilde{V}$, the 3-space $\langle F, v\rangle$ is characteristic, and hence the image of $\tilde{V}$ in $\tilde{Z}_+$ gives an open neighborhood of $(\vec{F}, P)$ contained in $\tilde{Z}^c$. This shows that $\tilde{Z}^c$ is open in $\tilde{Z}_+$ as desired. □

1.4.3. Remark. — The description of the Kähler cone of a hyper-Kähler manifold by Amerik and Verbitsky yields a necessary and sufficient condition for a positive three space P to be characteristic, see [**1**, Theorem 4.9].

Proposition 1.4.2 has the following consequences. Let $\mathfrak{g}_{-2}(X)$ and $\mathfrak{g}_2(X)$ be the span of the Λ_x and the L_x respectively, for $x \in H^2(X, \mathbb{Q})$ with the Lefschetz property. Let $\mathfrak{g}_0(X) := [\mathfrak{g}_2(X), \mathfrak{g}_{-2}(X)]$. Clearly, L_x and L_y commute for all $x, y \in H^2(X, \mathbb{Q})$.

1.4.4. Proposition. — *(i) For any $x, y \in H^2(X, \mathbb{Q})$ with the Lefschetz property, we have $[\Lambda_x, \Lambda_y] = 0$.*

(ii) We have $\mathfrak{g}_0(X) = \mathbb{Q} \cdot \theta \oplus \mathfrak{g}_0'(X)$, where $\mathfrak{g}_0'(X) = [\mathfrak{g}_0(X), \mathfrak{g}_0(X)]$. The Lie subalgebra $\mathfrak{g}_0'(X)$ acts on $H^\bullet(X, \mathbb{Q})$ by derivations.

(iii) We have $\mathfrak{g}(X) = \mathfrak{g}_{-2}(X) \oplus \mathfrak{g}_0(X) \oplus \mathfrak{g}_2(X)$.

Proof. — (i). Let $V \subset H^2(X, \mathbb{R})$ be the Zariski open subset consisting of the classes with the Lefschetz property. The expression $[\Lambda_x, \Lambda_y]$ is a rational function on $V \times V$ (see §1.2.1), and hence $[\Lambda_x, \Lambda_y] = 0$ defines a Zariski closed subset $W \subset V \times V$. Thanks to Proposition 1.4.2 and its proof, the subset of classes $(a, b) \in H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R})$ such that there exists a characteristic 3-space P containing both a and b is open for the Euclidean topology. By Theorem 1.3.8 this open subset is contained in W, and it is not empty since by assumption there exists some characteristic 3-space. We conclude that W is a Zariski closed subset of $V \times V$ of the same dimension; since V is connected, having complement of real codimension ≥ 2, we must have $W = V \times V$.

(ii). Let $\mathfrak{v}$ be the $\mathbb{R}$-Lie algebra generated by all Weil operators coming from some hyper-Kähler structure on M. The Lie algebra $\mathfrak{v}$ acts on the cohomology by derivations thanks to Corollary 1.3.9. By Theorem 1.3.8 and Proposition 1.3.5 we obtain $\mathfrak{v} \subset \mathfrak{g}_0(X) \otimes \mathbb{R}$ and $\mathfrak{v} = [\mathfrak{v}, \mathfrak{v}]$; moreover, if a and b are both contained in a characteristic 3-space P, we have $[L_a, \Lambda_b] \in \mathbb{R} \cdot \theta \oplus \mathfrak{v}$. Via the same argument as above, Proposition 1.4.2 then implies that $\mathfrak{g}_0(X) \otimes \mathbb{R} = \mathbb{R} \cdot \theta \oplus \mathfrak{v}$. It follows that $\mathfrak{g}_0(X) = \mathbb{Q} \cdot \theta \oplus \mathfrak{g}_0'(X)$ and that $\mathfrak{v}$ is defined over $\mathbb{Q}$ and coincides with $\mathfrak{g}_0'(X)$.

(iii). By (i), $\mathfrak{g}_{-2}(X)$ and $\mathfrak{g}_2(X)$ are abelian Lie algebras; by construction, we have $[\mathfrak{g}_2(X), \mathfrak{g}_{-2}(X)] = \mathfrak{g}_0(X)$. Let $\mathrm{G}_0'(X)$ be the simply connected algebraic group with Lie algebra $\mathfrak{g}_0'(X)$. Via integration of the representation of $\mathfrak{g}_0'(X)$ on the cohomology of X we obtain a representation of $\mathrm{G}_0'(X)$ on $H^\bullet(X, \mathbb{Q})$, which is by graded algebra automorphisms by (ii). By Lemma 1.2.4.(i), the adjoint action of an element $g \in \mathrm{G}_0'(X)$ maps an $\mathfrak{sl}_2$-triple (L_x, θ, Λ_x) to the $\mathfrak{sl}_2$-triple $(L_{g(x)}, \theta, \Lambda_{g(x)})$. It follows that the adjoint action of $\mathfrak{g}_0'(X)$ preserves $\mathfrak{g}_i(X)$ for $i = -2, 0, 2$. □

Consider now the subalgebra $A_2^\bullet \subset H^\bullet(X, \mathbb{Q})$ generated by $H^2(X, \mathbb{Q})$.

1.4.5. Lemma. — *The subalgebra $A_2^\bullet$ is a faithful and irreducible $\mathfrak{g}(X)$-module.*

Proof. — The subalgebra $A_2^\bullet$ is clearly stable under $\mathfrak{g}_2(X)$ and θ. Since $\mathfrak{g}_0'(X)$ acts by derivations, the subalgebra $A_2^\bullet$ is stable under the action of $\mathfrak{g}_0'(X)$ as well. Given $x \in H^2(X, \mathbb{Q})$ with the Lefschetz property and $x_1, x_2, \ldots, x_k \in H^2(X, \mathbb{Q})$, we have

$$\Lambda_x(x_1 \cdot x_2 \cdot \ldots \cdot x_k) = -[L_{x_1}, \Lambda_x](x_2 \cdot \ldots \cdot x_k) + L_{x_1}\Lambda_x(x_2 \cdot \ldots \cdot x_k).$$

If $k = 1$, obviously $\Lambda_x(x_1)$ belongs to $A_2^\bullet$. For $k > 1$, the first term belongs to $\mathfrak{g}_0(X) \cdot A_2^\bullet \subset A_2^\bullet$, and the second term is in $A_2^\bullet$ by induction hypothesis.

Therefore $A_2^\bullet$ is a $\mathfrak{g}(X)$-module. We claim that the induced map $\tau \colon \mathfrak{g}(X) \to \mathfrak{gl}(A_2^\bullet)$ is injective. Clearly, θ is not in the kernel of τ, and, since $\sum_i a_i L_{x_i} = L_{\sum_i a_i x_i}$, the map τ is injective on $\mathfrak{g}_2(X)$. By Proposition 1.2.9 the Lie algebra $\mathfrak{g}(X)$ is semisimple; therefore, the Killing form $K(g_1, g_2) = \mathrm{Tr}(\mathrm{ad}(g_1) \circ \mathrm{ad}(g_2))$ is non-degenerate. It follows that the form K identifies $\mathfrak{g}_2(X)$ with the dual of $\mathfrak{g}_{-2}(X)$; in particular, $\dim(\mathfrak{g}_{-2}(X)) = \dim(\mathfrak{g}_2(X))$. Since the image of τ is semisimple as well, the same argument implies that $\tau(\mathfrak{g}_{-2}(X))$ and $\tau(\mathfrak{g}_2(X))$ have the same dimension; hence, $\ker(\tau) \subset \mathfrak{g}_0'(X)$. Since $\mathfrak{g}_0'(X)$ acts on the cohomology by derivations, Lemma 1.2.4.(ii) implies that the kernel of τ is central in the semisimple Lie algebra $\mathfrak{g}_0'(X)$. This forces $\ker(\tau) = 0$. Thus $A_2^\bullet$ is a faithful $\mathfrak{g}(X)$-module. It is irreducible since it is generated by $H^0(X, \mathbb{Q})$. □

1.4.6. — By Corollary 1.2.10, the algebra $A_2^\bullet \subset H^\bullet(X,\mathbb{Q})$ is a Frobenius algebra of level $2n$, and, by Lemma 1.4.5, the restriction map identifies the Lie algebra $\mathfrak{g}(X)$ with $\mathfrak{g}(A_2^\bullet)$ (see Definition 1.2.3). Since by definition $A_2^\bullet$ is generated by $H^2(X,\mathbb{Q})$, there is a surjective morphism of algebras $\Phi\colon \mathrm{Sym}^\bullet(H) \to A_2^\bullet$.

The following result is due to Verbitsky [**83**], who however obtained it as a consequence of Theorem 1.2.12; Bogomolov has shown in [**12**] that this result is a consequence of the local Torelli theorem. We will give his argument below.

1.4.7. Theorem. — *Let $I \subset \mathrm{Sym}^\bullet(H)$ be the ideal generated by*

$$\{w^{n+1} \in \mathrm{Sym}^{n+1}(H) \mid w \in H, \quad q(w,w) = 0\}.$$

Then $I = \ker \Phi$, so that we have an isomorphism of graded algebras $A_2^\bullet \cong \mathrm{Sym}^\bullet(H)/I$.

Proof. — Thanks to the existence of the Beauville–Bogomolov form q and Fujiki's relation, see Theorem 1.1.7, the locus of those $x \in H_\mathbb{C}$ such that $x^{2n} = 0$ is a non-singular quadric hypersurface $D \subset \mathbb{P}(H_\mathbb{C})$. By the local Torelli theorem, there exists an open subset U of D (with respect to the Euclidean topology) such that any $y \in U$ is the period of some hyper-Kähler manifold Y. Since $H^{2n+2,0}(Y) = 0$, we have $y^{n+1} = 0$ for all $y \in U$. Then the locus D' of those $w \in H^2(X,\mathbb{C})$ such that $w^{n+1} = 0$ is a Zariski closed subset of D of the same dimension, and thus $D = D'$.

This proves that we have a surjective morphism of algebras $\mathrm{Sym}^\bullet(H)/I \to A_2^\bullet$. Bogomolov shows [**12**, Lemma 2.5] that the component of $\mathrm{Sym}^\bullet(H)/I$ in degree $4n$ is one dimensional, and hence it maps isomorphically onto A_2^{4n}; moreover, he shows that $\mathrm{Sym}^\bullet(H)/I$ is a Frobenius algebra. Therefore any of its ideals must contain the component in degree $4n$, and the map $\mathrm{Sym}^\bullet(H)/I \to A_2^\bullet$ must be an isomorphism. □

Conclusion of the proof of Theorem 1.2.12. — We consider $\tilde{H}$ as a Frobenius algebra as in §1.2.5. Then $\mathrm{Sym}^n(\tilde{H})$ is again a Frobenius algebra. The Lie algebra $\mathfrak{g}(\mathrm{Sym}^n(\tilde{H}))$ is identified with $\mathfrak{so}(\tilde{H})$ and its natural action on $\mathrm{Sym}^n(\tilde{H})$.

According to [**51**, Proposition 2.14] the Frobenius algebra $\mathrm{Sym}^\bullet(H)/I$ is isomorphic to the subalgebra $B \subset \mathrm{Sym}^n(\tilde{H})$ generated by H. Hence B is stable under the action of $\mathfrak{so}(\tilde{H})$ on $\mathrm{Sym}^n(\tilde{H})$ and the Lie algebra $\mathfrak{g}(B)$ coincides with $\mathfrak{so}(\tilde{H})$. The desired isomorphism $\varphi\colon \mathfrak{g}(X) \to \mathfrak{so}(\tilde{H})$ is induced by the inverse of the isomorphism of graded algebras $B \xrightarrow{\sim} A_2^\bullet$. □

1.4.8. Remark. — By [**51**, Proposition 2.14] the $\mathfrak{so}(H)$-representation on $A_2^\bullet$ is

$$A_2^{2k} = \begin{cases} \mathrm{Sym}^k H, & \text{for } 0 \leq k \leq n, \\ \mathrm{Sym}^{2n-k} H, & \text{for } n < k \leq 2n. \end{cases}$$

1.5. The integrated representation

In this section we study the representation obtained via integration of the LLV-Lie algebra action on the cohomology of a hyper-Kähler manifold.

1.5.1. — Let X be a hyper-Kähler manifold of dimension $2n$. Let H denote the quadratic space $H^2(X, \mathbb{Q})$ equipped with the Beauville–Bogomolov form and let $\tilde{H}$ denote the orthogonal direct sum of H with $U = \mathbb{Q} \cdot v \oplus \mathbb{Q} \cdot w$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$.

We consider the LLV-Lie algebra $\mathfrak{g}(X)$ of X and denote by $\mathrm{G}(X)$ the semisimple simply connected algebraic group with Lie algebra $\mathfrak{g}(X)$. We also let $\mathrm{G}_0(X) \subset \mathrm{G}(X)$ be the unique connected subgroup with Lie algebra $\mathfrak{g}_0(X)$. Theorem 1.2.12 yields an isomorphism

$$\tilde{\varphi} \colon \mathrm{G}(X) \xrightarrow{\sim} \mathrm{Spin}(\tilde{H}).$$

Since $\tilde{H} = H \oplus U$, we can view $\mathrm{Spin}(H)$ and $\mathrm{Spin}(U)$ as subgroups of $\mathrm{Spin}(\tilde{H})$.

1.5.2. Proposition. — *(i) We have* $\mathrm{G}_0(X) \cong \mathrm{CSpin}(H)$.

(ii) Via the isomorphism $\tilde{\varphi}$*, we have*

$$\tilde{\varphi} \colon \mathrm{G}_0(X) \xrightarrow{\sim} \mathrm{Spin}(H) \cdot \mathrm{Spin}(U) \ \subset \mathrm{Spin}(\tilde{H}).$$

Proof. — We will show that we have $\mathrm{Spin}(U) \cong \mathbb{G}_m$ and that the Lie algebra of $\mathrm{Spin}(U) \subset \mathrm{Spin}(\tilde{H})$ is $\mathbb{Q} \cdot \varphi(\theta)$. Since $\mathfrak{so}(\tilde{H}) = \mathbb{Q} \cdot \varphi(\theta) \oplus \mathfrak{so}(H)$, this will imply (ii). To show (i), we will show that $\mathrm{Spin}(H)$ and $\mathrm{Spin}(U)$ commute, and they intersect in $\mu_2 = \{\pm 1\}$. This will be sufficient, since $\mathrm{CSpin}(H) \cong \mathrm{Spin}(H) \times_{\mu_2} \mathbb{G}_m$.

With respect to the basis $\{v, -\frac{w}{2}\}$, the matrix of $\tilde{q}_{|U}$ is $\left(\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right)$. Let $\mathrm{Cl}(U)$ be the Clifford algebra on U. Then $\mathrm{Cl}(U)$ is identified with the algebra of 2 by 2 matrices with coefficients in $\mathbb{Q}$; an isomorphism is given by

$$v \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad -\frac{w}{2} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The even Clifford algebra $\mathrm{Cl}^+(U)$ consists of the diagonal matrices, while $\mathrm{Cl}^-(U)$ consists of those matrices with 0 on the diagonal. The spinor norm $\mathrm{Cl}(U)^\times \to \mathbb{Q}^\times$ is the determinant. Therefore $\mathrm{Spin}(U) \cong \mathbb{G}_m$ is the standard maximal torus of SL_2. The adjoint action of $\mathrm{Spin}(U)$ on $\tilde{H}$ is trivial on the summand H, and we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} v \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \lambda^{-2} v, \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} w \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \lambda^{2} w.$$

This implies that the Lie algebra of $\mathrm{Spin}(U) \subset \mathrm{Spin}(\tilde{H})$ is $\mathbb{Q}\cdot\theta \subset \mathfrak{so}(\tilde{H})$. Finally, since the Clifford algebra satisfies $\mathrm{Cl}(\tilde{H}) = \mathrm{Cl}(H) \otimes \mathrm{Cl}(U)$, we have $\mathrm{Spin}(H) \cap \mathrm{Spin}(U) = \mu_2$ and $\mathrm{Spin}(H)$ commutes with $\mathrm{Spin}(U)$. $\square$

1.5.3. — The action of $\mathfrak{g}(X)$ on $H^\bullet(X, \mathbb{Q})$ integrates to a representation ρ of $\mathrm{G}(X)$ on $H^\bullet(X, \mathbb{Q})$, which restricts to

$$\rho_0 \colon \mathrm{G}_0(X) \to \prod_j \mathrm{GL}(H^j(X, \mathbb{Q})).$$

We denote by $\rho_0^{(2)} \colon \mathrm{G}_0(X) \to \mathrm{GL}(H^2(X, \mathbb{Q}))$ its degree 2 component.

In what follows, we identify $\mathrm{G}(X)$ with $\mathrm{Spin}(\tilde{H})$ and $\mathrm{G}_0(X)$ with $\mathrm{CSpin}(H)$ via $\tilde{\varphi}$.

1.5.4. Remark. — By Remark 1.3.6, via the representation ρ the element $-1 \in \mathrm{CSpin}(H) \subset \mathrm{Spin}(\tilde{H})$ acts on $H^j(X, \mathbb{Q})$ as multiplication by $(-1)^j$. Combining this with Theorem 1.2.12 we deduce that the representation ρ is faithful if X has non-trivial cohomology in some odd degree, and that ρ has kernel $\mu_2 = \{\pm 1\}$ otherwise.

The connected center of the algebraic group $\mathrm{CSpin}(H)$ is the subgroup $\mathbb{G}_m$ of invertible scalars in the Clifford algebra, and we have short exact sequences of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{CSpin}(H) \xrightarrow{\pi} \mathrm{SO}(H) \longrightarrow 1$$

and

$$1 \longrightarrow \mathrm{Spin}(H) \longrightarrow \mathrm{CSpin}(H) \xrightarrow{\mathrm{Nm}} \mathbb{G}_m \longrightarrow 1$$

such that for all $z \in \mathbb{G}_m \subset \mathrm{CSpin}(H)$ we have $\mathrm{Nm}(z) = z^2$.

In addition to the representation ρ_0, we will consider a second, twisted, action of $\mathrm{CSpin}(H)$ on the cohomology of X.

1.5.5. Definition. — *The twisted LLV-representation R of* G_0 *on* $H^\bullet(X,\mathbb{Q})$ *is defined by the homomorphism*

$$R\colon\ \mathrm{CSpin}(H) \to \prod_j \mathrm{GL}(H^j(X,\mathbb{Q})), \quad R(g) = \mathrm{Nm}(g)^n \cdot \rho_0(g).$$

1.5.6. Remark. — The image of the differential of R is $\mathbb{Q}\cdot\theta' \oplus \mathfrak{so}(H) \subset \mathfrak{gl}(H^\bullet(X))$, where $\mathfrak{so}(H)$ is the semisimple part of $\mathfrak{g}_0(X)$ and θ' is multiplication by j on $H^j(X)$.

1.5.7. Lemma. — *Via the representation R, the action of* $\mathrm{CSpin}(H)$ *on* $H^\bullet(X,\mathbb{Q})$ *is an action by graded algebra automorphisms.*

Proof. — By Theorem 1.2.12, the semisimple part $\mathfrak{so}(H)$ of the Lie algebra of $\mathrm{CSpin}(H)$ acts on the cohomology algebra via derivations; therefore, the subgroup $\mathrm{Spin}(H) \subset \mathrm{CSpin}(H)$ acts on $H^\bullet(X,\mathbb{Q})$ by graded algebra automorphisms. Moreover for any $z \in \mathbb{G}_m$ and $y \in H^j(X,\mathbb{Q})$ we have $\rho_0(z)(y) = z^{j-2n}\cdot y$. Thus the factor $\mathrm{Nm}(z)^n = z^{2n}$ ensures that $\mathbb{G}_m \subset \mathrm{CSpin}(H)$ acts on $H^\bullet(X,\mathbb{Q})$ by algebra automorphisms as well. As $\mathrm{CSpin}(H) = \mathbb{G}_m \cdot \mathrm{Spin}(H)$, this concludes the proof. □

1.5.8. Remark. — The homomorphism

$$(\mathrm{Nm},\pi)\colon\ \mathrm{CSpin}(H) \to \mathbb{G}_m \times \mathrm{SO}(H)$$

is surjective with kernel μ_2. By Remark 1.5.4, the R-action on the even cohomology factors through (Nm,π). If $g \in \mathrm{CSpin}(H)$, then the degree 2 component $R^{(2)}(g)$ of $R(g)$ equals $\mathrm{Nm}(g)\cdot\pi(g)$, while for $\rho_0(g)$ we have $\rho_0^{(2)}(g) = \mathrm{Nm}(g)^{1-n}\cdot\pi(g)$.

The combination of this observation with Theorem 1.2.12 implies that the natural homomorphism $R(\mathrm{G}_0(X)) \to R^{(2)}(\mathrm{G}_0(X))$ is an isomorphism if the odd cohomology of X vanishes, and it has kernel μ_2 otherwise.

CHAPTER 2

THE MUMFORD–TATE CONJECTURE, MOTIVES AND FAMILIES

2.1. The Mumford–Tate conjecture

In this section we fix our notation for Hodge structures and Galois representations and give the statement of the Mumford–Tate conjecture. We refer to [**62**] for a detailed treatment of the subject. Throughout, $K \subset \mathbb{C}$ is a field which is finitely generated over $\mathbb{Q}$, with algebraic closure $\bar{K} \subset \mathbb{C}$, and ℓ is a fixed prime number.

2.1.1. — We denote by HS (resp. $\mathsf{HS}^{\mathrm{pol}}$) the category of $\mathbb{Q}$-Hodge structures (resp. polarizable $\mathbb{Q}$-Hodge structures). The category HS is an abelian Tannakian category, with fibre functor the forgetful functor to the category of $\mathbb{Q}$-vector spaces. This means that to any Tannakian subcategory $\mathsf{C} \subset \mathsf{HS}$ is attached a pro-algebraic group $\mathrm{MT}(\mathsf{C})$ whose category of $\mathbb{Q}$-representations is equivalent to C. It is defined as the group of tensor automorphisms of the fibre functor $f|_{\mathsf{C}}$. There is a notion of weights in HS, where a Hodge structure V is said to be pure of weight k if only terms with $p+q=k$ appear in the decomposition $V \otimes \mathbb{C} = \bigoplus_{p,q} V^{p,q}$. Given a Tannakian subcategory $\mathsf{C} \subset \mathsf{HS}$, the weights are given by a cocharacter $w\colon \mathbb{G}_m \to \mathrm{MT}(\mathsf{C})$.

Given $V \in \mathsf{HS}$, we let $\langle V \rangle \subset \mathsf{HS}$ be the Tannakian subcategory generated by V. The *Mumford–Tate* group $\mathrm{MT}(V)$ of V is by definition the group attached to $\langle V \rangle$ via the above procedure. It is an algebraic subgroup of $\mathrm{GL}(V)$; if V is polarizable, then the category $\langle V \rangle$ is semisimple, and hence $\mathrm{MT}(V)$ is reductive in this case.

2.1.2. — An alternative characterization of $\mathrm{MT}(V)$ is as follows. Let $\mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m)$ be the Deligne torus, that is, $\mathbb{C}^{\times}$ viewed as a real algebraic group. The Hodge structure on the vector space V is determined by a representation $h\colon \mathbb{S} \to \mathrm{GL}(V) \otimes \mathbb{R}$;

the Hodge decomposition is recovered as the isotypical decomposition for $h \otimes \mathbb{C}$. Then $\mathrm{MT}(V)$ is the smallest $\mathbb{Q}$-algebraic subgroup G of $\mathrm{GL}(V)$ with the property that h factors through $G \otimes \mathbb{R}$. Thus $\mathrm{MT}(V)$ is connected, since $\mathbb{S}$ is so.

2.1.3. — We denote by $\mathsf{Rep}_\ell(\mathrm{Gal}(\bar{K}/K))$ the category of continuous representations of the absolute Galois group of K on $\mathbb{Q}_\ell$-vector spaces. This $\mathbb{Q}_\ell$-linear category is also Tannakian: a natural fibre functor is given by the forgetful functor to $\mathbb{Q}_\ell$-vector spaces. Via the Tannakian formalism, to any ℓ-adic Galois representation W is associated an algebraic group $\mathcal{G}(W)$ over $\mathbb{Q}_\ell$, whose category of $\mathbb{Q}_\ell$-representations is equivalent to the Tannakian subcategory $\langle W \rangle \subset \mathsf{Rep}_\ell(\mathrm{Gal}(\bar{K}/K))$ generated by W. There is a notion of weights in $\mathsf{Rep}_\ell(\mathrm{Gal}(\bar{K}/K))$; we will not give details about this and refer to Deligne's paper [**24**].

More concretely, the group $\mathcal{G}(W) \subset \mathrm{GL}(W)$ is the Zariski closure of the image of the representation $\sigma \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(W)$. If K'/K is a finite field extension then we can see W as a $\mathrm{Gal}(\bar{K}/K')$ representation W'. Then $\mathcal{G}(W')$ is a subgroup of finite index of $\mathcal{G}(W)$. There exists a finite field extension $\hat{K}$ of K such that for the induced $\mathrm{Gal}(\bar{K}/\hat{K})$-module $\hat{W}$, the algebraic group $\mathcal{G}(\hat{W})$ is connected. In fact, consider the short exact sequence

$$1 \to \mathcal{G}(W)^0 \to \mathcal{G}(W) \to \Gamma \to 1,$$

where Γ is the group of connected components of $\mathcal{G}(W)$; then $\hat{K}/K$ is the field extension corresponding to the kernel of the composition $\mathrm{Gal}(\bar{K}/K) \to \mathcal{G}(W) \to \Gamma$.

2.1.4. — In both HS and $\mathsf{Rep}_\ell(\mathrm{Gal}(\bar{K}/K))$ we have Tate twists at our disposal. The Hodge structure $\mathbb{Q}(1)$ is the one-dimensional vector space $(2\pi i) \cdot \mathbb{Q} \subset \mathbb{C}$ with Hodge structure purely of type $(-1,-1)$. If n is a positive integer we define $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ and $\mathbb{Q}(-n) = \mathbb{Q}(1)^{\vee, \otimes n}$; given $V \in \mathsf{HS}$ we let $V(n)$ be the Hodge structure $V \otimes \mathbb{Q}(n)$.

In the ℓ-adic setting, we instead define the $\mathrm{Gal}(\bar{K}/K)$-module $\mathbb{Q}_\ell(1)$ as the one-dimensional representation corresponding to the ℓ-adic cyclotomic character $\xi_\ell \colon \mathrm{Gal}(\bar{K}/K) \to \mathbb{Z}_\ell^\times$ of K. Given a Galois representation $\sigma \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(W)$, we denote by $W(n)$ the twisted representation $\xi_\ell^n \cdot \sigma$.

2.1.5. — Let X be a smooth and projective variety over K.

– We denote by $H^i(X)$ the i-th rational singular (Betti) cohomology group $H^i(X(\mathbb{C}), \mathbb{Q})$ of the complex manifold $X(\mathbb{C})$; it carries a polarizable Hodge structure. We define $H^\bullet(X) \coloneqq \bigoplus_i H^i(X)$.
– We denote by $H^i_\ell(X)$ the i-th ℓ-adic étale cohomology group $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ of $X_{\bar{K}}$, which comes with an action of $\mathrm{Gal}(\bar{K}/K)$. We let $H^\bullet_\ell(X) \coloneqq \bigoplus_i H^i_\ell(X)$.

Artin proved [**7**, Exposé XI] that for any smooth and projective variety X over K and any integer i there is a canonical isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$\gamma_i \colon H^i(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^i_\ell(X).$$

We let $\gamma\colon H^\bullet \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^\bullet_\ell(X)$ be the isomorphism $\oplus_i \gamma_i$. The *Mumford–Tate conjecture* aims to compare the extra structure that we have on the two sides.

2.1.6. Conjecture. — *The isomorphism* $\gamma_*\colon \mathrm{GL}(H^\bullet(X)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{GL}(H^\bullet_\ell(X))$ *induced by Artin's isomorphism* γ *restricts to an isomorphism of algebraic groups*

$$\mathrm{MT}(H^\bullet(X)) \otimes \mathbb{Q}_\ell \cong \mathcal{G}(H^\bullet_\ell(X))^0.$$

The Mumford–Tate conjecture in degree i for X is the statement that the comparison isomorphism γ_i identifies $\mathrm{MT}(H^i(X)) \otimes \mathbb{Q}_\ell$ with $\mathcal{G}(H^i_\ell(X))^0$. We note that Conjecture 2.1.6 is equivalent to the Mumford–Tate conjecture in all degrees for X and all of its powers X^m.

The Mumford–Tate conjecture is a very difficult open problem - see [**62**] for a survey of known results. If $K \subset L \subset \mathbb{C}$ is a finitely generated field extension, the Mumford–Tate conjecture for X_L and X are equivalent, and this allows to formulate the conjecture for complex algebraic varieties, see [**63**, §1].

2.1.7. — Let X be a smooth and projective variety over K. A *Hodge class* on X is a rational cohomology class $\xi \in H^{2p}(X)(p)$ such that $\xi \otimes \mathbb{C}$ is of Hodge type $(0,0)$; equivalently, the action of $\mathrm{MT}(H^\bullet(X))$ on $H^{2p}(X)(p)$ fixes ξ. Similarly, a *Tate class* is a cohomology class $\xi \in H^{2p}_\ell(X)(p)$ which is fixed by $\mathcal{G}(H^\bullet_\ell(X))^0$.

Let $\mathrm{CH}^\bullet(X)$ be the rational Chow group of X. We have cycle class maps

$$\mathrm{cl}\colon \mathrm{CH}^\bullet(X_{\bar{K}}) \to H^{2\bullet}(X),$$
$$\mathrm{cl}_\ell\colon \mathrm{CH}^\bullet(X_{\bar{K}}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^{2\bullet}_\ell(X),$$

which are compatible with Artin's isomorphism γ. The image of cl (resp. of cl_ℓ) consists of Hodge (resp. Tate) classes. The *Hodge conjecture* predicts that any Hodge class lies in the image of cl, while the *Tate conjecture* predicts that any Tate class lies in the image of cl_ℓ. The Mumford–Tate conjecture would give a deep link between these two conjectures: if two out of the three statements

- the Mumford–Tate conjecture 2.1.6 for X,
- the Hodge conjecture for X and all of its powers X^k,
- the Tate conjecture for X and all of its powers X^k,

are true, then so is the third, see [**62**, §2.3].

2.2. Motives

A strong motivation for the Mumford–Tate conjecture comes from the theory of motives. The idea of motives goes back to Grothendieck. We briefly recall some categories of motives.

2.2.1. Chow motives. — Let K be a field, and let SmProj_K be the category of smooth projective varieties over K. Given a smooth and projective variety X over K we denote by $\mathrm{CH}^\bullet(X)$ its Chow group with rational coefficients. Given smooth and projective varieties X and Y over K of dimension d_X and d_Y respectively, a *correspondence* of degree k from X to Y is an element γ of $\mathrm{CH}^{d_X+k}(X \times Y)$. Then γ induces a map $\mathrm{CH}^\bullet(X) \to \mathrm{CH}^{\bullet+k}(Y)$ by the formula

$$\gamma_*(\beta) = \mathrm{pr}_{Y,*}(\gamma \cdot \mathrm{pr}_X^*(\beta)),$$

where $\mathrm{pr}_X \colon X \times Y \to X$ and $\mathrm{pr}_Y \colon X \times Y \to Y$ denote the projections.

The category CHM_K of Chow motives (with rational coefficients) over K is defined as follows:

- the objects of CHM_K are triples (X, p, n) such that $X \in \mathsf{SmProj}_K$, $p \in \mathrm{CH}^{d_X}(X \times X)$ is an idempotent correspondence (i.e. $p_* \circ p_* = p_*$) and n is an integer;
- the morphisms in CHM_K from (X, p, n) to (Y, q, m) are the correspondences $\gamma \in \mathrm{CH}^{d_X+m-n}(X \times Y)$ which satisfy $f_* \circ p_* = q_* \circ f_* = f_*$.

There is a natural functor

$$\mathfrak{h} \colon \mathrm{SmProj}_K^{op} \to \mathsf{CHM}_K;$$

the motive of a smooth and projective variety X over K is $h(X) = (X, \Delta_X, 0)$, where $\Delta_X \in \mathrm{CH}^{d_X}(X \times X)$ denotes the class of the diagonal.

The category CHM_K is a pseudo-abelian rigid symmetric tensor category, see [**5**]. The tensor product of two motives is defined via the fibre product over $\mathrm{Spec}(K)$; the unit motive is $\mathbb{1} = (\mathrm{Spec}(K), \Delta_{\mathrm{Spec}(K)}, 0)$. The Tate motive of weight $-2i$ is by definition the motive $\mathbb{1}(i) = (\mathrm{Spec}(K), \Delta_{\mathrm{Spec}(K)}, i)$. Given $\mathcal{M} \in \mathsf{CHM}_K$, we will use the notation $\mathcal{M}(i) = \mathcal{M} \otimes \mathbb{1}(i)$.

The pseudo-abelian tensor subcategory of CHM_K generated by a Chow motive $\mathcal{M}$ is the smallest full subcategory of CHM_K containing $\mathcal{M}$ that is stable under isomorphisms, direct sums, direct summands, tensor products and duality. We denote this subcategory by $\langle \mathcal{M} \rangle \subset \mathsf{CHM}_K$.

2.2.2. Grothendieck motives. — Consider now a field $K \subset \mathbb{C}$. Given varieties X and $Y \in \mathsf{SmProj}_K$, of dimension d_X and d_Y respectively, a *homological correpondence* of degree k from X to Y is an *algebraic* cohomology class $\alpha \in H^{2d_X+2k}(X \times Y)$, where $\alpha \in H^{2\bullet}(X)$ is called algebraic if it lies in the image of the cycle class map cl. The class $\alpha \in H^{2d_X+2k}(X \times Y)$ induces a linear map $\alpha_* \colon H^\bullet(X) \to H^{\bullet+2k}(Y)$ by

$$\alpha_*(\beta) = \mathrm{pr}_{Y,*}(\alpha \cdot \mathrm{pr}_X^*(\beta)),$$

where $\mathrm{pr}_X \colon X \times Y \to X$ and $\mathrm{pr}_Y \colon X \times Y \to Y$ denote the projections.

Replacing in the construction of Chow motives “correspondence” with “homological correspondence”, we obtain the category GRM_K of Grothendieck motives over K. This category is expected to be a semisimple abelian neutral Tannakian category; this led Grothendieck to formulate his *standard conjectures* [**39**], which are a special case of the Hodge conjecture and would ensure that GRM_K has enough morphisms. It is however very hard to construct algebraic cycles on varieties, and the good properties of GRM_K remain conditional to the validity of the standard conjectures.

2.2.3. André motives. — An unconditional theory of homological motives was later proposed by André [**4**], refining Deligne’s idea of absolute Hodge classes [**25**]. André introduces the notion of *motivated* correspondences, which roughly speaking are those induced by cohomology classes which can be constructed from algebraic cycles and the Hodge $\star$-operator; if the standard conjectures were true, then any motivated class would be algebraic.

Replacing algebraic classes with motivated ones in Grothendieck's construction we obtain the category of André motives over K, denoted by AM_K; this is a $\mathbb{Q}$-linear semisimple abelian Tannakian category. The motive of $X \in \mathsf{SmProj}_K$ is $(X, \Delta_X, 0)$, where $\Delta_X \in H^{2d_X}(X \times X)$ is the class of the diagonal. The Künneth projectors are given by motivated cycles; we will therefore write $\mathcal{H}^\bullet(X) = \bigoplus_i \mathcal{H}^i(X)$ to denote the motive of X. The tensor product of two motives is defined in the obvious way via the fibre product over $\mathrm{Spec}(K)$; the unit motive is $\mathsf{1} = (\mathrm{Spec}(K), \Delta_{\mathrm{Spec}(K)}, 0)$. The *Tate motive of weight* $-2i$ is by definition $\mathsf{1}(i) = (\mathrm{Spec}(K), \Delta_{\mathrm{Spec}\, K}, i)$; given $\mathcal{M} \in \mathsf{AM}_K$, we will use the notation $\mathcal{M}(i) = \mathcal{M} \otimes \mathsf{1}(i)$.

2.2.4. — The virtue of AM_K is that it works well with the Tannakian formalism. We have natural functors

$$\mathsf{SmProj}_K^{\mathrm{op}} \xrightarrow{\mathcal{H}^\bullet} \mathsf{AM}_K \xrightarrow{r} \mathsf{HS}^{\mathrm{pol}}$$

whose composition is $H^\bullet(X)$. The functor r is called the *realization functor*; to a motive $\mathcal{M} = (X, p, n)$ is attached the Hodge structure $p_*(H^\bullet(X))(n)$. The functor r is conservative, which means that a morphism of motives is an isomorphism if and only if its realization is so.

The composition of r with the forgetful functor to $\mathbb{Q}$-vector spaces gives a fibre functor on AM_K. Via the Tannakian formalism, to any motive $\mathcal{M} \in \mathsf{AM}_K$ is therefore attached an algebraic group

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{M}) \subset \mathrm{GL}(r(\mathcal{M})),$$

whose category of finite dimensional $\mathbb{Q}$-representations is equivalent to the Tannakian subcategory $\langle \mathcal{M} \rangle \subset \mathsf{AM}_K$ generated by $\mathcal{M}$. The group $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$ is called the *motivic Galois group* of $\mathcal{M}$, and the $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$-invariants in any tensor construction of $r(\mathcal{M})$ are precisely the motivated classes. Since AM_K is semisimple, $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$ is reductive.

2.2.5. — Assume that $K \subset K' \subset \mathbb{C}$ is a field extension. The base-change functor $\mathsf{SmProj}_K \to \mathsf{SmProj}_{K'}$ yields a functor $\mathsf{AM}_K \to \mathsf{AM}_{K'}$. Given $\mathcal{M} \in \mathsf{AM}_K$, we denote by $\mathcal{M}_{K'} \in \mathsf{AM}_{K'}$ the motive obtained from $\mathcal{M}$ via base-change. The motivic Galois group of $\mathcal{M}_{K'}$ is then a subgroup of finite index of $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$; moreover, there exists a finite field extension $\hat{K}/K$ such that for any extension K' as above containing $\hat{K}$ we have $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{K'}) \xrightarrow{\sim} \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\hat{K}})$. See [**62**, §3].

2.2.6. — For any $\mathcal{M} \in \mathsf{AM}_K$, we have a canonical inclusion $\mathrm{MT}(r(M)) \subset \mathrm{G}_{\mathrm{mot}}(M)$. This follows from the fact that all motivated cycles are Hodge classes: this means that for any tensor construction $T = (r(\mathcal{M}))^{\otimes m} \otimes (r(\mathcal{M}))^{\vee, \otimes n}(k)$, the invariants of the motivic Galois group are also invariants for the Mumford–Tate group. Since both groups are reductive, the claimed inclusion follows from [**25**, Proposition 3.1].

Over an algebraically closed field the converse should hold as well, as predicted by the Hodge conjecture.

2.2.7. Conjecture. — *Let $K \subset \mathbb{C}$ be an algebraically closed field. For any motive $\mathcal{M} \in \mathsf{AM}_K$, we have an equality in* $\mathrm{GL}(r(\mathcal{M}))$*:*

$$\mathrm{MT}(r(\mathcal{M})) = \mathrm{G}_{\mathrm{mot}}(\mathcal{M}).$$

Since Mumford–Tate groups are connected, also $\mathrm{G}_{\mathrm{mot}}(\mathcal{M})$ should be connected when the base field K is algebraically closed; this is not known in general.

Conjecture 2.2.7 predicts that all Hodge classes should be motivated. The most significant evidence towards this statement is André's result in [**4**] that Hodge classes are motivated on abelian varieties, which strengthens the previous result of Deligne [**25**] on absolute Hodge classes. To state this result, let $\mathsf{AM}_K^{\mathrm{ab}} \subset \mathsf{AM}_K$ be the Tannakian subcategory generated by the motives of all abelian varieties over K. A motive $\mathcal{M} \in \mathsf{AM}_K^{\mathrm{ab}}$ is callled an *abelian motive.*

2.2.8. Theorem **(**[**4**, Théorème 0.6.2]**).** — *Let $K \subset \mathbb{C}$ be an algebraically closed field. Conjecture 2.2.7 holds for all abelian motives $\mathcal{M} \in \mathsf{AM}_K^{\mathrm{ab}}$.*

2.2.9. — Let now $K \subset \mathbb{C}$ be a field finitely generated over $\mathbb{Q}$, and $\bar{K} \subset \mathbb{C}$ an algebraic closure of K. Fix a prime number ℓ. We then have a second realization functor

$$r_\ell \colon \mathsf{AM}_K \to \mathsf{Rep}_\ell(\mathrm{Gal}(\bar{K}/K)),$$

such that the composition $r_\ell \circ \mathcal{H}^\bullet$ is the functor $H_\ell^\bullet \colon \mathsf{SmProj}_K \to \mathsf{Rep}_\ell(\mathrm{Gal}(\bar{K}/K))$. Composing r_ℓ with the forgetful functor to $\mathbb{Q}_\ell$-vector spaces yields again a fibre functor on AM_K. Via the Tannakian formalism this determines for any motive $\mathcal{M}$ a $\mathbb{Q}_\ell$-algebraic group $\mathrm{G}_{\mathrm{mot},\ell}(\mathcal{M}) \subset \mathrm{GL}(r_\ell(\mathcal{M}))$.

Artin's comparison theorem gives a canonical isomorphism of vector spaces

$$\gamma \colon r(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} r_\ell(\mathcal{M}).$$

The induced isomorphism $\gamma_*\colon \mathrm{GL}(r(\mathcal{M})) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{GL}(r_\ell(\mathcal{M}))$ identifies the algebraic group $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}) \otimes \mathbb{Q}_\ell$ with $\mathrm{G}_{\mathrm{mot},\ell}(\mathcal{M})$.

Let K, $\bar{K}$ and ℓ be as above. The following statement is the *motivic Mumford–Tate conjecture.*

2.2.10. Conjecture. — *For any motive $\mathcal{M} \in \mathsf{AM}_K$, the comparison isomorphism γ induces isomorphisms of algebraic groups:*

$$\mathrm{MT}(r(\mathcal{M})) \otimes \mathbb{Q}_\ell \cong \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{K}}) \otimes \mathbb{Q}_\ell \cong \mathcal{G}(r_\ell(\mathcal{M}))^0.$$

We can see this conjecture as the conjunction of the statement that Hodge classes are motivated, that is, Conjecture 2.2.7 for $\mathcal{M}_{\bar{K}}$, with the statement that Tate classes are motivated, that is $\mathcal{G}(r_\ell(\mathcal{M}))^0 \cong \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{K}}) \otimes \mathbb{Q}_\ell$. Note that, by §2.2.5, we have $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\mathbb{C}}) \cong \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_{\bar{K}})$.

2.3. Relative André motives and monodromy

Another remarkable aspect of André motives is their behaviour in families. The results presented below are due to André [**4**] (based on Deligne [**23**]), formalized by Moonen [**62**, §4]. We work over the complex numbers.

2.3.1. — The starting point of the discussion is the deformation principle for motivated cycles due to André [**4**, Théorème 0.5].

2.3.2. Theorem. — *Let S be a connected and reduced complex variety and let $f\colon \mathfrak{X} \to S$ be a smooth and projective morphism. Let*

$$\xi \in H^0(S, R^{2i} f_* \mathbb{Q}(i)).$$

Assume that there exists $s_0 \in S$ such that the restriction $\xi_{s_0} \in H^{2i}(\mathfrak{X}_{s_0}, \mathbb{Q})(i)$ of ξ to $\mathfrak{X}_{s_0}$ is motivated. Then, for all $s \in S$, the class $\xi_s \in H^{2i}(\mathfrak{X}_s, \mathbb{Q})(i)$ is motivated.

The proof of the deformation principle is based on Deligne's theorem of the fixed part [**23**, Théorème 4.1.1]. Moonen introduced the following notion of families of motives, [**62**, Definition 4.3.3].

2.3.3. Definition. — *Let S be a non-singular and connected complex variety. An André motive over S is a triple $(\mathfrak{X}/S, p, n)$ where*

– *$f\colon \mathfrak{X} \to S$ is a smooth and projective morphism with connected fibers,*

– *p is a global section of $R^{2d}(f \times f)_*\mathbb{Q}_{\mathfrak{X}\times_S\mathfrak{X}}(d)$, where d is the relative dimension of f,*
– *n is an integer,*

such that for some $s \in S$ (equivalently, by Theorem 2.3.2, for any $s \in S$), the value $p(s) \in H^{2d}(\mathfrak{X}_s \times \mathfrak{X}_s)(d)$ is a motivated idempotent correspondence.

These objects, with morphisms defined via the usual formalism of correspondences, form a semisimple neutral Tannakian abelian category which we denote by AM_S. Obviously, the category of André motives over $\mathrm{Spec}(\mathbb{C})$ is nothing but $\mathsf{AM}_{\mathbb{C}}$.

2.3.4. — With S as above, we denote by $\mathsf{VHS}_S^{\mathrm{pol}}$ the category of polarized variations of $\mathbb{Q}$-Hodge structures over S, [**37**]. There is a natural realization functor:

$$r\colon\ \mathsf{AM}_S \xrightarrow{r} \mathsf{VHS}_S^{\mathrm{pol}}.$$

We call a variation $V/S \in \mathsf{VHS}_S^{\mathrm{pol}}$ *algebraic* if, possibly after restriction to some non-empty Zariski open subset U of S, it is a direct summand of a variation of the form $R^i f_*\mathbb{Q}(j)$ for some smooth projective morphism $f\colon \mathfrak{X} \to S$ and some integer j, cf. [**23**, Definition 4.2.4]. The image of the realization functor r is contained in the subcategory $\mathsf{VHS}_S^{\mathrm{a}}$ of algebraic variations of Hodge structures on S; this is a Tannakian and semisimple category, see [**23**, §4].

By construction, for any smooth projective morphism $f\colon \mathfrak{X} \to S$ with connected fibres and any integer i, we have a relative André motive $\mathcal{H}^i(\mathfrak{X}/S)$ over S whose realization is the variation $R^i f_*\mathbb{Q} \in \mathsf{VHS}_S^{\mathrm{a}}$.

2.3.5. — If $\mathcal{M}/S \in \mathsf{AM}_S$, we denote by $\mathcal{M}_s$ its fibre at a point $s \in S$; similarly, if $V/S \in \mathsf{VHS}_S^{\mathrm{pol}}$, the corresponding Hodge structure at the point s is denoted by V_s. We aim to study the families of motivic Galois groups $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_s)$ and Mumford–Tate groups $\mathrm{MT}(V_s)$ when s varies in S.

Given a variation $V/S \in \mathsf{VHS}_S^{\mathrm{pol}}$, we consider the monodromy representation $\pi_1(S,s) \to \mathrm{GL}(V_s)$ associated to the underlying local system. The *algebraic monodromy group* $\mathrm{G}_{\mathrm{mono}}(V/S)_s$ of V/S at $s \in S$ is by definition the Zariski closure in $\mathrm{GL}(V_s)$ of the image of the monodromy representation above. This group is not necessarily connected, but it becomes so after some finite étale cover of S. Deligne [**23**, Theorem 4.2.6] proved that $\mathrm{G}_{\mathrm{mono}}(V/S)_s$ is a semisimple $\mathbb{Q}$-algebraic group whenever V is an algebraic variation. Varying s in S, we obtain a local system of algebraic groups $\mathrm{G}_{\mathrm{mono}}(V/S)$ over S.

The theorem below summarizes several results from [**62**, §4.3].

2.3.6. Theorem. — *Let S be a non-singular and connected complex variety.*

(i) Let $V/S \in \mathsf{VHS}_S^{\mathrm{a}}$. There exists a local system of reductive algebraic groups $\mathrm{MT}(V/S) \subset \mathrm{GL}(V/S)$, the generic Mumford–Tate group of V/S, such that:
- *for all $s \in S$, we have $\mathrm{MT}(V_s) \subset \mathrm{MT}(V/S)_s$, and equality holds for very general (i.e. outside of a countable union of closed subvarieties) $s \in S$;*
- *we have inclusions of local systems of algebraic groups*

$$\mathrm{G}_{\mathrm{mono}}(V/S)^0 \subset \mathrm{MT}(V/S) \subset \mathrm{GL}(V/S).$$

(ii) Let $\mathcal{M}/S \in \mathsf{AM}_S$ and let $M/S \in \mathsf{VHS}_S^{\mathrm{a}}$ denote its realization. Then there exists a local system of reductive algebraic groups $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S) \subset \mathrm{GL}(M/S)$, called the generic motivic Galois group of $\mathcal{M}/S$, such that:
- *for all $s \in S$, we have $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_s) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S)_s$, and equality holds for very general $s \in S$;*
- *we have inclusions of local system of algebraic groups*

$$\mathrm{G}_{\mathrm{mono}}(M/S)^0 \subset \mathrm{MT}(M/S) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S) \subset \mathrm{GL}(M/S).$$

We refer to Theorems 4.1.2, 4.1.3, 4.3.6, and 4.3.9 in Moonen's survey [**62**]; part (*i*) is due to Deligne [**23**].

2.3.7. Remark. — For any $s \in S$, the inclusion $\mathrm{MT}(M_s) \subset \mathrm{MT}(M/S)_s$ (resp. $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}_s) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S)_s$) is an equality if and only if

$$\mathrm{G}_{\mathrm{mono}}(M/S)_s^0 \subset \mathrm{MT}(M_s) \qquad (\text{resp. } \mathrm{G}_{\mathrm{mono}}(M/S)_s^0 \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_s)).$$

We will need also the following consequence of Theorem 2.3.6, which is [**62**, Theorem 4.3.8].

2.3.8. Corollary. — *Let $\mathcal{M}/S \in \mathsf{AM}_S$, and let M/S denote its realization. Then, for all $s \in S$, we have*

$$\mathrm{G}_{\mathrm{mono}}(M/S)_s^0 \cdot \mathrm{MT}(M_s) = \mathrm{MT}(M/S)_s,$$
$$\mathrm{G}_{\mathrm{mono}}(M/S)_s^0 \cdot \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_s) = \mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S)_s.$$

Proof. — We follow Moonen's argument. By Theorem 2.3.6 we have

$$\mathrm{G}_{\mathrm{mono}}(M/S)^0_s \cdot \mathrm{G}_{\mathrm{mot}}(\mathcal{M}_s) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S)_s.$$

Since both sides are reductive, it suffices to compare their invariants in all tensor constructions $\mathcal{T}/S = (\mathcal{M}/S)^{\otimes m} \otimes (\mathcal{M}/S)^{\vee,\otimes n}(j)$ for integers m, n and j, by [**25**, Proposition 3.1].

Let T/S denote the realization of a tensor construction $\mathcal{T}/S$ as above. We fix a point $s \in S$. If $\xi_s \in T_s$ is invariant for the action of $\mathrm{G}_{\mathrm{mono}}(M/S)^0_s \cdot \mathrm{G}_{\mathrm{mot}}(M_s)$, then it is a motivated cohomology class which is monodromy invariant. By Theorem 2.3.2, we obtain a global section ξ of T/S such that $\xi_{s'}$ is motivated at any $s' \in S$. It follows that ξ_s is invariant for $\mathrm{G}_{\mathrm{mot}}(M/S)_s$. The proof of the assertion regarding the Mumford–Tate group is similar. □

2.3.9. Remark. — The category AM_S is a neutral Tannakian semisimple abelian category. Any point $s \in S$ determines a fibre functor on AM_S by mapping $\mathcal{M}/S$ to the $\mathbb{Q}$-vector space underlying the Hodge structure $r(\mathcal{M}_s)$; the corresponding Tannakian fundamental group is precisely the generic motivic Galois group $\mathrm{G}_{\mathrm{mot}}(\mathcal{M}/S)_s$ at s. Similarly, the point $s \in S$ determines a fibre functor on the Tannakian category $\mathsf{VHS}^{\mathrm{a}}_S$ by mapping V/S to the $\mathbb{Q}$-vector space V_s, and the corresponding Tannakian fundamental group is $\mathrm{MT}(V/S)_s$.

CHAPTER 3

CHOW MOTIVES OF MODULI SPACES

The results in this Chapter are joint work with Lie Fu and Ziyu Zhang [**29**].

3.1. Construction of symplectic varieties

The primary source of construction of higher dimensional holomorphic symplectic varieties is taking moduli spaces of stable sheaves on K3 or abelian surfaces.

Let S be a projective K3 or abelian surface. Consider the *algebraic Mukai lattice* $\widetilde{\mathrm{NS}}(S) = H^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S,\mathbb{Z})$, equipped with the following pairing: for any $\mathrm{v} = (r,l,s)$ and $\mathrm{v}' = (r,l,s')$ in $\widetilde{\mathrm{NS}}(S)$,

$$\langle \mathrm{v}, \mathrm{v}' \rangle := (l,l') - rs' - r's \in \mathbb{Z}.$$

To any coherent sheaf E on S is associated its Mukai vector $\mathrm{v}(E) \in \widetilde{\mathrm{NS}}(S)$, defined as $\mathrm{v}(E) = \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)}$. Given a primitive Mukai vector $\mathrm{v} \in \widetilde{\mathrm{NS}}(S)$ and a v-generic polarization H on S, there exists a smooth and projective moduli space $\mathcal{M}_H^{\mathrm{st}}(\mathrm{v})$ of H-stable sheaves on S with Mukai vector v, see [**65**], [**67**], [**92**], [**46**]. By [**64**], the moduli space $\mathcal{M}_H^{\mathrm{st}}(\mathrm{v})$ carries a holomorphic symplectic form, and, if not empty, it has dimension $2n = \mathrm{v}^2 + 2$. If S is a K3 surface, then the moduli space $\mathcal{M}_H^{\mathrm{st}}(\mathrm{v})$ is a hyper-Kähler manifold of $\mathrm{K3}^{[n]}$-type ([**42**]). If S is an abelian surface, there is an isotrivial fibration $\mathcal{M}_H^{\mathrm{st}}(\mathrm{v}) \to S \times \hat{S}$, where $\hat{S}$ is the dual abelian surface; the fibre of this fibration is a hyper-Kähler variety of Kum^{n-1}-type ([**92**]).

3.1.1. — Let $\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}$ be a smooth and projective moduli space as above. Let $\widetilde{H}(S,\mathbb{Z})$ be the Mukai lattice of S, that is $\widetilde{H}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$, with the pairing $\langle \mathrm{v}, \mathrm{v}' \rangle := (l, l') - rs' - r's$. A first relation ([**67**]) between $\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}$ and the surface S is that, when $\mathrm{v}^2 > 0$ we have a Hodge isometry

$$H^2(\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}, \mathbb{Z}) = \mathrm{v}^{\perp} \subset \widetilde{H}(S, \mathbb{Z}),$$

while if $\mathrm{v}^2 = 0$ we have $H^2(\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}, \mathbb{Z}) = \mathrm{v}^{\perp} / \langle v \rangle$. More fundamentally, Bülles has established the following relation between the Chow motives of $\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}$ and S.

***3.1.2. Theorem* ([15]).** — *Let S be a* K3 *or abelian surface. Let* v *be a primitive Mukai vector with $\mathrm{v}^2 \geq 0$, and let H be a* v*-generic polarization on S. Assume that the smooth and projective moduli space $\mathcal{M} = \mathcal{M}_H(\mathrm{v})^{\mathrm{st}}$ is not empty. Then the Chow motive of $\mathcal{M}$ belongs to the pseudo-abelian tensor subcategory $\langle \mathfrak{h}(S) \rangle \subset \mathsf{CHM}$.*

3.1.3. — When the Mukai vector v is not primitive, the moduli space $\mathcal{M}_H^{\mathrm{st}}(\mathrm{v})$ is not proper anymore. A natural compactification is given by the moduli space $\mathcal{M}_H(\mathrm{v})$ of H-semistable sheaves on S with Mukai vector v; the moduli space $\mathcal{M}_H(\mathrm{v})$ is proper but not necessarily non-singular. Its regular locus is the open subset $\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}$.

A *crepant* or *symplectic* resolution $f \colon \widetilde{\mathcal{M}} \to \mathcal{M}_H(\mathrm{v})$ is a birational morphism such that $f^*(\sigma)$ extends to a holomorphic symplectic form on $\widetilde{\mathcal{M}}$, where σ is the holomorphic symplectic form on $\mathcal{M}_H(\mathrm{v})^{\mathrm{st}}$. It is not often the case that a singular moduli space admits a symplectic resolution. However this happens in an important case thanks to O'Grady [**68**]. We will extend Bülles' result to O'Grady's symplectic resolutions.

3.2. Stable loci of moduli spaces

In this section, we generalize an argument of Bülles [**15**] to give a relationship between the motive of the (in general quasi-projective) moduli space of stable sheaves on a K3 or abelian surface and the motive of the surface.

Let S be an abelian or K3 surface. Given a Mukai vector $\mathrm{v} \in \widetilde{\mathrm{NS}}(S)$ and a v-generic polarization H, we can form the moduli space $\mathcal{M}^{\mathrm{st}}$ of H-stable sheaves with Mukai vector v. Let us recall the following result of Markman, [**53**], [**54**].

3.2.1. Theorem. — *Let $\mathcal{E}$ and $\mathcal{F}$ be two (twisted) universal families over $\mathcal{M}^{\mathrm{st}} \times S$. Then*

$$\Delta_{\mathcal{M}^{\mathrm{st}}} = c_{2m}(-\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*(\mathcal{E}), \pi_{23}^*(\mathcal{F}))) \in \mathrm{CH}^{2m}(\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}}),$$

where $2m$ *is the dimension of* $\mathcal{M}^{\mathrm{st}}$ *and* $\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*(\mathcal{E}), \pi_{23}^*(\mathcal{F}))$ *denotes the class of the complex* $R\pi_{13,*}(\pi_{12}^*(\mathcal{E})^\vee \otimes^{\mathbb{L}} \pi_{23}^*(\mathcal{F}))$ *in the Grothendieck group of* $\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}}$*, where* π_{ij}*'s are the natural projections from* $\mathcal{M}^{\mathrm{st}} \times S \times \mathcal{M}^{\mathrm{st}}$*.*

Pointer to references. — In [**53**, Theorem 1] the result is stated for the cohomology class, but the proof gives the equality in Chow groups. Indeed, in [**54**, Theorem 8], the statement is for Chow groups. Moreover, the assumption on the existence of a universal family can be dropped ([**54**, Proposition 24]): it suffices to replace in the formula the sheaves $\mathcal{E}$ and $\mathcal{F}$ by certain universal classes in the Grothendieck group $K_0(S \times \mathcal{M}^{\mathrm{st}})$ constructed in [**54**, Definition 26]. □

As a consequence, we obtain the following analogue of [**15**, (3), p.6]

3.2.2. Proposition. — *There exist finitely many integers* k_i *and algebraic cycles* $\gamma_i \in \mathrm{CH}^{e_i}(\mathcal{M}^{\mathrm{st}} \times S^{k_i})$ *and* $\delta_i \in \mathrm{CH}^{d_i}(S^{k_i} \times \mathcal{M}^{\mathrm{st}})$*, such that*

$$\Delta_{\mathcal{M}^{\mathrm{st}}} = \sum \delta_i \circ \gamma_i \in \mathrm{CH}^{2m}(\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}});$$

here $\dim \mathcal{M}^{\mathrm{st}} = 2m = e_i + d_i - 2k_i$ *for all* i*.*

Proof. — We follow the proof of [**15**, Theorem 1]. First of all, we observe that by Lieberman's formula (see [**5**, §3.1.4] and [**87**, Lemma 3.3] for a proof) the following two-sided ideal of $\mathrm{CH}^\bullet(\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}})$ (with respect to the ring structure given by the composition of correspondences)

$$I = \langle \beta \circ \alpha \mid \alpha \in \mathrm{CH}^\bullet(\mathcal{M}^{\mathrm{st}} \times S^k), \beta \in \mathrm{CH}^\bullet(S^k \times \mathcal{M}^{\mathrm{st}}), k \in \mathbb{N} \rangle \subseteq \mathrm{CH}^\bullet(\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}})$$

is closed under the intersection product, hence is a $\mathbb{Q}$-subalgebra of $\mathrm{CH}^\bullet(\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}})$. A computation similar to [**15**, (2), p.6] using the Grothendieck–Riemann–Roch theorem shows that

$$\mathrm{ch}(-[\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*(\mathcal{E}), \pi_{23}^*(\mathcal{F}))]) = -(\pi_{13})_*(\pi_{12}^*\alpha \cdot \pi_{23}^*\beta)$$

where

$$\alpha = \mathrm{ch}(\mathcal{E}^\vee) \cdot \pi_2^* \sqrt{\mathrm{td}(S)} \quad \text{and} \quad \beta = \mathrm{ch}(\mathcal{F}) \cdot \pi_2^* \sqrt{\mathrm{td}(S)}.$$

It follows that $\mathrm{ch}_n(-[\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*(\mathcal{E}), \pi_{23}^*(\mathcal{F}))]) \in I$ for any $n \in \mathbb{N}$. An induction argument then shows that $c_n(-[\mathcal{E}xt^!_{\pi_{13}}(\pi_{12}^*(\mathcal{E}), \pi_{23}^*(\mathcal{F}))]) \in I$ for each $n \in \mathbb{N}$. In particular, combined with Theorem 3.2.1, $\Delta_{\mathcal{M}^{\mathrm{st}}}$ is in I, which is equivalent to the conclusion. □

3.2.3. Remark. — In Proposition 3.2.2, if we let $\gamma = \oplus\gamma_i$ and $\delta = \oplus\delta_i$, we get the following morphisms of mixed Hodge structures.

$$H_c^*(\mathcal{M}^{\mathrm{st}}) \xrightarrow{\gamma} \bigoplus_i H^*(S^{k_i})(2k_i - e_i) \xrightarrow{\delta} H^*(\mathcal{M}^{\mathrm{st}}),$$

where the composition is precisely the comparison morphism from the compact support cohomology to the cohomology.

3.2.4. Remark. — In the case that S is an abelian surface, the moduli space $\mathcal{M}^{\mathrm{st}}$ is isotrivially fibered over $S\times\hat{S}$ (which is the Albanese fibration when $\mathcal{M}^{\mathrm{st}}$ is projective). We usually denote by $\mathcal{K}^{\mathrm{st}}$ the fibre of this fibration. The analogue of Theorem 3.2.1 seems to be unknown for $\mathcal{K}^{\mathrm{st}}$.

3.3. The motive of O'Grady's moduli spaces

In this section, we study the motive of O'Grady's 10-dimensional symplectic varieties [**68**]. Those are obtained as symplectic resolutions of certain singular moduli spaces of sheaves on K3 or abelian surfaces. We first recall the construction.

3.3.1. — Let S be a projective K3 surface or abelian surface, let $\mathrm{v}_0 \in \widetilde{\mathrm{NS}}(S)$ be a primitive Mukai vector with $\mathrm{v}_0^2 = 2$ and let $\mathrm{v} = 2\,\mathrm{v}_0$. Let H be a v_0-generic polarization on S. We write

$$\mathcal{M}^{\mathrm{st}} = \mathcal{M}_{S,H}(\mathrm{v})^{\mathrm{st}}$$

for the smooth and quasi-projective moduli space of H-stable sheaves on S with Mukai vector v, and

$$\mathcal{M} = \mathcal{M}_{S,H}(\mathrm{v})^{\mathrm{ss}}$$

for the (singular) moduli space of H-semistable sheaves with the same Mukai vector. By [**68**] and [**50**], there exists a symplectic resolution $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, which is a projective holomorphic symplectic manifold of dimension 10, not deformation equivalent to the fifth Hilbert schemes of the surface S. The manifolds so obtained are all deformation equivalent to each other by [**72**]. If S is a K3 surface, then $\widetilde{\mathcal{M}}$ is a hyper-Kähler manifold of OG10-type, while if S is an abelian surface there is an isotrivial fibration $\widetilde{\mathcal{M}} \to S \times \hat{S}$, whose fibre is a hyper-Kähler manifold of OG6-type ([**69**]).

3.3.2. Remark. — By [**47**], a symplectic resolution of the singular moduli space $\mathcal{M}$ exists only in O'Grady's case above.

3.3.3. — Let us briefly recall the geometry of $\mathcal{M}$. We follow the notation in [**68**], see also [**50**] and [**60**, §2]. The moduli space $\mathcal{M}$ admits a filtration

$$\mathcal{M} \supset \Sigma \supset \Omega$$

where

$$\Sigma = \operatorname{Sing}(\mathcal{M}) = \mathcal{M} \setminus \mathcal{M}^{\mathrm{st}} \cong \operatorname{Sym}^2(\mathcal{M}_{S,H}(\mathrm{v}_0))$$

is the singular locus of $\mathcal{M}$, which consists of strictly H-semistable sheaves, and

$$\Omega = \operatorname{Sing}(\Sigma) \cong \mathcal{M}_{S,H}(\mathrm{v}_0)$$

is the singular locus of Σ, hence the diagonal in $\operatorname{Sym}^2(\mathcal{M}_{S,H}(\mathrm{v}_0))$. Note that $\mathcal{M}_{S,H}(\mathrm{v}_0)$ is a smooth projective holomorphic symplectic fourfold, deformation equivalent to the Hilbert square of S.

In [**68**], O'Grady produced a symplectic resolution $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ in three steps. As the explicit geometry is used in the proof of our main result, we briefly recall his construction.

STEP 1. We blow up $\mathcal{M}$ along Ω, resulting in a space $\overline{\mathcal{M}}$ with an exceptional divisor $\overline{\Omega}$. The only singularity of $\overline{\mathcal{M}}$ is an A_1-singularity along the strict transform $\overline{\Sigma}$ of Σ. In fact, $\overline{\Sigma}$ is smooth, satisfying

$$\overline{\Sigma} \cong \operatorname{Hilb}^2(\mathcal{M}_{S,H}(\mathrm{v}_0)),$$

with the morphism $\overline{\Sigma} \to \Sigma$ being the corresponding Hilbert-Chow morphism, whose exceptional divisor is precisely the intersection of $\overline{\Omega}$ and $\overline{\Sigma}$ in $\overline{\mathcal{M}}$.

STEP 2. We blow up $\overline{\mathcal{M}}$ along $\overline{\Sigma}$ to obtain a (non-crepant) resolution $\widehat{\mathcal{M}}$ of $\mathcal{M}$. The exceptional divisor $\widehat{\Sigma}$ is thus a $\mathbb{P}^1$-bundle over $\overline{\Sigma}$. We denote by $\widehat{\Omega}$ the strict transform of $\overline{\Omega}$. Then $\widehat{\mathcal{M}}$ is a smooth projective compactification of $\mathcal{M}^{\mathrm{st}}$, with boundary

$$\partial\widehat{\mathcal{M}} = \widehat{\mathcal{M}} \setminus \mathcal{M}^{\mathrm{st}} = \widehat{\Omega} \cup \widehat{\Sigma}$$

being the union of two smooth hypersurfaces which intersect transversally.

STEP 3. Lastly, an extremal contraction of $\widehat{\mathcal{M}}$ contracts $\widehat{\Omega}$ as a $\mathbb{P}^2$-bundle to $\widetilde{\Omega}$, which is a 3-dimensional quadric bundle (more precisely, the relative Lagrangian Grassmannian fibration associated to the tangent bundle) over Ω. The space obtained is denoted by $\widetilde{\mathcal{M}}$, which is shown to be a symplectic resolution of $\mathcal{M}$.

3.3.4. Remark. — By the main result of Lehn–Sorger [**50**], O'Grady's symplectic resolution can also be obtained via a single blow-up of $\mathcal{M}$ along its (reduced) singular locus Σ. The exceptional divisor $\widetilde{\Sigma}$ is nothing else but the image of $\widehat{\Sigma}$ under the contraction in the third step described above, which is singular along $\widetilde{\Omega}$, the preimage of Ω. If we blow up $\widetilde{\mathcal{M}}$ along $\widetilde{\Omega}$, we will obtain again $\widehat{\mathcal{M}}$, with the exceptional divisor being $\widehat{\Omega}$ and the strict transform of $\widetilde{\Sigma}$ being $\widehat{\Sigma}$. In short, the order of blow-ups can be "reversed"; see the following commutative diagram [**60**, §2]:

$$\begin{tikzcd}
 & \mathrm{Bl}_{\widetilde{\Omega}}\widetilde{\mathcal{M}} = \widehat{\mathcal{M}} = \mathrm{Bl}_{\overline{\Sigma}}\overline{\mathcal{M}} \arrow[dl] \arrow[dr] & \\
\widetilde{\mathcal{M}} = \mathrm{Bl}_{\Sigma}\mathcal{M} \arrow[dr] & & \overline{\mathcal{M}} = \mathrm{Bl}_{\Omega}\mathcal{M} \arrow[dl] \\
 & \mathcal{M} &
\end{tikzcd}$$

3.3.5. — We will compute the Chow motives of the boundary components of $\widehat{\mathcal{M}}$, then describe the Chow motives of the resolutions $\widehat{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$, and prove the following result.

3.3.6. Theorem. — *The Chow motive of O'Grady symplectic resolution $\widetilde{\mathcal{M}}$ belongs to the pseudo-abelian tensor subcategory of* CHM *generated by the Chow motive of S.*

We start with an observation.

3.3.7. Lemma. — *Let X be a smooth projective variety. The Chow motive $\mathfrak{h}(\mathrm{Hilb}^2(X))$ belongs to $\langle \mathfrak{h}(X) \rangle \subset$ CHM, the pseudo-abelian tensor subcategory of* CHM *generated by $\mathfrak{h}(X)$.*

Proof. — Let $\dim X = n$. We denote by $\Delta_X \subseteq X \times X$ the diagonal. By [**52**, §9], we have

$$\mathfrak{h}\left(\mathrm{Bl}_{\Delta_X}(X \times X)\right) = \mathfrak{h}(X^2) \oplus \left(\oplus_{i=1}^{n-1} \mathfrak{h}(X)(-i)\right).$$

Since $\mathrm{Hilb}^2(X) = \mathrm{Bl}_{\Delta_X}(X \times X)/\mathbb{Z}_2$, its motive is the $\mathbb{Z}_2$-invariant part

$$\mathfrak{h}(\mathrm{Hilb}^2(X)) = \mathfrak{h}\left(\mathrm{Bl}_{\Delta_X}(X \times X)\right)^{\mathbb{Z}_2}$$

which is a direct summand of $\mathfrak{h}\left(\mathrm{Bl}_{\Delta_X}(X \times X)\right)$, and hence it is contained in the desired subcategory. $\square$

3.3.8. Lemma. — *The Chow motives* $\mathfrak{h}(\widehat{\Sigma})$, $\mathfrak{h}(\widehat{\Omega})$ *and* $\mathfrak{h}(\widehat{\Sigma}\cap\widehat{\Omega})$ *are all contained in the subcategory* $\langle\mathfrak{h}(S)\rangle \subset \mathsf{CHM}$.

Proof. — By O'Grady's construction, $\widehat{\Sigma}$ is a $\mathbb{P}^1$-bundle over $\overline{\Sigma} \cong \mathrm{Hilb}^2(\mathcal{M}_{S,H}(\mathrm{v}_0))$. It follows from [**52**, §7] that

$$\mathfrak{h}(\widehat{\Sigma}) = \mathfrak{h}(\overline{\Sigma}) \oplus \mathfrak{h}(\overline{\Sigma})(-1).$$

By Theorem 3.1.2, $\mathfrak{h}(\mathcal{M}_{S,H}(\mathrm{v}_0))$ lies in the pseudo-abelian tensor subcategory of Chow motives generated by $\mathfrak{h}(S)$. It follows from Lemma 3.3.7 that $\mathfrak{h}(\overline{\Sigma})$ is also in this subcategory, therefore so is $\mathfrak{h}(\widehat{\Sigma})$.

Again by O'Grady's construction, $\widehat{\Omega}$ is a $\mathbb{P}^2$-bundle over $\widetilde{\Omega}$. It follows that

$$\mathfrak{h}(\widehat{\Omega}) = \mathfrak{h}(\widetilde{\Omega}) \oplus \mathfrak{h}(\widetilde{\Omega})(-1) \oplus \mathfrak{h}(\widetilde{\Omega})(-2).$$

Moreover, since $\widetilde{\Omega}$ is a 3-dimensional quadric bundle over Ω, by [**86**, Remark 4.6] we have that

$$\mathfrak{h}(\widetilde{\Omega}) = \mathfrak{h}(\Omega) \oplus \mathfrak{h}(\Omega)(-1) \oplus \mathfrak{h}(\Omega)(-2) \oplus \mathfrak{h}(\Omega)(-3).$$

Since $\Omega \cong \mathcal{M}_{S,H}(\mathrm{v}_0)$, it follows from Bülles' Theorem 3.1.2 that $\mathfrak{h}(\Omega)$ belongs to the pseudo-abelian tensor subcategory of Chow motives generated by $\mathfrak{h}(S)$, hence the same is true for $\mathfrak{h}(\widetilde{\Omega})$ and $\mathfrak{h}(\widehat{\Omega})$.

Similarly, the intersection $\widehat{\Sigma} \cap \widehat{\Omega}$ is a smooth conic bundle over $\widetilde{\Omega}$, and, again by [**86**, Remark 4.6], its motive is in the tensor subcategory generated by that of $\widetilde{\Omega}$. One concludes as for $\widehat{\Omega}$. □

Here comes the key step of the proof.

3.3.9. Proposition. — *The Chow motive* $\mathfrak{h}(\widehat{\mathcal{M}})$ *belongs to* $\langle\mathfrak{h}(S)\rangle \subset \mathsf{CHM}$.

Proof. — By Proposition 3.2.2, we have

$$[\Delta_{\mathcal{M}^{\mathrm{st}}}] = \sum \delta_i \circ \gamma_i \in \mathrm{CH}^{10}(\mathcal{M}^{\mathrm{st}} \times \mathcal{M}^{\mathrm{st}}),$$

where $\gamma_i \in \mathrm{CH}^{e_i}(\mathcal{M}^{\mathrm{st}} \times S^{k_i})$ and $\delta_i \in \mathrm{CH}^{d_i}(S^{k_i} \times \mathcal{M}^{\mathrm{st}})$. Let $\widehat{\gamma}_i \in \mathrm{CH}^{e_i}(\widehat{\mathcal{M}} \times S^{k_i})$ and $\widehat{\delta}_i \in \mathrm{CH}^{d_i}(S^{k_i} \times \widehat{\mathcal{M}})$ be any closure of cycles representing γ_i and δ_i respectively. Then the support of the class

$$[\Delta_{\widehat{\mathcal{M}}}] - \sum \widehat{\delta}_i \circ \widehat{\gamma}_i \in \mathrm{CH}^{10}(\widehat{\mathcal{M}} \times \widehat{\mathcal{M}})$$

lies in the boundary $(\widehat{\mathcal{M}} \times \partial\widehat{\mathcal{M}}) \cup (\partial\widehat{\mathcal{M}} \times \widehat{\mathcal{M}})$. Hence, in $\mathrm{CH}^{10}(\widehat{\mathcal{M}} \times \widehat{\mathcal{M}})$, we have

$$[\Delta_{\widehat{\mathcal{M}}}] = \sum \widehat{\delta}_i \circ \widehat{\gamma}_i + Y_{\widehat{\Sigma}} + Y_{\widehat{\Omega}} + Z_{\widehat{\Sigma}} + Z_{\widehat{\Omega}},$$

for some algebraic cycles $Y_{\widehat{\Sigma}} \in \mathrm{CH}^9(\widehat{\mathcal{M}} \times \widehat{\Sigma})$, $Y_{\widehat{\Omega}} \in \mathrm{CH}^9(\widehat{\mathcal{M}} \times \widehat{\Omega})$, $Z_{\widehat{\Sigma}} \in \mathrm{CH}^9(\widehat{\Sigma} \times \widehat{\mathcal{M}})$ and $Z_{\widehat{\Omega}} \in \mathrm{CH}^9(\widehat{\Omega} \times \widehat{\mathcal{M}})$.

For each i, the cycles $\widehat{\gamma}_i$ and $\widehat{\delta}_i$ can be viewed as morphisms of motives

$$\mathfrak{h}(\widehat{\mathcal{M}}) \xrightarrow{\widehat{\gamma}_i} \mathfrak{h}(S^{k_i})(n_i) \xrightarrow{\widehat{\delta}_i} \mathfrak{h}(\widehat{\mathcal{M}}),$$

where $n_i = e_i - 10 = 2k_i - d_i$. On the other hand, denoting by $j_{\widehat{\Sigma}}$ and $j_{\widehat{\Omega}}$ the closed embedding of $\widehat{\Sigma}$ and $\widehat{\Omega}$ in $\hat{\mathcal{M}}$ respectively, we have morphisms of motives

$$\mathfrak{h}(\widehat{\mathcal{M}}) \xrightarrow{Y_{\widehat{\Sigma}}} \mathfrak{h}(\widehat{\Sigma}) \xrightarrow{(j_{\widehat{\Sigma}})_*} \mathfrak{h}(\widehat{\mathcal{M}}),$$

$$\mathfrak{h}(\widehat{\mathcal{M}}) \xrightarrow{Y_{\widehat{\Omega}}} \mathfrak{h}(\widehat{\Omega}) \xrightarrow{(j_{\widehat{\Omega}})_*} \mathfrak{h}(\widehat{\mathcal{M}}),$$

$$\mathfrak{h}(\widehat{\mathcal{M}}) \xrightarrow{j^*_{\widehat{\Sigma}}} \mathfrak{h}(\widehat{\Sigma})(-1) \xrightarrow{Z_{\widehat{\Sigma}}} \mathfrak{h}(\widehat{\mathcal{M}}),$$

$$\mathfrak{h}(\widehat{\mathcal{M}}) \xrightarrow{j^*_{\widehat{\Omega}}} \mathfrak{h}(\widehat{\Omega})(-1) \xrightarrow{Z_{\widehat{\Omega}}} \mathfrak{h}(\widehat{\mathcal{M}}).$$

The sum of all the above compositions is the identity of $\mathfrak{h}(\widehat{\mathcal{M}})$. Hence $\mathfrak{h}(\widehat{\mathcal{M}})$ is a direct summand of

$$\big(\oplus_i \mathfrak{h}(S^{k_i})(n_i)\big) \oplus \mathfrak{h}(\widehat{\Sigma}) \oplus \mathfrak{h}(\widehat{\Omega}) \oplus \mathfrak{h}(\widehat{\Sigma})(-1) \oplus \mathfrak{h}(\widehat{\Omega})(-1).$$

Combining this with Lemma 3.3.8, we finish the proof. □

Proof of Theorem 3.3.6. — Since $\widehat{\mathcal{M}}$ is a blow-up of $\widetilde{\mathcal{M}}$ along a smooth center, it follows from [**52**, §9] that $\mathfrak{h}(\widetilde{\mathcal{M}})$ is a direct summand of $\mathfrak{h}(\widehat{\mathcal{M}})$. Then the conclusion follows from Proposition 3.3.9 together with the fact that $\langle \mathfrak{h}(S) \rangle \subset \mathsf{CHM}$ is closed under taking direct summands. □

3.3.10. Corollary. — *The standard conjectures hold for all crepant resolutions $\widetilde{M}$ that appeared in Theorem 3.3.6.*

Proof. — In the situation of Theorem 3.3.6, $\widetilde{\mathcal{M}}$ is *motivated* by the surface S in the sense of Arapura [**6**], i.e. the Grothendieck motive of $\widetilde{\mathcal{M}}$ belongs to the pseudo-abelian tensor category generated by the Grothendieck motive of S. Since the Lefschetz standard conjecture holds for S, we can invoke Arapura's result [**6**, Lemma 4.2] to obtain the standard conjectures for $\widetilde{\mathcal{M}}$. □

3.3.11. Corollary. — *There are infinitely many projective hyper-Kähler varieties of* OG10-*type whose Chow motive is abelian.*

Proof. — By Theorem 3.3.6, it suffices to see that there are infinitely many projective K3 surfaces with abelian Chow motives. To this end, we can take for example the Kummer K3 surfaces or K3 surfaces with Picard number at least 19, by [**71**]. □

3.3.12. Remark. — When S is an abelian surface, the previously considered moduli spaces $\mathcal{M}^{\mathrm{st}}$, $\mathcal{M}$, $\widehat{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ are all isotrivally fibered over $S \times \hat{S}$. Let us denote the corresponding fibres by $\mathcal{K}^{\mathrm{st}}$, $\mathcal{K}$, $\widehat{\mathcal{K}}$ and $\widetilde{\mathcal{K}}$. Except for some special cases like generalized Kummer varieties (see [**33**]), Proposition 3.3.9 and Theorem 3.3.6 are unknown for those fibres in general; the missing ingredient is the analogue of Theorem 3.2.1, see Remark 3.2.4.

CHAPTER 4

HODGE THEORY OF HYPER-KÄHLER MANIFOLDS

4.1. The Kuga–Satake category of a K3-Hodge structure

Let V be a polarizable rational Hodge structure of K3-type, i.e. the Hodge decomposition has the form $V \otimes \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$, with $V^{2,0}$ and $V^{0,2}$ one dimensional. The Kuga–Satake construction [**22**] produces out of V an abelian variety $\mathrm{KS}(V)$, well-defined up to isogeny. The main point of this section is to characterize the Tannakian subcategory of Hodge structures generated by $\mathrm{KS}(V)$, which we call the *Kuga–Satake category* attached to V,

$$\mathsf{KS}(V) \coloneqq \langle H^1(\mathrm{KS}(V)) \rangle \subset \mathsf{HS}\,.$$

All cohomology groups are with rational coefficients and, given a Hodge structure W, we denote by $\langle W \rangle$ the Tannakian subcategory of HS generated by W. We first briefly review the classical construction.

4.1.1. — Let q be a polarization of V, and consider the Clifford algebra $\mathrm{Cl}(V, q)$. Deligne has shown in [**22**] that there is a natural way to induce a Hodge structure $\mathrm{Cl}(V, q) \otimes \mathbb{C} = \mathrm{Cl}(V, q)^{0,1} \oplus \mathrm{Cl}(V, q)^{0,1}$, which is polarizable and therefore is $H^1(\mathrm{KS}(V))$ for some abelian variety $\mathrm{KS}(V)$, well-defined up to isogeny. The key relation between V and $\mathrm{KS}(V)$ is the fact that the natural action of V on $\mathrm{Cl}(V, q)$ via left multiplication yields an embedding of Hodge structures

$$V(1) \hookrightarrow H^1(\mathrm{KS}(V)) \otimes H^1(\mathrm{KS}(V))^\vee.$$

Letting $\mathrm{GO}(V, q)$ denote the group of linear automorphisms of V preserving the polarization q up to scalar, the Hodge structure on V is given by a real representation

$h\colon \mathbb{S} \to \mathrm{GO}(V,q) \otimes \mathbb{R}$. There is a double cover $\phi\colon\ \mathrm{CSpin}(V,q) \to \mathrm{GO}(V,q)$, with kernel $\{\pm 1\}$. The key observation in Deligne's construction is that there exists a unique homomorphism $h'\colon \mathbb{S} \to \mathrm{CSpin}(V,q) \otimes \mathbb{R}$ such that $h = \phi \circ h'$. Via the natural action of $\mathrm{CSpin}(V,q)$ on the Clifford algebra, the representation h' induces the desired Hodge structure on $\mathrm{Cl}(V,q)$. The surjective homomorphism $\mathrm{MT}(H^1(\mathrm{KS}(V))) \to \mathrm{MT}(V)$ induced by the inclusion $\langle V \rangle \subset \langle H^1(\mathrm{KS}(V)) \rangle$ is the restriction of ϕ, and hence it is a double cover as well.

4.1.2. Remark. — The Kuga–Satake construction can be performed given any non-degenerate symmetric bilinear form q on V such that the restriction of $q \otimes \mathbb{R}$ to the subspace $(H^{2,0}(V) \oplus H^{0,2}(V)) \cap (V \otimes \mathbb{R})$ is positive definite and $q(\sigma,\sigma) = 0$ for any $\sigma \in H^{2,0}(V)$, see [**45**, §4, Remark 2.3]. If V is not polarizable, then the Kuga–Satake construction only yields a complex torus $\mathrm{KS}(V)$.

4.1.3. — Given a Tannakian subcategory $\mathsf{C} \subset \mathsf{HS}$ we denote by C^{ev} the full subcategory of C consisting of objects of *even* weight. Let $W \in \mathsf{HS}$, and let $w\colon \mathbb{G}_m \to \mathrm{MT}(W)$ be the weight cocharacter. We let $\iota := w(-1)$; it acts as -1 on any Hodge structure of odd weight in $\langle W \rangle$ and as the identity on $\langle W \rangle^{\mathrm{ev}}$. This means that, whenever $\langle W \rangle$ contains a Hodge structure of odd weight, the natural morphism of algebraic groups $\mathrm{MT}(W) \to \mathrm{MT}(\langle W \rangle^{\mathrm{ev}})$ is an isogeny of algebraic groups with kernel the order 2 central subgroup $\langle \iota \rangle$; in fact, ι is the only non-trivial element of $\mathrm{GL}(W)$ which acts trivially on $\mathrm{GL}(W \otimes W)$.

4.1.4. Definition. — *Let V be a polarizable Hodge structure of K3-type. A Kuga–Satake variety for V is an abelian variety A such that $\langle H^1(A) \rangle^{\mathrm{ev}} = \langle V \rangle$.*

***4.1.5. Lemma* (Equivalent definition).** — *An abelian variety A is a Kuga–Satake variety for V if and only if $V \in \langle H^1(A) \rangle$ and the induced surjective morphism* $\mathrm{MT}(H^1(A)) \to \mathrm{MT}(V)$ *is an isogeny of degree* 2.

Proof. — The only if part is already clear. Conversely, assume that A is an abelian variety such that $V \in \langle H^1(A) \rangle$ and that the induced surjection $\mathrm{MT}(H^1(A)) \to \mathrm{MT}(V)$ is an isogeny of degree 2. This morphism factors over $\mathrm{MT}(\langle H^1(A) \rangle^{\mathrm{ev}}) \to \mathrm{MT}(V)$, and it follows that the latter is an isomorphism. Hence, $\langle H^1(A) \rangle^{\mathrm{ev}} = \langle V \rangle$. □

By Lemma 4.1.5 and the discussion in §4.1.1, the abelian variety $\mathrm{KS}(V)$ is a Kuga–Satake variety for V in the sense of Definition 4.1.4. Kuga–Satake varieties are not unique, not even up to isogeny: for instance, we could apply the Kuga–Satake construction to the even Clifford algebra instead than to the full Clifford algebra. The main observation of this section is that however the corresponding Kuga–Satake category is indeed unique. Our argument relies on the Hodge maximality of polarizable Hodge structures of K3-type proven by Cadoret–Moonen, [**16**].

4.1.6. Theorem. — *Let V be a polarizable Hodge structure of* K3*-type. Then there exists a unique Tannakian subcategory $\mathsf{KS}(V)$ of $\mathsf{HS}^{\mathrm{pol}}$ such that*

$$\langle V\rangle = \mathsf{KS}(V)^{\mathrm{ev}} \subsetneq \mathsf{KS}(V).$$

We call $\mathsf{KS}(V)$ the Kuga–Satake category associated to V. If A is any Kuga–Satake variety for V, we have $\langle H^1(A)\rangle = \mathsf{KS}(V)$.

Let us first prove the following straightforward lemma. Consider Tannakian subcategories $\mathsf{C} \subset \mathsf{D}$ of HS. Assume that both contain some Hodge structure of odd weight. The inclusion of C in D induces surjective homomorphisms of pro-algebraic groups $q\colon \mathrm{MT}(\mathsf{D}) \to \mathrm{MT}(\mathsf{C})$ and $q^{\mathrm{ev}}\colon \mathrm{MT}(\mathsf{D}^{\mathrm{ev}}) \to \mathrm{MT}(\mathsf{C}^{\mathrm{ev}})$. Let π_{D}, resp. π_{C}, denote the double cover $\mathrm{MT}(\mathsf{D}) \to \mathrm{MT}(\mathsf{D}^{\mathrm{ev}})$, resp. $\mathrm{MT}(\mathsf{C}) \to \mathrm{MT}(\mathsf{C}^{\mathrm{ev}})$.

4.1.7. Lemma. — *In the above situation, the morphism $\pi_{\mathsf{D}}\colon \mathrm{MT}(\mathsf{D}) \to \mathrm{MT}(\mathsf{D}^{\mathrm{ev}})$ induces an isomorphism $\ker(q) \cong \ker(q^{\mathrm{ev}})$, and $\pi_{\mathsf{D}}^{-1}(\ker(q^{\mathrm{ev}})) = \langle\iota\rangle \times \ker(q)$.*

Proof. — Consider the commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \langle\iota\rangle \arrow[r] \arrow[d, "\cong"] & \mathrm{MT}(\mathsf{D}) \arrow[r, "\pi_{\mathsf{D}}"] \arrow[d, two heads, "q"] & \mathrm{MT}(\mathsf{D}^{\mathrm{ev}}) \arrow[r] \arrow[d, two heads, "q^{\mathrm{ev}}"] & 1 \\
1 \arrow[r] & \langle\iota\rangle \arrow[r] & \mathrm{MT}(\mathsf{C}) \arrow[r, "\pi_{\mathsf{C}}"] & \mathrm{MT}(\mathsf{C}^{\mathrm{ev}}) \arrow[r] & 1.
\end{tikzcd}$$

By the snake lemma, π_{D} induces an isomorphism $\ker(q) \cong \ker(q^{\mathrm{ev}})$. By assumption, $\iota \notin \ker(q)$ and ι is central in $\mathrm{MT}(\mathsf{D})$; we conclude that $\pi_{\mathsf{D}}^{-1}(\ker(q^{\mathrm{ev}})) = \langle\iota\rangle \times \ker(q)$. ☐

Proof of Theorem 4.1.6. — Assume given $W_1, W_2 \in \mathsf{HS}$, both of odd weight, and such that $\langle W_i\rangle^{\mathrm{ev}} = \langle V\rangle$, for $i = 1, 2$. We consider $W_1 \oplus W_2$. We have surjective homomorphisms $q_i\colon \mathrm{MT}(W_1 \oplus W_2) \to \mathrm{MT}(W_i)$, and a commutative diagram with exact rows

$$\begin{tikzcd}
\ker(q_1) \arrow[r] \arrow[d, "j"] & \mathrm{MT}(W_1 \oplus W_2) \arrow[r, two heads, "q_1"] \arrow[d, two heads, "q_2"] & \mathrm{MT}(W_1) \arrow[d, two heads] \\
\langle\iota_2\rangle \arrow[r] & \mathrm{MT}(W_2) \arrow[r, two heads] & \mathrm{MT}(V)
\end{tikzcd}$$

We claim that q_1 and q_2 are both isomorphisms. Equivalently, j is the trivial map. Indeed, if j is trivial then $\ker(q_1) = \ker(q_2)$, which implies $\langle W_1 \rangle = \langle W_2 \rangle \subset \mathsf{HS}$.

Assume by contradiction that there exists $\tau \in \ker(q_1)$ with $j(\tau) \neq 1$. Then, by construction, $\tau = (\mathrm{id}_{W_1}, -\mathrm{id}_{W_2}) \in \mathrm{GL}(W_1 \oplus W_2)$. Let $\mathsf{C} \subset \langle W_1 \oplus W_2 \rangle$ be the subcategory on which τ acts trivially. Then $\mathsf{C} \subset \mathsf{HS}$ is the Tannakian subcategory generated by W_1 and $\langle W_2 \rangle^{\mathrm{ev}}$; it follows that $\mathsf{C} = \langle W_1 \rangle$. Thus, the induced homomorphism $q_1 \colon \mathrm{MT}(W_1 \oplus W_2) \to \mathrm{MT}(W_1)$ is the quotient by $\langle \tau \rangle$. By Lemma 4.1.7

$$\mathrm{MT}(\langle W_1 \oplus W_2 \rangle^{\mathrm{ev}}) \to \mathrm{MT}(V)$$

is an isogeny of degree 2. Since Mumford–Tate groups are connected, this contradicts the Hodge maximality of V, see [**16**, Proposition 6.2]. □

Thanks to Theorem 2.2.8, we can lift Theorem 4.1.6 to the category of abelian André motives.

4.1.8. Corollary. — *Let $K \subset \mathbb{C}$ be an algebraically closed field. If $\mathcal{M} \in \mathsf{AM}_K^{\mathrm{ab}}$ is an abelian motive whose Hodge realization is of* K3*-type, then there exists a unique Tannakian subcategory $\mathsf{KS}(\mathcal{M})$ of $\mathsf{AM}_K^{\mathrm{ab}}$ such that*

$$\langle \mathcal{M} \rangle = \mathsf{KS}(\mathcal{M})^{\mathrm{ev}} \subsetneq \mathsf{KS}(\mathcal{M}).$$

Moreover, if A is any Kuga–Satake variety for the Hodge structure $r(\mathcal{M})$, we have $\langle \mathsf{KS}(\mathcal{M}) \rangle = \langle \mathcal{H}^1(A) \rangle$ in AM_K.

Proof. — Since $K \subset \mathbb{C}$ is algebraically closed, Theorem 2.2.8 implies that the realization functor r is fully faithful on $\mathsf{AM}_K^{\mathrm{ab}}$. This fact, together with Theorem 4.1.6, immediately gives the corollary. □

The above discussion leads us naturally to the following question about relations among different Kuga–Satake abelian varieties.

4.1.9. Question. — Assume that A and B are abelian varieties such that $\langle H^1(A) \rangle = \langle H^1(B) \rangle$ in $\mathsf{HS}^{\mathrm{pol}}$. Does this imply the existence of integers k, l, such that A is an isogeny factor of B^k, and viceversa B is an isogeny factor of A^l?

We close this section by mentioning that [**35**, Example 14] seems to suggest that in general the answer to the above question is negative.

4.2. Mumford–Tate groups of hyper-Kähler manifolds

The main theme of this section and the next is that the Mumford–Tate group of a hyper-Kähler manifold is controlled by the LLV-Lie algebra. As a consequence, $\mathrm{MT}(H^\bullet(X))$ is completely determined by $\mathrm{MT}(H^2(X))$. Similar results have already appeared in the literature in various forms, cf. [**51**], [**48**], [**36**].

Let X be a complex hyper-Kähler manifold. We denote by $H^\bullet(X) = \bigoplus_i H^i(X)$ the rational singular cohomology of X. We also let $H^+(X) = \bigoplus_i H^{2i}(X)$. The natural inclusions of $H^2(X)$ into $H^+(X)$ and $H^\bullet(X)$ induce surjective morphisms of Mumford–Tate groups:

$$\pi_2^+ \colon \mathrm{MT}(H^+(X)) \to \mathrm{MT}(H^2(X));$$

$$\pi_2 \colon \mathrm{MT}(H^\bullet(X)) \to \mathrm{MT}(H^2(X)).$$

Let $\iota \in \mathrm{GL}(H^\bullet(X))$ act on each $H^i(X)$ via the multiplication by $(-1)^i$.

4.2.1. Proposition. — *The notation is as above.*

(i) The morphism π_2^+ is an isomorphism. In particular, the Hodge structure $H^+(X)$ belongs to the tensor subcategory of HS *generated by $H^2(X)$.*

(ii) If X has non-trivial odd cohomology, the morphism π_2 is an isogeny with kernel $\langle\iota\rangle \simeq \mathbb{Z}/2\mathbb{Z}$. If X is projective, for any Kuga–Satake variety A for $H^2(X)$ we have $\langle H^\bullet(X)\rangle = \langle H^1(A)\rangle$ in HS.

For instance, A could be the abelian variety obtained via the Kuga–Satake construction from $H^2(X)$ equipped with the Beauville–Bogomolov form (Remark 4.1.2).

The above proposition is a consequence of the properties of the LLV-representation. We denote by H the $\mathbb{Q}$-vector space $H^2(X)$ equipped with the Beauville–Bogomolov form. Recall (Definition 1.5.5) that we introduced the twisted LLV-representation

$$R \colon \mathrm{G}_0(X) \cong \mathrm{CSpin}(H) \to \prod_i \mathrm{GL}(H^i(X)).$$

4.2.2. Lemma. — *The Mumford–Tate group $\mathrm{MT}(H^\bullet(X))$ is contained in the image of the representation R.*

Proof. — The Hodge structure on $H^\bullet(X)$ is determined by a real representation $h \colon \mathbb{S} \to \mathrm{GL}(H^\bullet(X)) \otimes \mathbb{R}$ of the Deligne torus $\mathbb{S}$. The Lie algebra $\mathfrak{s}$ of this torus is mapped by h to the two dimensional real Lie subalgebra of $\mathfrak{gl}(H^\bullet(X)) \otimes \mathbb{R}$ spanned by θ' and W, where θ' is multiplication by j on each $H^j(X)$ and W is the Weil operator, i.e. the endomorphism of $H^\bullet(X)\otimes\mathbb{R}$ whose complexification acts on $H^{p,q}(X)$ as multiplication by $i(p-q)$.

Denote by G the image of R, and let $\mathfrak{g}\subset\mathfrak{gl}(H^\bullet(X))$ be the Lie algebra of G. By Remark 1.5.6, we have $\theta'\in\mathfrak{g}$; by Theorem 1.3.8, the Weil operator W belongs to $\mathfrak{g}\otimes\mathbb{R}$. It follows that the image of the differential of h is contained in $\mathfrak{g}\otimes\mathbb{R}$. Therefore, we have $h(\mathbb{S})\subset G\otimes\mathbb{R}$. By definition of the Mumford–Tate group (§2.1.2), we conclude that $\mathrm{MT}(H^\bullet(X))\subset G$. □

Proof of Proposition 4.2.1. — (i). By Lemma 1.5.7, the representation R is by graded algebra automorphism. Let $R^+\colon \mathrm{G}_0(X)\to\prod_i\mathrm{GL}(H^{2i}(X))$ be the induced representation on the even part $H^+(X)$ of the cohomology. Lemma 4.2.2 implies that $\mathrm{MT}(H^+(X))\subset\mathrm{Im}(R^+)$. The morphism π_2^+ is the restriction of the natural projection $\mathrm{pr}_2^+\colon\prod_i\mathrm{GL}(H^{2i}(X))\to\mathrm{GL}(H^2(X))$. By Remark 1.5.8, the restriction of pr_2^+ to $\mathrm{Im}(R^+)$ is injective; hence its restriction to the subgroup $\mathrm{MT}(H^+(X))$ is also injective, and thus π_2^+ is an isomorphism.

(ii). Assume that the odd cohomology of X is non-trivial. This time the restriction of $\mathrm{pr}_2\colon\prod_i\mathrm{GL}(H^i(X))\to\mathrm{GL}(H^2(X))$ to $\mathrm{Im}(R)$ has kernel the order 2 subgroup generated by $\iota=R(-1)$, by Remark 1.5.8. Since ι is clearly in $\mathrm{MT}(H^\bullet(X))$, Lemma 4.2.2 implies that the kernel of $\pi_2\colon\mathrm{MT}(H^\bullet(X))\to\mathrm{MT}(H^2(X))$ is $\langle\iota\rangle\cong\mathbb{Z}/2\mathbb{Z}$.

Finally, let X be projective and let A be a Kuga–Satake abelian variety for $H^2(X)$. Then $\langle H^1(A)\rangle\subset\mathsf{HS}^{\mathrm{pol}}$ is the unique Tannakian subcategory such that

$$\langle H^2(X)\rangle=\langle H^1(A)\rangle^{\mathrm{ev}}\subsetneq\langle H^1(A)\rangle,$$

by Theorem 4.1.6. It is therefore enough to show that $\langle H^\bullet(X)\rangle\subset\mathsf{HS}$ also has this property. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{MT}(H^\bullet(X)) \arrow[rr, two heads, "\pi_2"] \arrow[dr, two heads, "\pi^{\mathrm{ev}}"'] & & \mathrm{MT}(H^2(X)) \\
& \mathrm{MT}(\langle H^\bullet(X)\rangle^{\mathrm{ev}}). \arrow[ur, two heads, "\pi_2^{\mathrm{ev}}"'] &
\end{tikzcd}$$

We have just seen that π_2 is an isogeny of degree 2, and we know that π^{ev} is also an isogeny of degree 2, see §4.1.3; we conclude that π_2^{ev} is an isomorphism. Since, by assumption, $\langle H^\bullet(X)\rangle$ contains Hodge structures of odd weight, we have

$$\langle H^2(X)\rangle=\langle H^\bullet(X)\rangle^{\mathrm{ev}}\subsetneq\langle H^\bullet(X)\rangle,$$

and hence $\langle H^\bullet(X)\rangle=\langle H^1(A)\rangle$ in $\mathsf{HS}^{\mathrm{pol}}$. □

4.3. The H^2 determines the full Hodge structure

The next result, which is a slight generalization of a result proven by Soldatenkov in [**80**], makes precise that the total Hodge structure of a hyper-Kähler variety is determined by its component in degree 2.

In what follows, we will say that two hyper-Kähler manifolds X_1 and X_2 are $H^\bullet$-*equivalent* if there exists an isomorphism of graded algebras $H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ which is an isometry in degree 2 with respect to the Beauville–Bogomolov pairings. Since the algebra $H^\bullet(X_i)$ and the Beauville–Bogomolov form on $H^2(X_i)$ only depend on the topology of X_i, deformation equivalent manifolds are $H^\bullet$-equivalent. It seems unknown whether, conversely, $H^\bullet$-equivalent varieties are deformation equivalent; this holds for the known deformation types, though, since they are distinguished by their second Betti number.

4.3.1. Theorem. — *Let X_1 and X_2 be $H^\bullet$-equivalent complex projective hyper-Kähler manifolds. Assume given a Hodge isometry $f\colon H^2(X_1) \xrightarrow{\sim} H^2(X_2)$. Then there exists an isomorphism of graded algebras $F\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ which is an isomorphism of Hodge structures.*

Proof of Theorem 4.3.1. — By assumption, there exists an isomorphism of graded algebras $\Psi\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ whose degree 2 component $\psi\colon H^2(X_1) \xrightarrow{\sim} H^2(X_2)$ is an isometry. We construct the required isomorphism of graded algebras $F\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ as follows. We have $\psi^{-1} \circ f \in \mathrm{O}(H^2(X_1))(\mathbb{Q})$. We may assume that $\psi^{-1} \circ f$ has determinant 1, for if it has determinant -1 we can choose an ample line bundle on X_1 with first Chern class $e \in H^2(X_1)$ and replace f with the Hodge isometry given by $e \mapsto -f(e)$ and $v \mapsto f(v)$, for any $v \in \langle e\rangle^\perp$.

The morphism $\pi\colon \mathrm{CSpin}(H^2(X_1)) \to \mathrm{SO}(H^2(X_1))$ is surjective on $\mathbb{Q}$-points. Indeed, by Hilbert's theorem 90, the short exact sequence

$$1 \to \mathbb{G}_m \to \mathrm{CSpin}(H^2(X_1)) \to \mathrm{SO}(H^2(X_1)) \to 1$$

yields a short exact sequence

$$1 \to \mathbb{Q}^\times \to \mathrm{CSpin}(H^2(X_1))(\mathbb{Q}) \to \mathrm{SO}(H^2(X_1))(\mathbb{Q}) \to 1,$$

see [**78**, Chapter X, §1].

Therefore, there exists some $g \in \mathrm{CSpin}(H^2(X_1))(\mathbb{Q})$ such that $\pi(g) = \psi^{-1} \circ f$. By Lemma 1.5.7, $R(g)$ is a graded algebra automorphism of $H^\bullet(X_1)$, and we define the isomorphism of graded algebras

$$F \coloneqq \Psi \circ R(g) \colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2).$$

By Remark 1.5.8, the degree 2 component of F is a multiple of the Hodge isometry f; by Proposition 4.3.3 below, F is an isomorphism of Hodge structures. □

4.3.2. ***Remark.*** — The assumption that X_1 and X_2 are projective is only used to change the sign of the determinant of $\psi^{-1} \circ f$. It would have been sufficient to assume that $\mathrm{Pic}(X_i) \neq 0$.

In the above proof, we used the following result.

4.3.3. ***Proposition.*** — *Let X_1 and X_2 be complex hyper-Kähler manifolds. Let $F \colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ be an isomorphism of graded algebras and assume that the degree 2 component $F^{(2)} \colon H^2(X_1) \xrightarrow{\sim} H^2(X_2)$ is an isomorphism of Hodge structures. Then F is an isomorphism of Hodge structures.*

Proof. — If $\mathbb{S}$ denotes the Deligne torus, the total Hodge structure on $H^\bullet(X_i)$ corresponds to a real representation $h_i \colon \mathbb{S} \to \prod_j \mathrm{GL}(H^j(X_i)) \otimes \mathbb{R}$. By definition, h_i factors through $\mathrm{MT}(H^\bullet(X_i))(\mathbb{R})$. By Proposition 4.2.1, the group $\mathrm{MT}(H^\bullet(X_i))$ is contained in the image of the twisted LLV-representation $R \colon \mathrm{G}_0(X_i) \to \prod_j \mathrm{GL}(H^j(X_i))$. By Lemma 1.2.4.($i$), the induced isomorphism $F_* \colon \mathrm{GL}(H^\bullet(X_1)) \xrightarrow{\sim} \mathrm{GL}(H^\bullet(X_2))$ restricts to an isomorphism of LLV-Lie algebras $\mathfrak{g}(X_1) \xrightarrow{\sim} \mathfrak{g}(X_2)$. As F preserves the cohomological grading, F_* restricts to an isomorphism $R(\mathrm{G}_0(X_1)) \xrightarrow{\sim} R(\mathrm{G}_0(X_2))$.

We have to show that the diagram

$$\begin{tikzcd}
 & \mathbb{S} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
R(\mathrm{G}_0(X_1))(\mathbb{R}) \arrow[rr, "F_*"] \arrow[d, "\mathrm{pr}_1"] & & R(\mathrm{G}_0(X_2))(\mathbb{R}) \arrow[d, "\mathrm{pr}_2"] \\
R^{(2)}(\mathrm{G}_0(X_1))(\mathbb{R}) \arrow[rr, "F_*^{(2)}"] & & R^{(2)}(\mathrm{G}_0(X_2))(\mathbb{R})
\end{tikzcd}$$

is commutative. By Remark 1.5.8, the morphism $\mathrm{pr}_2 \colon R(\mathrm{G}_0(X_2)) \to R^{(2)}(\mathrm{G}_0(X_2))$ is either an isomorphism or a central isogeny of degree 2; let C be the kernel. Since $F^{(2)}$ is an isomorphism of Hodge structures, we have $F_*^{(2)} \circ \mathrm{pr}_1 \circ h_1 = \mathrm{pr}_2 \circ h_2$. Hence, there is a morphism $\xi \colon \mathbb{S} \to C$ such that $F_* \circ h_1 = \xi \cdot h_2$. But $\mathbb{S}$ is connected and C is finite, so ξ is trivial and F is an isomorphism of Hodge structures. □

CHAPTER 5

DEFECT GROUPS OF HYPER-KÄHLER VARIETIES

5.1. The defect group

A hyper-Kähler variety X over a field $K \subset \mathbb{C}$ is a smooth and projective variety over K such that the complex manifold associated to $X_{\mathbb{C}}$ is a hyper-Kähler manifold. Proposition 4.2.1 leads us to formulate the following conjecture about the motives of hyper-Kähler varieties.

5.1.1. Conjecture. — *Let $K \subset \mathbb{C}$ be an algebraically closed field. Let X be a hyper-Kähler variety over K, and let $\mathcal{H}^\bullet(X) = \bigoplus_i \mathcal{H}^i(X) \in \mathsf{AM}_K$ be its motive. Then:*

(i) the even part $\mathcal{H}^+(X) = \bigoplus_i \mathcal{H}^{2i}(X)$ of the motive of X belongs to the Tannakian category $\langle \mathcal{H}^2(X) \rangle \subset \mathsf{AM}_K$;

(ii) if X has non-trivial cohomology in some odd degree, then, for any Kuga–Satake variety A for the Hodge structure $H^2(X)$, the motive $\mathcal{H}^\bullet(X)$ belongs to the Tannakian category $\langle \mathcal{H}^1(A) \rangle \subset \mathsf{AM}_K$.

In any case, the motive of X is abelian.

Note that the results in Chapter 3 provide evidence towards the conjecture above. We will make use of the following fundamental result due to André [**3**].

5.1.2. Theorem. — *Let X be a hyper-Kähler variety over a field $K \subset \mathbb{C}$. Assume that $b_2(X) > 3$. Then the motive $\mathcal{H}^2(X_{\bar{K}})$ is abelian. If K is finitely generated over $\mathbb{Q}$, the motivic Mumford–Tate conjecture holds for $\mathcal{H}^2(X)$.*

This result implies the Tate conjecture for divisors on hyper-Kähler varieties with $b_2 > 3$. The assumption on the second Betti number in Theorem 5.1.2 ensures that X admits non-trivial deformations; this is crucial for André's argument. By Hodge theory, any hyper-Kähler variety X satisfies $b_2(X) \geq 3$, and conjecturally ([**10**]) we should always have $b_2(X) > 3$, but this is not known at present time. Moonen [**63**] refined André's method and proved the Tate and Mumford–Tate conjectures for divisors for varieties with H^2 of K3-type and that admit a non-isotrivial deformation.

5.1.3. — Let $K \subset \mathbb{C}$ be an algebraically closed field. To any hyper-Kähler variety X over K with $b_2(X) > 3$, we now attach its *defect group* $P(X)$, an algebraic group which measures the failure of Conjecture 5.1.1 for X.

Let $\mathcal{H}^\bullet(X) \in \mathsf{AM}_K$ be the motive of X, and let $\mathcal{H}^+(X) = \bigoplus_i \mathcal{H}^{2i}(X)$ denote the even part of $\mathcal{H}^\bullet(X)$. The inclusions of $\mathcal{H}^2(X)$ in $\mathcal{H}^+(X)$ and $\mathcal{H}^\bullet(X)$ determine surjective morphisms of motivic Galois groups,

$$\pi_{2,\mathrm{mot}} \colon \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \longrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X)),$$
$$\pi^+_{2,\mathrm{mot}} \colon \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X)) \longrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X)).$$

Let $\iota \in \mathrm{GL}(H^\bullet(X))$ be multiplication by $(-1)^j$ on $H^j(X)$.

5.1.4. Definition. — *Let X be a hyper-Kähler variety over the algebraically closed field $K \subset \mathbb{C}$ with $b_2(X) > 3$.*

(i) The even defect group $P^+(X) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X))$ is the kernel

$$P^+(X) := \ker\Big(\pi^+_{2,\mathrm{mot}} \colon \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X))\Big).$$

If the odd cohomology of X is trivial, the defect group $P(X) \subset \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$ is by definition $P^+(X)$.

(ii) Assume that X has non-trivial cohomology in odd degree. Let

$$\widetilde{P}(X) := \ker\Big(\pi_{2,\mathrm{mot}} \colon \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X))\Big).$$

The defect group $P(X)$ of X is by definition the quotient $\widetilde{P}(X)/\langle\iota\rangle$. We will sometimes call $\widetilde{P}(X)$ the extended defect group of X.

5.1.5. Remark. — Note that ι belongs to $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$, since it fixes all motivated cycles in $\langle H^\bullet(X)\rangle \subset \mathsf{HS}$. Hence the defect group is well defined.

In light of Conjecture 5.1.1, we expect the defect group to be always trivial, see Conjecture 5.2.5 below. We will prove this for all hyper-Kähler varieties of known deformation type. A much weaker expectation is that, in presence of odd cohomology, the short exact sequence defining the defect group should split: $\widetilde{P}(X) = \langle \iota \rangle \times P(X)$.

5.1.6. — The following is the key property of the defect group. Recall (§2.2.5) that the motivic Galois group $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$ contains naturally the Mumford–Tate group $\mathrm{MT}(H^\bullet(X))$. Our result shows that the defect group is a complement.

5.1.7. Theorem. — *Let $K \subset \mathbb{C}$ be an algebraically closed field, and let X be a hyper-Kähler variety over K such that $b_2(X) > 3$. Then, the even defect group $P^+(X)$ is a direct complement of* $\mathrm{MT}(H^+(X))$ *inside* $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X))$ *:*

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X)) = \mathrm{MT}(H^+(X)) \times P^+(X).$$

In presence of non-trivial cohomology in odd degree, the extended defect group $\widetilde{P}(X)$ is a complement of $\mathrm{MT}(H^\bullet(X))$ *in* $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$*, in the sense that*

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) = \mathrm{MT}(H^\bullet(X)) \cdot \widetilde{P}(X),$$

the subgroups $\widetilde{P}(X)$ and $\mathrm{MT}(H^\bullet(X))$ *commute and intersect in $\langle \iota \rangle$.*

5.1.8. Remark. — Since in presence of non-trivial cohomology in odd degree the defect group is defined as $P(X) = \widetilde{P}(X)/\langle \iota \rangle$, in this case we have

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))/\langle \iota \rangle = \mathrm{MT}(H^\bullet(X))/\langle \iota \rangle \times P(X).$$

Proof of Theorem 5.1.7. — We first treat the statement about the even defect group. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X)) \arrow[r, two heads, "\pi^+_{2,\mathrm{mot}}"] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X)) \\ \mathrm{MT}(H^+(X)) \arrow[r, two heads, "\pi^+_2"] \arrow[u, hook, "i_+"] & \mathrm{MT}(H^2(X)) \arrow[u, hook, "i_2"'] \end{tikzcd}$$

Here, i_+ and i_2 denote the natural inclusions; π_2^+ and i_2 are isomorphisms due to Proposition 4.2.1 and Theorem 5.1.2 respectively. We deduce that $s = i_+ \circ (i_2 \circ \pi_2^+)^{-1}$ is a section of $\pi^+_{2,\mathrm{mot}}$. The image of s is $\mathrm{MT}(H^+(X))$.

As $P^+(X)$ is defined as the kernel of the map $\pi^+_{2,\mathrm{mot}}$, the group $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X))$ is generated by the subgroups $P^+(X)$ and $\mathrm{MT}(H^+(X))$, which intersect trivially. In order to show that $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X)) = \mathrm{MT}(H^+(X)) \times P^+(X)$, we are left to prove that $P^+(X)$ and $\mathrm{MT}(H^+(X))$ commute. By Lemma 4.2.2, it suffices to show that $P^+(X)$ commutes with the image of the twisted LLV-representation R^+ (Definition 1.5.5). The action of $P^+(X)$ on $H^+(X)$ is by graded algebra automorphisms: clearly $P^+(X)$ preserves the cohomological grading and its action is compatible with the algebra structure since the cup-product is induced by an algebraic correspondence, namely, by the class of the image of the diagonal embedding $X \hookrightarrow X \times X \times X$. By Lemma 1.2.4.(*ii*), the action of $P^+(X)$ commutes with the image of R^+.

In presence of non-trivial cohomology in odd degree, the proof is similar. As above, the algebraic group $\widetilde{P}(X)$ acts on $H^\bullet(X)$ by graded algebra automorphisms trivial on $H^2(X)$; hence, its action commutes with the LLV-representation R on the cohomology. Since, by Lemma 4.2.2, the Mumford–Tate group is contained in the image of R, the subgroups $\widetilde{P}(X)$ and $\mathrm{MT}(H^\bullet(X))$ commute. Since the odd cohomology of X is non-trivial, ι is the only non-trivial element in the image of R which is trivial on $H^2(X)$, by Remark 1.5.8, and hence $\widetilde{P}(X) \cap \mathrm{MT}(H^\bullet(X)) = \langle \iota \rangle$.

Finally, consider the commutative diagram

$$\begin{tikzcd} \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \arrow[r, "\pi_{2,\mathrm{mot}}"] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X)) \\ \mathrm{MT}(H^\bullet(X)) \arrow[u, hook, "i"'] \arrow[r, "\sim"] & \mathrm{MT}(H^1(A)) \arrow[u, "2:1"] \end{tikzcd}$$

where A is any Kuga–Satake abelian variety for $H^2(X)$; the horizontal map on the bottom is an isomorphism by Proposition 4.2.1.(*ii*), while the vertical homomorphism on the right hand side is an isogeny of degree 2, by Corollary 4.1.8. As $\widetilde{P}(X)$ is by definition the kernel of $\pi_{2,\mathrm{mot}}$, we have $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) = \mathrm{MT}(H^\bullet(X)) \cdot \widetilde{P}(X)$. □

5.1.9. Remark. — The inclusion $\mathcal{H}^+(X) \subset \mathcal{H}^\bullet(X)$ yields a quotient homomorphism $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X))$. It induces a surjective homomorphism $\widetilde{P}(X) \to P^+(X)$ which factors through a quotient $P(X) \to P^+(X)$, because, by definition, $P(X) = \widetilde{P}(X)/\langle \iota \rangle$ and ι is trivial on $H^+(X)$.

5.1.10. — As apparent from the definition, in presence of non-trivial cohomology in odd degree, the defect group does not, a priori, act on $H^\bullet(X)$, but only after passing to a cover of degree 2. We expect that it should always be possible to lift the defect group to a subgroup of $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$.

5.1.11. Lemma. — *In presence of non-trivial odd cohomology, the following are equivalent:*

(i) the short exact sequence

$$1 \to \langle \iota \rangle \to \widetilde{P}(X) \to P(X) \to 1$$

splits: $\widetilde{P}(X) = \langle \iota \rangle \times P(X)$*;*

(ii) for any Kuga–Satake variety A *for* $H^2(X)$*, we have*

$$\langle \mathcal{H}^1(A) \rangle \subset \langle \mathcal{H}^\bullet(X) \rangle \subset \mathsf{AM}_K\,.$$

Proof. — Assume that (i) holds. We identify $P(X)$ with a subgroup of $\widetilde{P}(X)$ such that $\widetilde{P}(X) = \langle \iota \rangle \times P(X)$. Since $\iota \in \mathrm{MT}(H^\bullet(X))$, Theorem 5.1.7 then implies that $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) = \mathrm{MT}(H^\bullet(X)) \times P(X)$. The projection $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \twoheadrightarrow \mathrm{MT}(H^\bullet(X))$ corresponds to a Tannakian subcategory $\mathsf{C} \subset \langle \mathcal{H}^\bullet(X) \rangle$; by construction, we have $\mathrm{G}_{\mathrm{mot}}(\mathsf{C}) \cong \mathrm{MT}(H^\bullet(X))$. This implies that $r(\mathsf{C}) = \langle H^\bullet(X) \rangle \subset \mathsf{HS}^{\mathrm{pol}}$, where r is the realization functor.

It follows that $\mathrm{G}_{\mathrm{mot}}(\mathsf{C}) = \mathrm{MT}(r(\mathsf{C}))$: any Hodge class in C is motivated. By Proposition 4.2.1.(ii), the category $r(\mathsf{C}) \subset \mathsf{HS}^{\mathrm{pol}}$ is the Kuga–Satake category attached to $H^2(X)$ (see Theorem 4.1.6). It follows that C consists of abelian motives. Then, by Corollary 4.1.8, we have $\mathsf{C} = \langle \mathcal{H}^1(A) \rangle$ for any Kuga–Satake variety A for $H^2(X)$.

Assume conversely that (ii) holds. Let A be a Kuga–Satake abelian variety for $H^2(X)$, and define

$$P'(X) \coloneqq \ker\Big(\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \xrightarrow{\pi_{A,\mathrm{mot}}} \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^1(A))\Big).$$

Clearly, $P'(X)$ is contained in the extended defect group $\widetilde{P}(X)$. Moreover $\iota \notin P'(X)$, since ι is multiplication by -1 on $H^1(A)$. Theorem 5.1.7 then implies that $P'(X)$ and $\mathrm{MT}(H^\bullet(X))$ commute and have trivial intersection. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \arrow[r, two heads, "\pi_{A,\mathrm{mot}}"] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^1(A)) \\
\mathrm{MT}(H^\bullet(X)) \arrow[u, hook, "i"] \arrow[r, two heads, "\pi_A"] & \mathrm{MT}(H^1(A)) \arrow[u, hook, "i_A"']
\end{tikzcd}$$

The morphisms π_A and i_A are isomorphisms by Proposition 4.2.1.(ii) and Theorem 2.2.8 respectively. We deduce the existence of a section of $\pi_{A,\mathrm{mot}}$ with image $\mathrm{MT}(H^\bullet(X))$. It follows that

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) = \mathrm{MT}(H^\bullet(X)) \times P'(X),$$

and therefore $\widetilde{P}(X) = \langle \iota \rangle \times P'(X)$. $\square$

5.1.12. Remark. — By the uniqueness of the Kuga–Satake category, the group $P'(X)$ defined in the above proof does not depend on the choice of the Kuga–Satake variety A. If the equivalent conditions of Lemma 5.1.11 are satisfied, we identify the defect group $P(X)$ with the kernel of the homomorphism $\pi_{A,\mathrm{mot}}\colon \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^1(A))$, for any Kuga–Satake abelian variety A for $H^2(X)$. Then the defect group becomes a direct complement of the Mumford–Tate group, as in the even case.

5.1.13. — From the Tannakian point of view, Theorem 5.1.7 has the following consequence. We denote by $\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K)$ the pro-algebraic group attached via Tannaka duality to the whole category AM_K. Tannakian subcategories of AM_K correspond to quotients of $\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K)$; we let $\pi_X\colon \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$ be the surjective homomorphism corresponding to $\langle \mathcal{H}^\bullet(X)\rangle \subset \mathsf{AM}_K$. Theorem 5.1.7 gives a natural Tannakian subcategory of $\langle \mathcal{H}^\bullet(X)\rangle$, corresponding to the quotient homomorphism

$$\pi'_X\colon \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K) \to P(X).$$

This is the subcategory of $\langle \mathcal{H}^\bullet(X)\rangle$ consisting of motives whose Hodge realization is a trivial Hodge structure, isomorphic to $\mathbb{Q}(0)^{\oplus k}$ for some $k \geq 0$. We refer to it as the *Hodge-trivial part* of $\langle \mathcal{H}^\bullet(X)\rangle \subset \mathsf{AM}_K$; conjecturally, any motive in this category should be a sum of copies of the unit motive $\mathbb{1} \in \mathsf{AM}_K$. By [**25**, Proposition 3.1] there exists a tensor construction

$$\mathcal{T} = \mathcal{H}^\bullet(X)^{\otimes n} \otimes \mathcal{H}^\bullet(X)^{\vee,\otimes m}(j) \ \in \mathsf{AM}_K,$$

with Hodge realization denoted by T, such that $P(X)$ acts faithfully on the subspace $W = T^{\mathrm{MT}(H^\bullet(X))} \subset T$ of Hodge classes. Then W is a faithful $P(X)$-module, and hence it is the Hodge realization of a submotive $\mathcal{W} \subset \mathcal{T}$. For any motive $\mathcal{W}$ obtained in this way, the Tannakian category corresponding to π'_X is $\langle \mathcal{W}\rangle \subset \mathsf{AM}_K$.

5.2. What does the defect group measure?

Let $K \subset \mathbb{C}$ be an algebraically closed field. Theorem 5.1.7 has the following consequence, which justifies the name "defect group".

5.2.1. Corollary. — *For any hyper-Kähler variety X over K with $b_2(X) > 3$, the following conditions are equivalent:*

(i^+) The even defect group $P^+(X)$ is trivial.

(ii^+) The even André motive $\mathcal{H}^+(X)$ lies in the Tannakian subcategory of AM_K generated by $\mathcal{H}^2(X)$.

(iii^+) The motive $\mathcal{H}^+(X)$ is abelian.

(iv^+) Conjecture 2.2.7 holds for $\mathcal{H}^+(X)$: $\mathrm{MT}(H^+(X)) = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X))$.

Similarly, in presence of non-trivial cohomology in odd degree, we have the following equivalent conditions:

(i) The defect group $P(X)$ is trivial (equivalently, the extended defect group $\widetilde{P}(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$).

(ii) The André motive $\mathcal{H}^\bullet(X)$ lies in the Tannakian subcategory of AM_K generated by $\mathcal{H}^1(A)$, where A is any Kuga–Satake abelian variety associated to $H^2(X)$.

(iii) The motive $\mathcal{H}^\bullet(X)$ is abelian.

(iv) Conjecture 2.2.7 holds for $\mathcal{H}^\bullet(X)$: $\mathrm{MT}(H^\bullet(X)) = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$.

Proof. — We first treat the even motive. It follows immediately from Theorem 5.1.7 that (i^+) and (iv^+) are equivalent.

(i^+) implies (ii^+): by the definition of $P^+(X)$, if it is trivial, then the natural surjection $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X))$ is an isomorphism. Then (ii^+) follows from Tannaka duality. The implication from (ii^+) to (iii^+) is due to the fact that $\mathcal{H}^2(X)$ is an abelian motive, by Theorem 5.1.2. Finally, (iii^+) implies (iv^+) by Theorem 2.2.8.

If the odd cohomology of X is trivial, we are done. Otherwise, the proof of the second statement is similar: the equivalence of (i) and (iv) is an immediate consequence of Theorem 5.1.7; it is obvious that (ii) implies (iii), and (iii) implies (iv) by Theorem 2.2.8. Finally, let us show how (i) implies (ii). If $P(X)$ is trivial and A is any Kuga–Satake abelian variety for $H^2(X)$, Lemma 5.1.11 implies that $\mathcal{H}^1(A) \in \langle \mathcal{H}^\bullet(X) \rangle$. By Remark 5.1.12, the defect group is then identified with the kernel of $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \to \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^1(A))$. Hence this homomorphism is an isomorphism, which, by Tannaka duality, means that $\langle \mathcal{H}^\bullet(X) \rangle = \langle \mathcal{H}^1(A) \rangle \subset \mathsf{AM}_K$. $\square$

5.2.2. — Assume now that $K \subset \mathbb{C}$ is a field which is finitely generated over $\mathbb{Q}$, with algebraic closure $\bar{K} \subset \mathbb{C}$, and let ℓ be a fixed prime number. The notion of defect group helps us in studying the Mumford–Tate conjecture for hyper-Kähler varieties.

5.2.3. Proposition. — *Let X be a hyper-Kähler variety over K, and assume that $b_2(X) > 3$.*

(i) *If $P^+(X_{\bar{K}})$ is finite, then the Mumford–Tate conjecture 2.1.6 holds for the even part of the cohomology of X, i.e. the isomorphism $H^+(X) \otimes \mathbb{Q}_\ell \cong H^+_\ell(X)$ induces an isomorphism $\mathrm{MT}(H^+(X)) \otimes \mathbb{Q}_\ell \cong \mathcal{G}(H^+_\ell(X))^0$. Moreover, the even defect group $P^+(X_{\bar{K}})$ is trivial if and only if the motivic Mumford–Tate conjecture 2.2.10 holds for the motive $\mathcal{H}^+(X)$.*

(ii) *Similarly, if $P(X_{\bar{K}})$ is finite, then the Mumford–Tate conjecture 2.1.6 holds for X; moreover, the defect group $P(X_{\bar{K}})$ is trivial if and only if the motivic Mumford–Tate conjecture 2.2.10 holds for the motive $\mathcal{H}^\bullet(X)$.*

Proof. — Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{MT}(H^+(X)) \otimes \mathbb{Q}_\ell \arrow[r, hook, "\sim"] \arrow[d, "\wr"'] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X_{\bar{K}}))^0 \otimes \mathbb{Q}_\ell \arrow[d, two heads] & \mathcal{G}(H^+_\ell(X))^0 \arrow[l, hook'] \arrow[d, two heads] \\
\mathrm{MT}(H^2(X)) \otimes \mathbb{Q}_\ell \arrow[r, "\sim"] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(X_{\bar{K}})) \otimes \mathbb{Q}_\ell & \mathcal{G}(H^2_\ell(X))^0 \arrow[l, "\sim"']
\end{tikzcd}$$

The two horizontal morphisms on the bottom are isomorphisms due to Theorem 5.1.2, the vertical map on the left is an isomorphism thanks to Proposition 4.2.1, and the top left horizontal arrow is an isomorphism by Theorem 5.1.7 since $P^+(X_{\bar{K}})$ is a finite group by assumption. Then all arrows in the diagram are isomorphisms, and hence

$$\mathcal{G}(H^+_\ell(X))^0 \cong \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X_{\bar{K}}))^0 \otimes \mathbb{Q}_\ell \cong \mathrm{MT}(H^+(X)) \otimes \mathbb{Q}_\ell.$$

If $P^+(X_{\bar{K}})$ is trivial, the group $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X_{\bar{K}}))$ is connected, and we conclude that the motivic Mumford–Tate conjecture 2.2.10 holds for $\mathcal{H}^+(X)$. Conversely, assume that the motivic Mumford–Tate conjecture holds for $\mathcal{H}^+(X)$. Then, in particular, we have $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X_{\bar{K}})) = \mathrm{MT}(H^+(X))$, which forces the triviality of the even defect group $P^+(X_{\bar{K}})$ by Theorem 5.1.7. This proves (i).

Assume now that the odd cohomology of X is non-trivial and that $P(X_{\bar{K}})$ is finite. It then follows that also the extended defect group $\widetilde{P}(X_{\bar{K}})$ is finite. Then, by Theorem 5.1.7, the algebraic groups $\mathrm{MT}(H^\bullet(X))$ and $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))$ have the same dimension, and hence the inclusion $\mathrm{MT}(H^\bullet(X)) \hookrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X))^0$ is an isomorphism.

The even defect group $P^+(X_{\bar{K}})$ is also finite, being a quotient of $P(X_{\bar{K}})$ (see Remark 5.1.5). We consider another commutative diagram

$$\begin{tikzcd}
\mathrm{MT}(H^\bullet(X)) \otimes \mathbb{Q}_\ell \arrow[r, hook, "\sim"] \arrow[d] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X_{\bar{K}}))^0 \otimes \mathbb{Q}_\ell \arrow[d, two heads] & \mathcal{G}(H^\bullet_\ell(X))^0 \arrow[l, hook'] \arrow[d, two heads] \\
\mathrm{MT}(H^+(X)) \otimes \mathbb{Q}_\ell \arrow[r, "\sim"] & \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(X_{\bar{K}}))^0 \otimes \mathbb{Q}_\ell & \mathcal{G}(H^+_\ell(X))^0 \arrow[l, "\sim"']
\end{tikzcd}$$

The horizontal arrows on the bottom are isomorphisms due to (i); the top left horizontal map is an isomorphism by the above discussion, while the leftmost vertical arrow is an isogeny due to Proposition 4.2.1. It follows that the other vertical maps are isogenies as well. We deduce that $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X_{\bar{K}}))^0 \otimes \mathbb{Q}_\ell$ and $\mathcal{G}(H^\bullet_\ell(X))^0$ are connected algebraic groups of the same dimension; hence,

$$\mathcal{G}(H^\bullet_\ell(X))^0 \hookrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X_{\bar{K}}))^0 \otimes \mathbb{Q}_\ell$$

has to be an isomorphism.

If $P(X_{\bar{K}})$ is actually trivial, then $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X_{\bar{K}})) = \mathrm{MT}(H^\bullet(X))$ is connected, and the motivic Mumford–Tate conjecture holds for $\mathcal{H}^\bullet(X)$. Conversely, the motivic Mumford–Tate conjecture for $\mathcal{H}^\bullet(X)$ implies that $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X_{\bar{K}})) = \mathrm{MT}(H^\bullet(X))$, and then $P(X_{\bar{K}})$ has to be trivial by Theorem 5.1.7. □

5.2.4. — By the above results, Conjecture 5.1.1, the Mumford–Tate conjecture 2.1.6 and Conjecture 2.2.7 for hyper-Kähler varieties with $b_2 > 3$ are all equivalent to the following conjecture.

5.2.5. Conjecture. — *The defect group $P(X)$ of any hyper-Kähler variety X over an algebraically closed field $K \subset \mathbb{C}$ with $b_2(X) > 3$ is trivial.*

In the next Chapter, we will prove this conjecture for all hyper-Kähler varieties of known deformation type.

5.3. Deformation invariance of defect groups

A remarkable property of defect groups is their deformation invariance.

5.3.1. Theorem. — *Let S be a smooth and connected complex variety, and let $\mathfrak{X} \to S$ be a smooth projective morphism with fibres hyper-Kähler varieties $\mathfrak{X}_s$ such that $b_2(\mathfrak{X}_s) > 3$. Then, for any $s, s' \in S$, the defect groups $P(\mathfrak{X}_s)$ and $P(\mathfrak{X}_{s'})$ are isomorphic; similarly, the even defect groups $P^+(\mathfrak{X}_s)$ and $P^+(\mathfrak{X}_{s'})$ are isomorphic.*

Proof. — The family $\mathfrak{X} \to S$ determines motives in AM_S (see Definition 2.3.3):

$$\mathcal{H}^\bullet(\mathfrak{X}/S) = \bigoplus_i \mathcal{H}^i(\mathfrak{X}/S), \quad \mathcal{H}^+(\mathfrak{X}/S) = \bigoplus_i \mathcal{H}^{2i}(\mathfrak{X}/S),$$

with Hodge realization

$$H^\bullet(\mathfrak{X}/S) = \bigoplus_i H^i(\mathfrak{X}/S), \quad H^+(\mathfrak{X}/S) = \bigoplus_i H^{2i}(\mathfrak{X}/S),$$

respectively. Here, $H^i(\mathfrak{X}/S)$ denotes the variation of Hodge structures $R^i f_* \mathbb{Q}_{\mathfrak{X}}$. Upon taking an étale cover of S, we may assume that the algebraic monodromy group $\mathrm{G}_{\mathrm{mono}}(H^\bullet(\mathfrak{X}/S))$ is connected.

We prove first the invariance of the even defect group. Since $\mathcal{H}^2(\mathfrak{X}/S) \subset \mathcal{H}^+(\mathfrak{X}/S)$, we have a natural morphism of generic motivic Galois groups

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}/S)) \twoheadrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^2(\mathfrak{X}/S)).$$

Let $P^+(\mathfrak{X}/S)$ denote the kernel of this morphism. By Theorem 2.3.6, for a very general point $s_0 \in S$, we have the equalities $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}/S))_{s_0} = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}_{s_0}))$ and $\mathrm{MT}(H^+(\mathfrak{X}/S))_{s_0} = \mathrm{MT}(H^+(\mathfrak{X}_{s_0}))$. Hence, $P^+(\mathfrak{X}/S)_{s_0} = P^+(\mathfrak{X}_{s_0})$ and

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}/S))_{s_0} = \mathrm{MT}(H^+(\mathfrak{X}/S))_{s_0} \times P^+(\mathfrak{X}/S)_{s_0},$$

by Theorem 5.1.7. Since the monodromy group is connected, it is a subgroup of $\mathrm{MT}(H^+(\mathfrak{X}/S))$. Therefore, it commutes with $P^+(\mathfrak{X}/S)$; in other words, $P^+(\mathfrak{X}/S)$ is a constant local system of algebraic groups over S. We obtain a splitting

$$\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}/S)) = \mathrm{MT}(H^+(\mathfrak{X}/S)) \times P^+(\mathfrak{X}/S)$$

of local systems of algebraic groups over S. For any $s \in S$, the inclusion of $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}_s))$ into $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}/S))_s$ is the direct product of inclusions

$$\mathrm{MT}(H^+(\mathfrak{X}_s)) \hookrightarrow \mathrm{MT}(H^+(\mathfrak{X}/S))_s \text{ and } P^+(\mathfrak{X}_s) \hookrightarrow P^+(\mathfrak{X}/S)_s.$$

It is enough to show that for all $s \in S$, the equality $P^+(\mathfrak{X}_s) = P^+(\mathfrak{X}/S)_s$ holds. By Corollary 2.3.8, for all $s \in S$, we have

$$\mathrm{G}_{\mathrm{mono}}(H^+(\mathfrak{X}/S))_s \cdot \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}_s)) = \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^+(\mathfrak{X}/S))_s.$$

But we know that $\mathrm{G}_{\mathrm{mono}}(H^+(\mathfrak{X}/S))_s$ is contained in

$$\mathrm{MT}(H^+(\mathfrak{X}/S))_s \times \{1\} \subset \mathrm{MT}(H^+(\mathfrak{X}/S))_s \times P^+(\mathfrak{X}/S)_s,$$

and therefore

$$\begin{aligned}\mathrm{G_{mono}}(H^+(\mathfrak{X}/S))_s \cdot \mathrm{G_{mot}}(\mathcal{H}^+(\mathfrak{X}_s)) = \\ &= \mathrm{G_{mono}}(H^+(\mathfrak{X}/S))_s \cdot (\mathrm{MT}^+(\mathfrak{X}_s) \times P^+(\mathfrak{X}_s)) \\ &= (\mathrm{G_{mono}}(H^+(\mathfrak{X}/S))_s \cdot \mathrm{MT}^+(\mathfrak{X}_s)) \times P^+(\mathfrak{X}_s) \\ &= \mathrm{MT}^+(H^+(\mathfrak{X}/S))_s \times P^+(\mathfrak{X}_s),\end{aligned}$$

which forces $P^+(\mathfrak{X}_s) = P^+(\mathfrak{X}/S)_s$.

In presence of non-trivial odd cohomology, the proof is similar. The inclusion $\mathcal{H}^2(\mathfrak{X}/S) \subset \mathcal{H}^\bullet(\mathfrak{X}/S)$ gives a surjective homomorphism of generic motivic Galois groups

$$\mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}/S)) \twoheadrightarrow \mathrm{G_{mot}}(\mathcal{H}^2(\mathfrak{X}/S));$$

let $\widetilde{P}(\mathfrak{X}/S)$ denote its kernel. Then $\widetilde{P}(\mathfrak{X}/S)$ is a local system of algebraic groups over S, and it contains a central sub-local system $\langle\iota\rangle_S$ with fibre $\langle\iota\rangle \cong \mathbb{Z}/2\mathbb{Z}$. We denote by $P(\mathfrak{X}/S)$ the quotient local system $\widetilde{P}(\mathfrak{X}/S)/\langle\iota\rangle_S$. By Theorem 2.3.6, at a very general point $s_0 \in S$ the fibre $\widetilde{P}(\mathfrak{X}/S)_{s_0}$ (resp. $P(\mathfrak{X}/S)_{s_0}$) is the extended defect group $\widetilde{P}(\mathfrak{X}_{s_0})$ (resp. the defect group $P(\mathfrak{X}_{s_0})$) of the fibre $\mathfrak{X}_{s_0}$.

Reasoning as above, from Theorem 2.3.6 and Theorem 5.1.7 we obtain that $\widetilde{P}(\mathfrak{X}/S)$ is a constant local system over S and $\mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}/S)) = \mathrm{MT}(H^\bullet(\mathfrak{X}/S)) \cdot \widetilde{P}(\mathfrak{X}/S)$, with $\widetilde{P}(\mathfrak{X}/S)$ and $\mathrm{MT}(H^\bullet(\mathfrak{X}/S))$ commuting with each other and intersecting in $\langle\iota\rangle_S$. For any $s \in S$, the inclusion of $\mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}_s))$ into $\mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}/S))_s$ is given by inclusions

$$\mathrm{MT}(H^\bullet(\mathfrak{X}_s)) \hookrightarrow \mathrm{MT}(H^\bullet(\mathfrak{X}/S))_s \quad \text{and} \quad \widetilde{P}(\mathfrak{X}_s) \hookrightarrow \widetilde{P}(\mathfrak{X}/S)_s.$$

It suffices to show that $\widetilde{P}(\mathfrak{X}_s) = \widetilde{P}(\mathfrak{X}/S)_s$ for any $s \in S$. We apply Corollary 2.3.8 as in the even case: for any $s \in S$, we have

$$\mathrm{G_{mono}}(H^\bullet(\mathfrak{X}/S))_s \cdot \mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}_s)) = \mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}/S))_s;$$

on the other hand, $\mathrm{G_{mono}}(H^\bullet(\mathfrak{X}/S))_s \cdot \mathrm{G_{mot}}(\mathcal{H}^\bullet(\mathfrak{X}_s)) \subset \mathrm{MT}(H^\bullet(\mathfrak{X}/S))_s \cdot \widetilde{P}(\mathfrak{X}_s)$, which forces $\widetilde{P}(\mathfrak{X}_s) = \widetilde{P}(\mathfrak{X}/S)_s$. □

As the proof shows, for a family $\mathfrak{X} \to S$ as above with connected mondromy, the defect groups of the fibres arrange into a constant local system $P(\mathfrak{X}/S)$ of algebraic groups over S. The Tannakian interpretation of Theorem 5.1.7 leads to the following more precise result, which says that, in such a family, the Hodge-trivial part (see §5.1.13) of the categories $\langle \mathcal{H}^\bullet(\mathfrak{X})_s) \rangle \subset \mathsf{AM}_{\mathbb{C}}$ does not depend on $s \in S$.

5.3.2. Corollary. — *Let $\mathfrak{X} \to S$ be as above and assume that the monodromy group $\mathrm{G}_{\mathrm{mono}}(H^\bullet(\mathfrak{X}/S))$ is connected. Let a, b be points of S. Choose a continuous path γ from a to b, and let $\Xi\colon P(\mathfrak{X}_a) \to P(\mathfrak{X}_b)$ be the isomorphism obtained via parallel transport along γ in the local system $P(\mathfrak{X}/S)$. Then Ξ does not depend on the choice of γ and the diagram*

$$\begin{tikzcd} & P(\mathfrak{X}_a) \arrow[dd, "\Xi"] \\ \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_{\mathbb{C}}) \arrow[ur, two heads, "\pi'_{\mathfrak{X}_a}"] \arrow[dr, two heads, "\pi'_{\mathfrak{X}_b}"'] & \\ & P(\mathfrak{X}_b) \end{tikzcd}$$

is commutative. The analogous statement holds for the even defect groups.

Proof. — Since $\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_{\mathbb{C}}) \twoheadrightarrow P^+(X)$ factors over the quotient $P(X) \to P^+(X)$ by Remark 5.1.5, the statement for defect groups implies that for even defect groups. The local system $P(\mathfrak{X}/S)$ is constant, so Ξ does not depend on the choice of γ.

Consider a motive $\mathcal{T}/S := \mathcal{H}^\bullet(\mathfrak{X}/S)^{\otimes m} \otimes \mathcal{H}^\bullet(\mathfrak{X}/S)^{\vee,\otimes n} \otimes 1_S(j) \in \mathsf{AM}_S$, for integers m, n, j. Let T/S denote the realization of $\mathcal{T}/S$. For any $s \in S$ we let $W_s \subset T_s$ be the subspace of invariants for the generic Mumford–Tate group $\mathrm{MT}(T/S)_s$ at s; this yields a subvariation of Hodge structures $W/S \subset T/S$. Moreover, since $\mathrm{MT}(T/S)_s$ is normal in $\mathrm{G}_{\mathrm{mot}}(\mathcal{T}/S)_s$ by Theorem 5.1.7, the variation W/S is the Hodge realization of a submotive $\mathcal{W}/S \subset \mathcal{T}/S$ over S.

The motive $\mathcal{W}/S$ is a constant motive over S. Indeed, let us denote by $\mathcal{D}$ the motive $\mathcal{W}_b$, and let $\mathcal{D}/S$ be the constant motive over S with fibre $\mathcal{D}$; let D/S be the realization of $\mathcal{D}/S$. Then $\mathrm{id}_b\colon W_b \to D_b$ is monodromy invariant and obviously an isomorphism of motives. By Theorem 2.3.2, this morphism extends to an isomorphism $\mathcal{W}/S \cong \mathcal{D}/S$ in AM_S. It follows that the isomorphism $\Psi\colon W_a \to W_b$ given by parallel transport along γ in the local system W/S is the realization of an isomorphism of motives $\mathcal{W}_a \cong \mathcal{W}_b$. Hence, the induced isomorphism $\Psi_*\colon \mathrm{GL}(W_a) \to \mathrm{GL}(W_b)$ fits into a commutative diagram

$$\begin{tikzcd}
 & \mathrm{G}_{\mathrm{mot}}(\mathcal{W}_a) \arrow[dd, "\Psi_*"] \\
\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_{\mathbb{C}}) \arrow[ur, two heads] \arrow[dr, two heads] & \\
 & \mathrm{G}_{\mathrm{mot}}(\mathcal{W}_b)
\end{tikzcd}$$

Note that since the generic Mumford–Tate group acts trivially on W_s by construction, the group $\mathrm{G}_{\mathrm{mot}}(\mathcal{W}_s)$ is a quotient of the defect group $P(\mathfrak{X}_s)$.

We now choose the tensor construction T/S in such a way that the action of $P(\mathfrak{X}_s)$ on the subspace W_s is faithful; in this case we have $\mathrm{G}_{\mathrm{mot}}(\mathcal{W}_s) = P(\mathfrak{X}_s)$ for all points $s \in S$, and the homomorphism $\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_{\mathbb{C}}) \twoheadrightarrow \mathrm{G}_{\mathrm{mot}}(\mathcal{W}_s)$ is identified with the projection $\pi'_{\mathfrak{X}_s} \colon \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_{\mathbb{C}}) \to P(\mathfrak{X}_s)$. Moreover, $P(\mathfrak{X}/S) \subset \mathrm{GL}(W/S)$ is a sub-local system of algebraic groups, and therefore the isomorphism $\Xi \colon P(\mathfrak{X}_a) \to P(\mathfrak{X}_b)$ obtained via parallel transport along γ in the local system $P(\mathfrak{X}/S)$ is the restriction of the isomorphism $\Psi_* \colon \mathrm{GL}(\mathcal{W}_a) \to \mathrm{GL}(\mathcal{W}_b)$ to $P(\mathfrak{X}_a)$. This concludes the proof. $\square$

CHAPTER 6

APPLICATIONS

6.1. Invariance of the defect group in a deformation class

The following is the key to our applications.

6.1.1. Theorem. — *Let X_1, X_2 be deformation equivalent (in the complex analytic sense) projective hyper-Kähler manifolds. Assume that $b_2(X) > 6$. Then there exist:*

- *finitely many connected and non-singular complex varieties S_i, $i = 1, \dots, N$;*
- *for each $i = 1, \dots, N$, a smooth and projective family $\mathfrak{X}^i \to S_i$ with fibres hyper-Kähler varieties;*
- *for $i = 1, \dots, N$, points $a_i, b_i \in S_i$ together with birational maps*

$$X_1 \dashrightarrow \mathfrak{X}^1_{a_1}, \qquad \mathfrak{X}^i_{b_i} \dashrightarrow \mathfrak{X}^{i+1}_{a_{i+1}}, \text{ for } i = 1, \dots, N-1, \qquad \mathfrak{X}^N_{b_N} \dashrightarrow X_2.$$

This theorem is proven in Appendix A. Combining it with Theorem 5.3.1, we obtain the following result.

6.1.2. Corollary. — *Let Y and Y' be deformation equivalent projective complex hyper-Kähler manifolds with $b_2 > 6$. Then their (even) defect groups are isomorphic:*

$$P(Y) \cong P(Y') \quad (P^+(Y) \cong P^+(Y')).$$

Thus, Conjecture 5.1.1 and the equivalent statements in Corollary 5.2.1 hold for Y if and only if they hold for Y'.

Proof. — Riess [**76**] proved that birationally equivalent hyper-Kähler varieties have isomorphic Chow motives. In particular, they have isomorphic defect groups. Given smooth projective families $\mathfrak{X}^i \to S_i$ and points a_i, b_i as in Theorem 6.1.1, the conclusion follows by repeated application of Theorem 5.3.1. $\square$

6.1.3. Remark. — With notation and assumptions as in the above proof, choose a continuous path γ_i in S_i from a_i to b_i, $i = 1, \ldots, N$. We may assume that the monodromy groups $\mathrm{G}_{\mathrm{mono}}(H^\bullet(\mathfrak{X}^i/S_i))$ are connected. Let $\Xi_i \colon P(\mathfrak{X}^i_{a_i}) \to P(\mathfrak{X}^i_{b_i})$ be the isomorphism induced via parallel transport along γ_i in the local system $P(\mathfrak{X}^i/S_i)$, and let $\Xi\colon P(Y) \to P(Y')$ be the composition $\Xi = \Xi_N \circ \cdots \circ \Xi_1$. Then, by repeated application of Corollary 5.3.2, the isomorphism Ξ fits into a commutative diagram

$$\begin{tikzcd} & P(Y) \arrow[dd, "\Xi"] \\ \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_{\mathbb{C}}) \arrow[ur, two heads, "\pi''_Y"] \arrow[dr, two heads, "\pi''_{Y'}"'] & \\ & P(Y') \end{tikzcd}$$

This implies that the subcategories $\langle \mathcal{H}^\bullet(Y) \rangle$ and $\langle \mathcal{H}^\bullet(Y') \rangle$ of $\mathsf{AM}_{\mathbb{C}}$ share the same Hodge-trivial part (see §5.1.13).

6.2. Motives of known hyper-Kähler varieties

In this section we use the defect group to establish Conjecture 5.2.5 and the Mumford–Tate conjecture for all known hyper-Kähler varieties. These results have appeared in the joint work with Lie Fu and Ziyu Zhang [**29**].

In what follows, we will say that a hyper-Kähler variety X over a field $K \subset \mathbb{C}$ is *known* if the base change $X_{\mathbb{C}}$ is deformation equivalent (in the complex analytic sense) to one of the known examples. This means that $X_{\mathbb{C}}$ is either a K3 surface, or of one of the deformation types $\mathrm{K3}^{[n]}$, Kum^n, OG10 or OG6. The second Betti number of X is 22 if X is a K3 surface, and 23, 7, 24 and 8 for X of type $\mathrm{K3}^{[n]}$, Kum^n, OG10 and OG6 respectively.

6.2.1. Theorem. — *Let X be a known hyper-Kähler variety over an algebraically closed field $K \subset \mathbb{C}$. Then its defect group $P(X)$ is trivial. Hence, the motive of X is abelian and satisfies Conjectures 5.1.1 and 2.2.7.*

Proof. — Since K is algebraically closed, for any hyper-Kähler variety X over K we have an isomorphism $\mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X)) \cong \mathrm{G}_{\mathrm{mot}}(\mathcal{H}^\bullet(X_{\mathbb{C}}))$, see §2.2.5. Thanks to Corollary 6.1.2 and Corollary 5.2.1, it suffices to find a projective complex hyper-Kähler manifold with abelian André motive in each of the known deformation classes.

- For K3 surfaces, this follows immediately from Theorem 5.1.2.
- For the $\mathrm{K3}^{[n]}$-type, de Cataldo–Migliorini [**17**] described the motive of a Hilbert scheme on a K3 surface in terms of the motive of the surface. The André motive of such a Hilbert scheme is abelian, because motives of K3 surfaces are so. Alternatively, we could have used Bülles' result (Theorem 3.1.2).
- For the Kum^n-type, a motivic decomposition of a generalized Kummer variety associated to an abelian surface was obtained in [**90**] and [**33**, Corollary 6.3] using the work of de Cataldo–Migliorini [**18**] on semi-small resolutions. This decomposition shows that the motive of the generalized Kummer variety belongs to the category generated by the motive of the abelian surface.
- For the OG10-type, we use Theorem 3.3.6.
- Finally, for the OG6-type, as observed by Soldatenkov in [**79**], we can use a construction from [**60**] which describes a hyper-Kähler variety in this deformation class as the quotient of a hyper-Kähler variety of $\mathrm{K3}^{[3]}$-type by a birational involution (with well-understood indeterminacy loci). Since varieties of $\mathrm{K3}^{[n]}$-type have abelian motive, this yields a variety of OG6-type with abelian motive.

□

6.2.2. — Let now $K \subset \mathbb{C}$ be a field finitely generated over $\mathbb{Q}$, let $\bar{K} \subset \mathbb{C}$ be the algebraic closure of K, and let ℓ be a prime number. We define the category

$$\mathsf{HK}_K^{\mathrm{known}} \subset \mathsf{AM}_K$$

as the Tannakian subcategory generated by the motives of all known hyper-Kähler varieties over K. This category contains already the motive of cubic fourfolds, as their motives belong to the category generated by the motives of their Fano varieties of lines (see for example [**49**]). Very likely, $\mathsf{HK}_K^{\mathrm{known}}$ also contains the motive of some interesting Fano varieties whose cohomology is of K3-type, for instance, Gushel–Mukai varieties [**40**] [**66**], Debarre–Voisin Fano varieties [**21**] and many more, see [**26**].

6.2.3. Theorem. — *The motivic Mumford–Tate Conjecture 2.2.10 holds for any motive $\mathcal{M} \in \mathsf{HK}_K^{\text{known}}$. In particular, if Y is any smooth and projective variety over K such that $\mathcal{H}^\bullet(Y) \in \mathsf{HK}_K^{\text{known}}$, then the Hodge and Tate conjectures for any power Y^m are equivalent.*

Proof. — For any hyper-Kähler variety X over K of known deformation type, the defect group $P(X_{\bar{K}})$ is trivial, by Theorem 6.2.1. Therefore, Proposition 5.2.3 implies that the motivic Mumford–Tate conjecture holds for $\mathcal{H}^\bullet(X)$.

To conclude the proof, we have to establish the motivic Mumford–Tate conjecture for $\mathcal{H}^\bullet(Y)$, where $Y = X_1 \times \cdots \times X_k$ is any product of known hyper-Kähler varieties over K. With Y as above, we can find a finitely generated field extension L of K such that the motives $\mathcal{H}^\bullet(X_{i,L})$ are abelian. Since we have already shown that the Mumford–Tate conjecture holds for the factors, the motivic Mumford–Tate conjecture holds for $\mathcal{H}^\bullet(Y_L)$ by a result of Commelin [**20**, Theorem 10.3]. But then it holds for $\mathcal{H}^\bullet(Y)$ as well, since the Mumford–Tate conjecture is insensitive to finitely generated field extensions. □

6.2.4. Remark. — Thanks to [**20**], we can put even more generators in the category $\mathsf{HK}_K^{\text{known}}$ to obtain new evidence for the Mumford–Tate conjecture. Since the conjecture is known to hold for

- (i) geometrically simple abelian varieties of prime dimension, by Tankeev [**81**],
- (ii) abelian varieties of dimension g with trivial endomorphism ring over $\bar{K}$ such that $2g$ is neither a k-th power for some odd $k > 1$ nor of the form $\binom{2k}{k}$ for some odd $k > 1$, thanks to Pink [**73**],

we deduce that the Mumford–Tate conjecture holds for any product of varieties in (i) and (ii) above and hyper-Kähler varieties of the known deformation types. See Moonen [**63**] for more potential examples.

6.3. The $\mathcal{H}^2$ determines the full motive

Even when we do not know that the defect group is trivial, we can still show that the motive of a hyper-Kähler variety X is determined by its degree 2 component, analogously to what happens for the Hodge theory of X, cf. Theorem 4.3.1.

6.3.1. Theorem. — *Let $K \subset \mathbb{C}$ be an algebraically closed field, and let X_1 and X_2 be hyper-Kähler varieties over K such that $b_2(X_i) > 6$. Assume that $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent and that $f\colon H^2(X_1) \xrightarrow{\sim} H^2(X_2)$ is a Hodge isometry. If the odd cohomology of X_i is not trivial, assume further that there exists a Kuga–Satake abelian variety A_i for $H^2(X_i)$ such that $\mathcal{H}^1(A_i) \subset \langle \mathcal{H}^\bullet(X_i)\rangle$. Then, there exists an isomorphism of graded algebras $F\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ which is the realization of an isomorphism of motives $\mathcal{H}^\bullet(X_1) \xrightarrow{\sim} \mathcal{H}^\bullet(X_2)$ in AM_K.*

6.3.2. Remark. — The condition that $\langle \mathcal{H}^\bullet(X)\rangle$ contains the motive of a Kuga–Satake abelian variety is deformation invariant, by Corollary 6.1.2 and Lemma 5.1.11.

Proof. — Applying Theorem 6.1.1 we find finitely many smooth projective families $\mathfrak{X}^i \to S_i$ of hyper-Kähler varieties over non-singular and connected complex varieties S_i, for $i = 1, \dots, N$, and points $a_i, b_i \in S_i$ with birational maps

$$X_{1,\mathbb{C}} \dashrightarrow \mathfrak{X}^1_{a_1}; \quad \mathfrak{X}^i_{b_i} \dashrightarrow \mathfrak{X}^{i+1}_{a_{i+1}}, \text{ for } i = 1, \dots, N-1; \quad \mathfrak{X}^N_{b_N} \dashrightarrow X_{2,\mathbb{C}}.$$

We may and will assume that the monodromy groups $\mathrm{G}_{\mathrm{mono}}(H^\bullet(\mathfrak{X}^i/S_i))$ are connected. For $i = 1, \dots, N$, we choose a path γ_i in S_i from a_i to b_i, and we define $\Psi\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ as the composition of the isomorphisms $\Psi_i\colon H^\bullet(\mathfrak{X}^i_{a_i}) \xrightarrow{\sim} H^\bullet(\mathfrak{X}^i_{b_i})$ obtained via parallel transport along γ_i and the isomorphisms $H^\bullet(\mathfrak{X}^i_{b_i}) \cong H^\bullet(\mathfrak{X}^{i+1}_{a_{i+1}})$ induced by the birational maps $\mathfrak{X}^i_{b_i} \dashrightarrow \mathfrak{X}^{i+1}_{a_{i+1}}$. We denote by $\psi\colon H^2(X_1) \xrightarrow{\sim} H^2(X_2)$ the isometry induced by Ψ.

We construct the isomorphism of graded algebras $F\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2)$ as in the proof of Theorem 4.3.1: we may assume that $\psi^{-1} \circ f \in \mathrm{SO}(H^2(X_1))(\mathbb{Q})$; by Hilbert's Theorem 90, the morphism $\pi\colon \mathrm{CSpin}(H^2(X_1)) \to \mathrm{SO}(H^2(X_1))$ is surjective on $\mathbb{Q}$-points and hence we find $g \in \mathrm{CSpin}(H^2(X_1))(\mathbb{Q})$ such that $\pi(g) = \psi^{-1} \circ f$. By Lemma 1.5.7, $R(g)$ is an automorphism of the graded algebra $H^\bullet(X_1)$, and we define

$$F \coloneqq \Psi \circ R(g)\colon H^\bullet(X_1) \xrightarrow{\sim} H^\bullet(X_2).$$

It is an isomorphism of graded algebras. By Remark 1.5.8, the degree 2 component of F is a multiple of the Hodge isometry f; hence, F is an isomorphism of Hodge structures by Proposition 4.3.3.

We claim that F is the realization of an isomorphism $\mathcal{H}^\bullet(X_1) \xrightarrow{\sim} \mathcal{H}^\bullet(X_2)$ of motives. Denoting by $F_*\colon \mathrm{GL}(H^\bullet(X_1)) \xrightarrow{\sim} \mathrm{GL}(H^\bullet(X_2))$ the induced isomorphism, we have to prove that F_* is $\mathrm{G_{mot}}(\mathsf{AM}_K)$-equivariant, i.e. F_* restricts to an isomorphism $F_*\colon \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_1)) \xrightarrow{\sim} \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_2))$ which fits in a commutative diagram

$$\begin{tikzcd} & \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_1)) \arrow[dd, "F_*"] \\ \mathrm{G_{mot}}(\mathsf{AM}_K) \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_2)) \end{tikzcd}$$

Theorem 5.1.7 (and, in presence of non-trivial cohomology in odd degree, Remark 5.1.12, thanks to our extra assumption) gives

$$\mathrm{G_{mot}}(\mathcal{H}^\bullet(X_i)) = P(X_i) \times \mathrm{MT}(H^\bullet(X_i)).$$

The isomorphism F_* preserves this decomposition. Indeed, since F is an isomorphism of Hodge structure, F_* maps $\mathrm{MT}(H^\bullet(X_1))$ isomorphically onto $\mathrm{MT}(H^\bullet(X_2))$. Moreover, the automorphism $R(g)$ of $H^\bullet(X_1)$ commutes with $P(X_1)$ since the defect group commutes with the LLV-representation, and $\Psi_*\colon \mathrm{GL}(H^\bullet(X_1)) \xrightarrow{\sim} \mathrm{GL}(H^\bullet(X_2))$ restricts to an isomorphism $\Xi\colon P(X_1) \xrightarrow{\sim} P(X_2)$, because Ψ is the composition of parallel transport operators along smooth projective families and isomorphisms of motives.

Hence F_* restricts to an isomorphism $F_*\colon \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_1)) \xrightarrow{\sim} \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_2))$ which is the direct product of

$$\Xi\colon P(X_1) \xrightarrow{\sim} P(X_2) \quad \text{and} \quad F_*\colon \mathrm{MT}(H^\bullet(X_1)) \xrightarrow{\sim} \mathrm{MT}(H^\bullet(X_2)).$$

Therefore, it is enough to show that the two diagrams

$$\begin{tikzcd} & P(X_1) \arrow[dd, "\Xi"] \\ \mathrm{G_{mot}}(\mathsf{AM}_K) \arrow[ur, "\pi_1'"] \arrow[dr, "\pi_2'"'] & \\ & P(X_2) \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{MT}(H^\bullet(X_1)) \arrow[dd, "F_*"] \\ \mathrm{G_{mot}}(\mathsf{AM}_K) \arrow[ur, "\pi_1''"] \arrow[dr, "\pi_2''"'] & \\ & \mathrm{MT}(H^\bullet(X_2)) \end{tikzcd}$$

are commutative. Since $\mathrm{G_{mot}}(\mathcal{H}^\bullet(X_i)) \cong \mathrm{G_{mot}}(\mathcal{H}^\bullet(X_{i,\mathbb{C}}))$, the left triangle is commutative by Remark 6.1.3.

For the right one, we proceed as follows. If the odd cohomology of X_i is trivial, the quotient π_i'' corresponds to the Tannakian subcategory $\langle \mathcal{H}^2(X_i)\rangle \subset \langle \mathcal{H}^\bullet(X_i)\rangle$, while otherwise π_i'' corresponds to $\langle \mathcal{H}^1(A_i)\rangle \subset \langle \mathcal{H}^\bullet(X_i)\rangle$, for any Kuga–Satake abelian variety A_i for $H^2(X_i)$. In any case, π_i'' corresponds to a subcategory of abelian motives. It follows that the homomorphism π_i'' factors through $\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K) \to \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K^{\mathrm{ab}})$. Denoting by $\mathsf{HS}^{\mathrm{ab}} \subset \mathsf{HS}^{\mathrm{pol}}$ the Tannakian subcategory generated by the Hodge structures of abelian varieties, Theorem 2.2.8 gives $\mathrm{MT}(\mathsf{HS}^{\mathrm{ab}}) = \mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K^{\mathrm{ab}})$. But then the diagram

$$\begin{tikzcd}
 & \mathrm{MT}(H^\bullet(X_1)) \arrow[dd, "F_*"] \\
\mathrm{G}_{\mathrm{mot}}(\mathsf{AM}_K^{\mathrm{ab}}) = \mathrm{MT}(\mathsf{HS}^{\mathrm{ab}}) \arrow[ur, "\pi_1'"] \arrow[dr, "\pi_2'"'] & \\
 & \mathrm{MT}(H^\bullet(X_2))
\end{tikzcd}$$

is commutative, since F is an isomorphism of Hodge structures. □

6.3.3. — Consider now a field $K \subset \mathbb{C}$ which is finitely generated over $\mathbb{Q}$, and fix a prime number ℓ. Theorem 6.3.1 has the following consequence for the Galois representations on the cohomology of hyper-Kähler varieties. If X is a hyper-Kähler variety over K, the Beauville–Bogomolov form yields a $\mathbb{Q}_\ell$-valued non-degenerate symmetric bilinear pairing on $H^2_\ell(X)$ via the comparison isomorphism $H^\bullet(X) \otimes \mathbb{Q}_\ell \cong H^\bullet_\ell(X)$.

6.3.4. Corollary. — *Let X_1, X_2 be hyper-Kähler varieties with $b_2(X_i) > 6$ over K such that $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent (in the complex analytic sense). If X_i has non-trivial cohomology in odd degree assume further that $\langle \mathcal{H}^\bullet(X_{i,\bar{K}})\rangle$ contains the motive of a Kuga–Satake abelian variety A_i for $H^2(X_i)$. Assume that $f\colon H^2_\ell(X_1) \xrightarrow{\sim} H^2_\ell(X_2)$ is a $\mathrm{Gal}(\bar{K}/K)$-equivariant isometry. Then, there exist a finite field extension K'/K and a $\mathrm{Gal}(\bar{K}/K')$-equivariant isomorphism of graded algebras $F\colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$.*

Proof. — By Theorem 5.1.2, the motivic Mumford–Tate Conjecture 2.2.10 holds for the motives $\mathcal{H}^2(X_1)$ and $\mathcal{H}^2(X_2)$. Hence, there exists a finite field extension K' of K such that the isometry f is the ℓ-adic realization of an isomorphism of motives $\mathcal{H}^2(X_1) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} \mathcal{H}^2(X_2) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ in $\mathsf{AM}_{K'} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Theorem 6.3.1 yields an isomorphism $F\colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$ of graded algebras which, up to further replacing K' with a finite extension, is the realization of an isomorphism of motives over K' with $\mathbb{Q}_\ell$-coefficients. Hence, F is $\mathrm{Gal}(\bar{K}/K')$-equivariant. □

6.4. Galois representations from hyper-Kähler varieties

Assuming the validity of the Mumford–Tate conjecture we can sharpen Corollary 6.3.4 and obtain a more precise statement about the Galois representations on the cohomology of hyper-Kähler varieties. Thanks to Theorem 6.2.3, the result applies to all known hyper-Kähler varieties. As a consequence, we show that the Galois representation on the cohomology of a hyper-Kähler variety over a finite field is determined by the representation on the second cohomology group.

6.4.1. — Let K_1, K_2 be subfields of $\mathbb{C}$, finitely generated over $\mathbb{Q}$, and consider hyper-Kähler varieties X_1, X_2 over K_1 and K_2 respectively.

6.4.2. Definition. — *We say that X_1 and X_2 are $H^\bullet_\ell$-equivalent if there exists an isomorphism of graded algebras $H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$ which is an isometry in degree 2.*

Note that if $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are deformation equivalent (in the complex analytic sense), then X_1 and X_2 are $H^\bullet_\ell$-equivalent, since in this case the complex manifolds associated to $X_{1,\mathbb{C}}$ and $X_{2,\mathbb{C}}$ are homeomorphic, and both the graded algebra $H^\bullet_\ell(X_i) \cong H^\bullet(X_i) \otimes \mathbb{Q}_\ell$ and the Beauville–Bogomolov form only depend on the topology of $X_{i,\mathbb{C}}$ as a complex manifold.

With notation as above, we let $\sigma_i \colon \mathrm{Gal}(\bar{K}_i/K_i) \to \mathrm{GL}(H^\bullet_\ell(X_i))$ be the Galois representation on the cohomology of X_i.

6.4.3. Proposition. — *Assume that X_1 and X_2 are $H^\bullet_\ell$-equivalent. Assume that $\Gamma \subset \mathrm{Gal}(\bar{K}_1/K_1)$ is a subgroup and that we have a homomorphism $\epsilon \colon \Gamma \to \mathrm{Gal}(\bar{K}_2/K_2)$; we let Γ act on $H^\bullet_\ell(X_1)$ via σ_1 and on $H^\bullet_\ell(X_2)$ via $\epsilon \circ \sigma_2$. If there exists an isometry $f \colon H^2_\ell(X_1) \xrightarrow{\sim} H^2_\ell(X_2)$ which is Γ-equivariant, then there exists an isomorphism $F \colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$ of graded algebras whose degree 2 component is again Γ-equivariant.*

Proof. — The argument is the same as the one given in the proof of Theorems 4.3.1 and 6.3.1. We recall it once again. Since X_1 and X_2 are $H^\bullet_\ell$-equivalent, there exists an isomorphism of graded algebras $\Psi \colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$ which is an isometry in degree 2. Let ψ denote this isometry; we may assume that $\psi^{-1} \circ f \in \mathrm{SO}(H^2_{X_1})(\mathbb{Q}_\ell)$. Since the morphism $\pi \colon \mathrm{CSpin}(H^2(X_1)) \to \mathrm{SO}(H^2(X_1))$ is surjective on $\mathbb{Q}_\ell$-points, we find $g \in \mathrm{CSpin}(H^2(X_1))(\mathbb{Q}_\ell)$ such that $\pi(g) = \psi^{-1} \circ f$. We then define

$$F \coloneqq \Psi \circ R(g) \colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2).$$

Then F is a graded algebra isomorphism by Lemma 1.5.7, and its degree 2 component is Γ-equivariant, since it is a multiple of f. □

6.4.4. Theorem. — *Assume that X_1 and X_2 are $H^\bullet_\ell$-equivalent and that the Mumford–Tate conjecture 2.1.6 holds for both of them. Let $\Gamma \subset \mathrm{Gal}(\bar{K}_1/K_1)$ be a subgroup, let $\epsilon \colon \Gamma \to \mathrm{Gal}(\bar{K}_2/K_2)$ be a homomorphism and let $f \colon H^2_\ell(X_1) \xrightarrow{\sim} H^2_\ell(X_2)$ be a Γ-equivariant isometry. Then, there exist a subgroup $\Gamma' \subset \Gamma$ of finite index and a Γ'-equivariant isomorphism of graded algebras $F \colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$.*

Proof. — Replacing K_i by a finite field extension if necessary, we may assume that $\mathcal{G}(H^\bullet_\ell(X_i))$ is connected for $i = 1, 2$. Since the Mumford–Tate conjecture holds for X_i, the representation $\sigma_i \colon \mathrm{Gal}(\bar{K}_i/K_i) \to \mathrm{GL}(H^\bullet_\ell(X_i))$ factors through the $\mathbb{Q}_\ell$-points of the image of the LLV-representation $R \colon \mathrm{G}_0(X_i) \to \prod_j \mathrm{GL}(H^j(X_i))$, by Lemma 4.2.2.

Applying Proposition 6.4.3 we find an isomorphism $F \colon H^\bullet_\ell(X_1) \xrightarrow{\sim} H^\bullet_\ell(X_2)$ of graded algebras whose degree 2 component $F^{(2)}$ is Γ-equivariant. Now the argument is the same as in the proof of Proposition 4.3.3. We consider the isomorphism $F_* \colon \mathrm{GL}(H^\bullet_\ell(X_1)) \xrightarrow{\sim} \mathrm{GL}(H^\bullet_\ell(X_2))$ given by $A \mapsto FAF^{-1}$, and the analogous isomorphism $F^{(2)}_* \colon \mathrm{GL}(H^2_\ell(X_1)) \xrightarrow{\sim} \mathrm{GL}(H^2_\ell(X_2))$. By Lemma 1.2.4.($i$), the isomorphism F_* restricts to an isomorphism $R(\mathrm{G}_0(X_1))(\mathbb{Q}_\ell) \xrightarrow{\sim} R(\mathrm{G}_0(X_2))(\mathbb{Q}_\ell)$.

We consider the diagram

$$\begin{tikzcd}
 & \Gamma \arrow[dl, hook'] \arrow[dr, "\epsilon"] & \\
\mathrm{Gal}(\bar{K}_1/K_1) \arrow[d, "\sigma_1"] & & \mathrm{Gal}(\bar{K}_2/K_2) \arrow[d, "\sigma_2"] \\
R(\mathrm{G}_0(X_1))(\mathbb{Q}_\ell) \arrow[rr, "F_*"] \arrow[d, "\mathrm{pr}_1"] & & R(\mathrm{G}_0(X_2))(\mathbb{Q}_\ell) \arrow[d, "\mathrm{pr}_2"] \\
R^{(2)}(\mathrm{G}_0(X_1))(\mathbb{Q}_\ell) \arrow[rr, "F^{(2)}_*"] & & R^{(2)}(\mathrm{G}_0(X_2))(\mathbb{Q}_\ell)
\end{tikzcd}$$

We have to show that, up to replacing Γ by one of its subgroups of finite index, this diagram commutes. Since $F^{(2)}$ is Γ-equivariant, we have

$$F^{(2)}_* \circ \mathrm{pr}_1 \circ \sigma_1 = \mathrm{pr}_2 \circ \sigma_2 \circ \epsilon.$$

By Remark 1.5.8, the homomorphism $\mathrm{pr}_2\colon R(\mathrm{G}_0(X_2)) \to R^{(2)}(\mathrm{G}_0(X_2))$ is either an isomorphism or a central isogeny of degree 2; let C be its kernel. Then there exists a homomorphism $\chi\colon \Gamma \to C(\mathbb{Q}_\ell)$ such that $F_* \circ \sigma_1(\gamma) = \chi(\gamma)\cdot\sigma_2(\gamma)$ for any $\gamma \in \Gamma$. The kernel $\Gamma' \subset \Gamma$ of χ is a subgroup of finite index, and F is Γ'-equivariant. □

6.4.5. Remark. — Note that in the above proof we have only used that $\mathcal{G}(H^\bullet_\ell(X_i))^0 \subset \mathrm{MT}(H^\bullet(X_i))(\mathbb{Q}_\ell)$, so we only need one of the two inclusions predicted by the Mumford–Tate conjecture.

6.4.6. — We apply Theorem 6.4.4 to the study of Galois representations on the cohomology of hyper-Kähler varieties over finite fields. We will consider the following situation. Let k be a finite field with algebraic closure $\bar{k}$, and let Z_1 and Z_2 be smooth projective varieties over k. We assume that there exist hyper-Kähler varieties X_1 and X_2 over fields of characteristic 0 which lift Z_1 and Z_2. More precisely, we assume that there exist:

- normal integral domains $R_i \subset \mathbb{C}$ essentially of finite type over $\mathbb{Z}$ with fraction fields K_i of characteristic 0;
- smooth and projective morphisms $\mathfrak{X}_i \to \mathrm{Spec}(R_i)$ whose generic fibres X_i are hyper-Kähler varieties;
- homomorphisms $R_i \to k$ together with isomorphisms $\mathfrak{X}_i \otimes_{R_i} k \cong Z_i$ of k-schemes.

We let ℓ be a prime number invertible in k and consider $H^\bullet_\ell(Z_i) := \bigoplus_j H^j_{\text{ét}}(Z_{i,\bar{k}}, \mathbb{Q}_\ell)$. By the smooth and proper base-change theorems we have an isomorphism of graded algebras $H^\bullet_\ell(X_i) \cong H^\bullet_\ell(Z_i)$. Via this isomorphism, the Beauville–Bogomolov form induces a non-degenerate symmetric bilinear form on $H^2_\ell(Z_i)$ with values in $\mathbb{Q}_\ell$.

6.4.7. Remark. — A priori, the bilinear form that we obtain on $H^2_\ell(Z_i)$ depends on the choices of R_i and $\mathfrak{X}_i$. However, by [**34**, Remark 4.12], the formula

$$\alpha \mapsto \int_{X_i} \alpha^2 \wedge \sqrt{\mathrm{td}(X_i)}$$

defines a non-degenerate quadratic form on $H^2(X_i)$ which is a non-zero multiple of the Beauville–Bogomolov form. The form induced on $H^2_\ell(Z_i)$ via base change is given by $\alpha \mapsto \int_{Z_i} \alpha^2 \wedge \sqrt{\mathrm{td}(Z_i)}$, and it is thus independent from the choices of R_i and $\mathfrak{X}_i$.

6.4.8. Theorem. — *With notations and assumptions as above, assume that X_1 and X_2 are $H^\bullet_\ell$-equivalent, and that the Mumford–Tate conjecture holds for both of* *them. Let* $f\colon H^2_\ell(Z_1) \xrightarrow{\sim} H^2_\ell(Z_2)$ *be a* $\mathrm{Gal}(\bar{k}/k)$*-equivariant isometry. Then, there exist a finite field extension* k' *of* k *and a* $\mathrm{Gal}(\bar{k}/k')$*-equivariant isomorphism of graded algebras* $F\colon H^\bullet_\ell(Z_1) \xrightarrow{\sim} H^\bullet_\ell(Z_2)$*.*

Proof. — Let $|k| = p^r$, and let $\mathrm{Fr}_k \in \mathrm{Gal}(\bar{k}/k)$ be the Frobenius automorphism. With notations as in §6.4.6, let $\mathfrak{m}_i \subset R_i$ be the kernel of $R_i \to k$; let $|R_i/\mathfrak{m}_i| = p^{r/a_i}$ and denote by $\phi_i \in \mathrm{Gal}(\bar{K}_i/K_i)$ a Frobenius element at $\mathfrak{m}_i$, for $i = 1, 2$.

By construction, we have isomorphisms of groups $\langle \phi_i^{a_i} \rangle \cong \langle \mathrm{Fr}_k \rangle$ (both isomorphic to $\mathbb{Z}$) such that the action of ϕ^{a_i} on $H^\bullet_\ell(Z_i)$ via the base-change isomorphism $H^\bullet_\ell(X_i) \cong H^\bullet_\ell(Z_i)$ corresponds to that of Fr_k.

Let now $\Gamma = \langle \phi_1^{a_1} \rangle \subset \mathrm{Gal}(\bar{K}_1/K)$ and let $\epsilon\colon \Gamma \to \mathrm{Gal}(\bar{K}_2/K_2)$ be the homomorphism such that $\phi_1^{a_1} \mapsto \phi_2^{a_2}$. By Theorem 6.4.4, there exists an integer m and an isomorphism $H^\bullet_\ell(Z_1) \to H^\bullet_\ell(Z_2)$ of graded algebras which is Fr_k^m-equivariant. □

CHAPTER 7

COHOMOLOGICAL RIGIDITY OF HYPER-KÄHLER MANIFOLDS

7.1. A conjecture

In §6.2 we used a deformation argument to show the triviality of the defect group for the known hyper-Kähler varieties. This argument cannot however be used to prove that the defect group is trivial in general, since this would require some knowledge on a specific example in each deformation class, which amounts to achieve a topological classification of hyper-Kähler manifolds. This is a notoriously difficult problem.

In this chapter we present a different approach towards at least the finiteness of defect groups; by Proposition 5.2.3 this would suffice to prove the Mumford–Tate conjecture. The idea is to exploit the constraints imposed on the defect group by the LLV-representation and the algebra structure on the cohomology. We propose a conjecture on the cohomology algebras of hyper-Kähler manifolds, which would imply that the defect group is finite.

7.1.1. — Let X be a hyper-Kähler manifold. Let $\mathrm{Aut}(H^\bullet(X)) \subset \prod_i \mathrm{GL}(H^i(X))$ be the group of graded algebra automorphisms of the rational cohomology $H^\bullet(X)$ of X. We aim to study the subgroup $\mathrm{Aut}_0(H^\bullet(X)) \subset \mathrm{Aut}(H^\bullet(X))$ of automorphisms that act trivially on $H^2(X)$.

Let H denote $H^2(X)$ equipped with the Beauville–Bogomolov form. By Lemma 1.5.7, the LLV-representation yields a morphism $R\colon \mathrm{Spin}(H) \to \mathrm{Aut}(H^\bullet(X))$, and the kernel of the induced representation $\mathrm{Spin}(H) \to \mathrm{GL}(H^2(X))$ is generated by $-1 \in \mathrm{Spin}(H)$, by Remark 1.5.4. If we denote by $\mathrm{Aut}_+(H^\bullet(X)) \subset \mathrm{Aut}(H^\bullet(X))$

the subgroup of those automorphisms acting on $H^2(X)$ by isometries of determinant 1, both $\mathrm{Aut}_0(H^\bullet(X))$ and the image of R are contained in $\mathrm{Aut}_+(H^\bullet(X))$.

7.1.2. Lemma. — *We have* $\mathrm{Aut}_+(H^\bullet(X)) = R(\mathrm{Spin}(H)) \cdot \mathrm{Aut}_0(H^\bullet(X))$*; the subgroups* $\mathrm{Aut}_0(H^\bullet(X))$ *and* $R(\mathrm{Spin}(H))$ *commute and intersect in* $\langle R(-1)\rangle$*. If X has trivial cohomology in odd degree then* $\mathrm{Aut}_+(H^\bullet(X)) = R(\mathrm{Spin}(H)) \times \mathrm{Aut}_0(H^\bullet(X))$*.*

Proof. — By definition, we have a short exact sequence

$$1 \to \mathrm{Aut}_0(H^\bullet(X)) \to \mathrm{Aut}_+(H^\bullet(X)) \xrightarrow{\pi} \mathrm{SO}(H) \to 1.$$

The restriction of π to the subgroup $R(\mathrm{Spin}(H))$ is an isogeny onto $\mathrm{SO}(H)$. Therefore $\mathrm{Aut}_+(H^\bullet(X)) = R(\mathrm{Spin}(H)) \cdot \mathrm{Aut}_0(H^\bullet(X))$. We have already remarked that -1 is the only non-trivial element of $\mathrm{Spin}(H)$ which acts trivially on $H^2(X)$. By Lemma 1.2.4.(*ii*), the action of $\mathrm{Aut}_0(H^\bullet(X))$ commutes with the LLV-representation. The last assertion follows since $R(-1)$ acts as $(-1)^j$ on $H^j(X)$. □

The following example shows that $\mathrm{Aut}_0(H^\bullet(X))$ is not finite in general.

7.1.3. Example. — Let X be a hyper-Kähler fourfold of Kum^2-type. Its Betti numbers are

$$1 \quad 0 \quad 7 \quad 8 \quad 108 \quad 8 \quad 7 \quad 0 \quad 1.$$

Note that X has non zero odd Betti numbers: $b_3(X) = b_5(X) = 8$. By [**51**], there is a subspace $U \subset H^4(X)$ such that:

- the LLV-Lie algebra $\mathfrak{g}(X)$ acts trivially on U;
- the dimension of U is 80;
- there is a decomposition $H^4(X) = U \oplus \mathrm{Sym}^2(H^2(X))$ which is orthogonal with respect to the intersection pairing on $H^4(X)$.

The algebra $H^\bullet(X)$ is generated by $H^2(X), H^3(X)$ and U. The image of the product map $H^2(X) \otimes H^2(X) \to H^4(X)$ is $\mathrm{Sym}^2(H^2(X)) \subset H^4(X)$.

The product map $H^2(X) \otimes U \to H^6(X)$ is zero, because U is a trivial $\mathfrak{g}(X)$-representation and hence $x \cdot u = L_x(u) = 0$ for any $x \in H^2(X)$ and $u \in U$. The product map $H^3(X) \otimes U \to H^7(X)$ is also zero. If $H' \subset H^\bullet(X)$ is the subalgebra generated by $H^2(X)$ and $H^3(X)$, we then have

$$H^\bullet(X) = H' \oplus U,$$

and the product map $H' \otimes U \to H^\bullet(X)$ is zero.

It follows that the action of $\mathrm{Aut}_0(H^\bullet(X))$ on $H^4(X)$ preserves the subspace U. The intersection product on $H^4(X)$ restricts to a non-degenerate pairing ϕ on U. The image of $\mathrm{Aut}_0(H^\bullet(X)) \to \mathrm{GL}(U)$ is contained in $\mathrm{O}(U,\phi)$. By the above, for any isometry $g \in \mathrm{O}(U,\phi)$, the linear automorphism $G\colon H^\bullet(X) \to H^\bullet(X)$ defined as $G = (\mathrm{id}_{H'} \oplus g)$ is a graded algebra automorphism. Hence, $\mathrm{O}(U,\phi) \subset \mathrm{Aut}(H^\bullet(X))$.

7.1.4. — Let $\mathrm{Aut}(X)$ be the group of automorphisms of the hyper-Kähler manifold X. Mapping an automorphism $g\colon X \to X$ to the pull-back automorphism $(g^{-1})^*\colon H^\bullet(X) \to H^\bullet(X)$ of the cohomology algebra, we obtain a homomorphism

$$\nu\colon \mathrm{Aut}(X) \to \mathrm{Aut}(H^\bullet(X)).$$

We define Γ as the kernel of the induced representation $\nu^{(2)}\colon \mathrm{Aut}(X) \to \mathrm{GL}(H^2(X))$. Huybrechts [**42**, Proposition 9.1] proved that Γ is a finite group, and Hassett–Tschinkel [**41**, Theorem 2.1] have shown that it is deformation invariant.

Our cohomological rigidity conjecture is the following statement.

7.1.5. Conjecture. — *The commutator*

$$\mathrm{Aut}_0(H^\bullet(X))^\Gamma := \{g \in \mathrm{Aut}_0(H^\bullet(X)) \mid g(h^{-1})^* = (h^{-1})^* g \text{ for any } h \in \Gamma\}$$

is a finite group.

If Y is a hyper-Kähler manifold deformation equivalent to X, the conjecture holds for X if and only if it holds for Y. Note that, since Γ is a group of automorphisms, the defect group $P(X)$ is a subgroup of $\mathrm{Aut}_0(H^\bullet(X))^\Gamma$; hence, thanks to Proposition 5.2.3, the above conjecture would imply the Mumford–Tate conjecture.

7.1.6. — The group Γ has been computed for each of the known deformation types.

- If X is of $\mathrm{K3}^{[n]}$-type, Beauville has shown [**8**, Proposition 10] that Γ is trivial.
- For X of Kum^n-type, the group Γ has been calculated by Boissiére–Nieper-Wisskirken–Sarti [**13**, Corollary 5], and we have

  $$\Gamma = (\mathbb{Z}/(n+1)\mathbb{Z})^4 \rtimes \mathbb{Z}/2\mathbb{Z},$$

  where $\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbb{Z}/(n+1)\mathbb{Z})^4$ via ± 1. Oguiso [**70**] has shown that Γ acts faithfully on $H^\bullet(X)$.
- Mongardi–Wandel [**61**] computed Γ for the O'Grady deformation types: for$X$ of OG10-type, Γ is trivial, while for X of OG6-type, we have $\Gamma = (\mathbb{Z}/2\mathbb{Z})^8$.

7.1.7. Remark. — It does not seem generally known whether

$$\nu\colon \operatorname{Aut}(X) \to \operatorname{Aut}(H^\bullet(X))$$

is injective; note however that by the above ν is injective for X of $\mathrm{K3}^{[n]}$, OG10 and Kum^n-type.

We now verify the conjecture for the fourfolds of Example 7.1.3.

7.1.8. Proposition. — *Let X be a hyper-Kähler fourfold of* Kum^2*-type. Then Conjecture 7.1.5 holds for X.*

Proof. — Let $U \subset H^4(X)$ be as in Example 7.1.3. The cohomology algebra $H^\bullet(X)$ is generated by $H^2(X)$, $H^3(X)$ and U; therefore, any algebra automorphism is determined by its action on these subspaces. The group $\operatorname{Aut}_0(H^\bullet(X))$ acts trivially on $H^2(X)$; it follows that

$$\operatorname{Aut}_0(H^\bullet(X)) \subset \operatorname{GL}(H^3(X)) \times \operatorname{GL}(U).$$

It suffices to show that $\operatorname{Aut}_0(H^\bullet(X))^\Gamma$ acts on $H^3(X)$ and U via a finite group.

The action of $\operatorname{Aut}_0(H^\bullet(X))$ on $H^3(X)$ already factors through $\mathbb{Z}/2\mathbb{Z}$. In fact, under the representation R, the subspace $H^3(X)$ is the absolutely irreducible 8-dimensional spin representation of $\operatorname{Spin}(H)$ ([**51**, §4.6]). Since the action of $\operatorname{Aut}_0(H^\bullet(X))$ commutes with the representation R, any $g \in \operatorname{Aut}_0(H^\bullet(X))$ acts on $H^3(X)$ as multiplication by a scalar. Any $x \in H^2(X)$ with the Lefschetz property yields a non-degenerate pairing ϕ_x on $H^3(X)$ via $\phi_x(\alpha,\beta) = \int_X x \cdot \alpha \cdot \beta$, which is preserved by $\operatorname{Aut}_0(H^\bullet(X))$. It follows that g is multiplication by ± 1 on $H^3(X)$.

By Example 7.1.3, the action of the whole $\operatorname{Aut}_0(H^\bullet(X))$ on U does not factor through a finite group. We have $\Gamma = (\mathbb{Z}/3\mathbb{Z})^4 \rtimes \mathbb{Z}/2\mathbb{Z}$. Hasset–Tschinkel [**41**] have shown that, as a representation of $(\mathbb{Z}/3\mathbb{Z})^4 \subset \Gamma$, the subspace $U \subset H^4(X)$ is identified with the complement of the trivial representation in the regular representation of $(\mathbb{Z}/3\mathbb{Z})^4$. Equivalently, $U \otimes \mathbb{C}$ is the sum of the 80 distinct irreducible non-trivial representations of $(\mathbb{Z}/3\mathbb{Z})^4$. Any element of $\operatorname{Aut}_0(H^\bullet(X))^\Gamma \otimes \mathbb{C}$ must preserve this decomposition. By Proposition 1.2.9, the intersection product ϕ on U is definite. Since $\operatorname{Aut}_0(H^\bullet(X))^\Gamma$ preserves the non-degenerate form ϕ on U, it will preserve its hermitian extension to $U \otimes \mathbb{C}$. This implies that the image of $\operatorname{Aut}(H^\bullet(X))^\Gamma$ in $\operatorname{GL}(U) \otimes \mathbb{C}$ is contained in a subgroup of $(\mathbb{Z}/2\mathbb{Z})^{80}$. □

7.2. A decomposition of the cohomology

As we have seen in Proposition 7.1.8, the action of the LLV-Lie algebra is very useful in the investigation of Conjecture 7.1.5. Following Markman [**55**] we use the Lie algebra $\mathfrak{g}(X)$ to introduce a minimal subspace which generates the cohomology algebra of a hyper-Kähler manifold X.

7.2.1. — Let $2n = \dim X$. There is a $\mathfrak{g}(X)$-invariant non-degenerate bilinear form ϕ on $H^\bullet(X)$, given by $\phi(\alpha,\beta) = (-1)^k \int_X \alpha\cdot\beta$ for α of degree $2k+2n$ or $2k+2n+1$. By Proposition 1.2.9, the restriction of ϕ to any $\mathfrak{g}(X)$-submodule $V \subset H^\bullet(X)$ is non-degenerate.

For any integer $i \geq 0$, let $A_i^\bullet(X) \subset H^\bullet(X)$ be the subalgebra generated by $\bigoplus_{j\leq i} H^j(X)$, and let $\widetilde{A}_i^\bullet(X)$ be the $\mathfrak{g}(X)$-submodule of $H^\bullet(X)$ generated by $A_i^\bullet$. Since ϕ restricts to a non-degenerate pairing on $\widetilde{A}_i^\bullet(X)$, for all i, we obtain an orthogonal decomposition

$$H^\bullet(X) = \widetilde{A}_i^\bullet(X) \oplus \widetilde{A}_i^\bullet(X)^\perp$$

into $\mathfrak{g}(X)$-submodules. We now define, for $i > 2$, the subspace

$$C^i(X) \coloneqq \widetilde{A}_{i-2}^\bullet(X)^\perp \cap H^i(X).$$

In other words, $C^i(X)$ consists of the cohomology classes in $H^i(X)$ which are orthogonal with respect to ϕ to all products of cohomology classes of degree lower than i. We let $C^0(X) = H^0(X)$, $C^1(X) = 0$ and $C^2(X) = H^2(X)$. Note that $C^i(X) = 0$ for $i > 2n$. We introduce the notation $C^\bullet(X) = \bigoplus_i C^i(X)$.

7.2.2. Proposition. — *(i) For all i, we have a $\mathfrak{g}_0(X)$-invariant decomposition*

$$H^i(X) = A_{i-2}^i(X) \oplus C^i(X).$$

(ii) The $\mathfrak{g}_0(X)$-module $C^\bullet(X)$ generates the algebra $H^\bullet(X)$.

(iii) For all i, the subspace $C^i(X)$ is stable under the action of $\mathrm{Aut}_0(H^\bullet(X))$.

Proof. — Statement (i) is [**55**, Lemma 4.6], while (ii) is clear. For (iii), note that $\mathrm{Aut}_0(H^\bullet(X))$ stabilizes each $A_i^\bullet(X)$ and preserves the pairing ϕ. Since $\mathrm{Aut}_0(H^\bullet(X))$ commutes with the LLV-Lie algebra, it acts on the $\mathfrak{g}(X)$-module $\tilde{A}_i^\bullet(X)$. Hence, $\tilde{A}_i^\bullet(X)^\perp$ is also stable under the action of $\mathrm{Aut}_0(H^\bullet(X))$. □

7.2.3. — We now establish Conjecture 7.1.5 for the deformation types $\mathrm{K3}^{[n]}$ and OG10. These are the two known cases in which Γ is trivial (§7.1.6).

7.2.4. Theorem. — *Let X be a hyper-Kähler manifold of deformation type* $\mathrm{K3}^{[n]}$ *or* OG10. *Then Conjecture 7.1.5 holds for X.*

Proof. — Assume first that X is of $\mathrm{K3}^{[n]}$-type. Let $C^\bullet(X) \subset H^\bullet(X)$ be the subspace given by Proposition 7.2.2. Since X has trivial odd cohomology, $C^i(X) = 0$ for odd i. Let H denote $H^2(X)$ equipped with the Beauville–Bogomolov form; the semisimple part of $\mathfrak{g}_0(X)$ is $\mathfrak{so}(H)$ (see Theorem 1.2.12).

Specializing to the case in which X is a moduli space of stable sheaves on a K3 surface, Markman [**55**, Lemma 4.8] has shown that the $\mathfrak{so}(H)$-module C^{2i} is a quotient of the representation $H^2(X) \oplus T$, where T is a trivial one dimensional representation.

Thus, any irreducible $\mathfrak{g}_0(X)$-module $V \subset C^{2i}$ is absolutely irreducible and appears with multiplicity one. Moreover, for any $x \in H^2(X)$ with the Lefschetz property, the formula $\phi_x(\alpha, \beta) := \phi(\alpha, x^{2n-2i} \cdot \beta)$ defines a non-degenerate bilinear form on V. Since the action of $\mathrm{Aut}_0(H^\bullet(X))$ commutes with that of $\mathfrak{g}_0(X)$, any V as above is stable under $\mathrm{Aut}_0(H^\bullet(X))$; moreover, $\mathrm{Aut}_0(H^\bullet(X))$ preserves the form ϕ_x. It follows that the action of $\mathrm{Aut}_0(H^\bullet(X))$ on V factors through $\{\pm 1\}$. As $C^\bullet(X)$ generates $H^\bullet(X)$, the group $\mathrm{Aut}_0(H^\bullet(X))$ acts faithfully on it; we conclude that $\mathrm{Aut}_0(H^\bullet(X)) \subset (\mathbb{Z}/2\mathbb{Z})^m$ is a finite group.

Assume now that X is of OG10-type, and let $C^\bullet(X) \subset H^\bullet(X)$ be the subspace given by Proposition 7.2.2. The LLV-representation on $H^\bullet(X)$ was calculated in [**36**, Theorem 1.1]. By their result, if $i \geq 3$, we have $C^i(X) = 0$ except for $i = 6$, in which case $C^6(X)$ is the unique non-trivial $\mathfrak{so}(H)$-subrepresentation of $\mathrm{Sym}^2(H^2(X))$. This is an absolutely irreducible representation of $\mathfrak{so}(H)$. Any $x \in H^2(X)$ with the Lefschetz property yields a non-degenerate bilinear form ϕ_x on $C^6(X)$, given by $\phi_x(\alpha, \beta) = (\alpha, x^4 \cdot \beta)$. Since $\mathrm{Aut}_0(H^\bullet(X))$ preserves the form ϕ_x, we have $\mathrm{Aut}_0(H^\bullet(X)) \subset \mathbb{Z}/2\mathbb{Z}$. $\square$

7.2.5. Remark. — We are not yet able to prove Conjecture 7.1.5 for the remaining known deformation types OG6 and Kum^n, for $n > 2$. In these cases the group Γ is not trivial, and we lack a full understanding of its action on the generators $C^\bullet(X)$ of the cohomology. On the positive side, the LLV-representation on their cohomology has been described in [**36**], and [**57**] contains strong results on the cohomology of generalized Kummer varieties.

7.3. Interpretation via monodromy

We now present a slightly weaker version of Conjecture 7.1.5 which involves Markman's monodromy group [**56**].

Let X be a hyper-Kähler manifold, and let $\mathfrak{X} \to S$ be a smooth and proper holomorphic map of connected complex manifolds such that $\mathfrak{X}_s = X$ for some $s \in S$. For any loop γ in S based at s, we have a *monodromy operator* $\gamma^* \colon H^\bullet(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z})$ induced by parallel transport along γ. The monodromy group $\mathrm{Mon}(X)$ of a hyper-Kähler manifold X is defined as the subgroup of $\mathrm{GL}(H^\bullet(X, \mathbb{Z}))$ generated by the monodromy operators coming from all families $\mathfrak{X} \to S$ as above. The action of $\mathrm{Mon}(X)$ on $H^\bullet(X)$ is by graded algebra automorphisms. Moreover, since the Beauville–Bogomolov form is deformation invariant, $\mathrm{Mon}(X)$ acts on $H^2(X)$ by isometries.

7.3.1. Conjecture. — *The commutator*

$$\mathrm{Aut}_0(H^\bullet(X))^{\mathrm{Mon}(X)} := \{g \in \mathrm{Aut}_0(H^\bullet(X)) \mid gh = hg \textit{ for any } h \in \mathrm{Mon}(X)\}$$

is a finite group.

This statement would be a consequence of Conjecture 7.1.5. In fact, by [**84**, Theorems 3.4, 3.5] (see also [**85**]), the image $\mathrm{Mon}^2(X)$ of $\mathrm{Mon}(X)$ in $\mathrm{GL}(H^2(X, \mathbb{Z}))$ is a subgroup of $\mathrm{O}(H^2(X, \mathbb{Z}))$ of finite index, and the kernel of $\mathrm{Mon}(X) \to \mathrm{Mon}^2(X)$ is a finite group. Moreover, Verbitsky has shown [**84**, Remark 7.5] that this kernel is identified with the image of Γ in $\mathrm{Aut}_0(H^\bullet(X))$. Thus, $\mathrm{Aut}_0(H^\bullet(X))^{\mathrm{Mon}(X)}$ is a subgroup of $\mathrm{Aut}_0(H^\bullet(X))^\Gamma$, and hence Conjecture 7.1.5 implies Conjecture 7.3.1.

Note however that Conjecture 7.3.1 does not immediately imply the Mumford–Tate conjecture. The reason is that it is not clear that the defect group commutes with the whole group $\mathrm{Mon}(X)$. In fact, $P(X)$ commutes with the subgroup of $\mathrm{Mon}(X)$ of elements whose action on $H^2(X)$ has determinant 1, but it is known [**56**] that in general $\mathrm{Mon}^2(X)$ contains also some isometry with determinant -1.

APPENDIX A

PROJECTIVE FAMILIES OF HYPER-KÄHLER VARIETIES

In this appendix we prove the following result.

A.0.1. ***Theorem.*** — *Let X_1, X_2 be deformation equivalent complex projective hyper-Kähler varieties. Assume that $b_2(X) > 6$. Then there exist:*

- *finitely many connected and non-singular complex varieties S_i, for $i = 1, \dots, N$;*
- *for each $i = 1, \dots, N$, a smooth and projective family $\mathfrak{X}^i \to S_i$ with fibres hyper-Kähler varieties;*
- *for $i = 1, \dots, N$, points $a_i, b_i \in S_i$ together with birational maps*

$$X_1 \dashrightarrow \mathfrak{X}^1_{a_1}, \qquad \mathfrak{X}^i_{b_i} \dashrightarrow \mathfrak{X}^{i+1}_{a_{i+1}},\ \textit{for } i = 1, \dots, N-1, \qquad \mathfrak{X}^N_{b_N} \dashrightarrow X_2.$$

If X_1 and X_2 satisfy the condition in the Theorem, we will write $X_1 \sim X_2$.

A.1. Polarized hyper-Kähler varieties

We start by recalling some facts on polarized hyper-Kähler varieties.

A.1.1. — For any hyper-Kähler manifold X, the cone of those $x \in H^{1,1}(X, \mathbb{R})$ such that $(x, x) > 0$ has two connected components; the positive cone is the component containing the Kähler cone. Here, we denote by $(\cdot, \cdot)$ the Beauville–Bogomolov pairing. Equivalently, the positive cone consists of $x \in H^{1,1}(X, \mathbb{R})$ such that $(x, x) > 0$ and $(x, \omega) > 0$ for a Kähler class ω on X, see [**42**].

We denote by $\mathrm{NS}^+(X) \subset \mathrm{NS}(X)$ the intersection of the positive cone with the Néron-Severi group; if $h \in \mathrm{NS}(X)$ is such that $(h, h) > 0$, exactly one among h and $-h$ belongs to $\mathrm{NS}^+(X)$. If X is projective, the ample cone $\mathrm{Amp}(X) \subset \mathrm{NS}^+(X)$

is the open cone consisting of classes representing ample divisors. In [**2**], Amerik–Verbitsky described the ample cone in terms of what they call MBM classes; this notion is equivalent to that of wall divisors introduced by Mongardi [**59**]. These classes are the analogue of -2-classes for K3 surfaces. We will need the following result, proven in [**2**]. Let $\mathrm{MBM}(X) \subset \mathrm{NS}(X)$ be the subset of MBM classes on X.

A.1.2. Theorem. — *(i) Let X be a projective hyper-Kähler manifold. The ample cone is one of the connected components of*

$$\mathrm{NS}^+(X) \setminus \bigcup_{z \in \mathrm{MBM}(X)} z^\perp.$$

In particular, if $\mathrm{NS}(X)$ does not contain any MBM class, then

$$\mathrm{Amp}(X) = \mathrm{NS}^+(X).$$

(ii) Fix a deformation class of hyper-Kähler manifolds with $b_2 \geq 5$. There exists a positive integer N, depending only on the deformation class, such that for any projective X of the given deformation type, every MBM class z on X satisfies

$$-N < (z, z) < 0.$$

A.1.3. — Let X be a hyper-Kähler manifold and let Λ be a lattice isometric to $H^2(X, \mathbb{Z})$ equipped with the Beauville–Bogomolov form. Let

$$\Omega = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (x, x) = 0,\ (x, \bar{x}) > 0\}$$

be the period domain. Fix a connected component $\mathfrak{M}$ of the moduli space of Λ-marked hyper-Kähler manifolds containing X (for some choice of a marking). The period map $\mathcal{P}\colon \mathfrak{M} \to \Omega$ is surjective with discrete fibres, and each fibre represents a birational class of marked hyper-Kähler manifolds, by the Torelli theorem (Theorem 1.1.11).

By Huybrechts' projectivity criterion [**42**], the hyper-Kähler manifold X is projective if and only if $\mathrm{NS}(X)$ contains a class h with $(h, h) > 0$. For any positive class $h \in \Lambda$, we have a hypersurface

$$\Omega_{h^\perp} = \{x \in \Omega \mid (x, h) = 0\} \subset \Omega.$$

Any point in $\Omega_{h^\perp}$ represents a class of birational Λ-marked hyper-Kähler varieties (Y, τ) such that $\tau^{-1}(h)$ is an integral $(1,1)$ class on Y. The period space $\Omega_{h^\perp}$ has two connected components; we denote by $\Omega^+_{h^\perp}$ the component parametrizing those $(Y, \tau) \in \mathfrak{M}$ such that $\tau^{-1}(h)$ belongs to the positive cone. Thus, the locus in Ω of the periods of $(Z, \phi) \in \mathfrak{M}$ with Z projective is the subset

$$\bigcup_{h \in \Lambda,\ (h,h)>0} \Omega^{+}_{h^\perp}.$$

Let $\Omega^{\mathrm{a}}_{h^\perp} \subset \Omega^{+}_{h^\perp}$ be the subset of those period points x such that there exists $(Y, \tau) \in \mathfrak{M}$ with period x on which $\tau^{-1}(h)$ is an ample class. Then $\Omega^{\mathrm{a}}_{h^\perp}$ is an open, connected and dense subset of $\Omega^{+}_{h^\perp}$, by [**56**, Corollary 7.3].

A.1.4. — A *polarization type* for hyper-Kähler manifolds with $H^2(X, \mathbb{Z}) \cong \Lambda$ is a $\mathrm{O}(\Lambda)$-orbit $\bar{h}$ in Λ. By work of Viehweg [**88**], there exists a coarse moduli space $\mathcal{F}_{\bar{h}}$, parametrizing $\bar{h}$-polarized hyper-Kähler varieties of the chosen deformation type; $\mathcal{F}_{\bar{h}}$ is a non-singular quasi-projective variety. Fix a representative $h \in \bar{h}$ and let $\mathcal{F}^0_{\bar{h}} \subset \mathcal{F}_{\bar{h}}$ be a connected component. By [**56**, Theorem 8.4], see also [**38**, §1], we can find a torsion free arithmetic subgroup $\Gamma \subset \mathrm{O}(h^\perp)$ acting freely on $\Omega^{+}_{h^\perp}$, and an embedding

$$\mathcal{F}^0_{\bar{h}} \hookrightarrow \Gamma \backslash \Omega^{+}_{h^\perp}.$$

By [**14**], $\Gamma \backslash \Omega^{+}_{h^\perp}$ is a non-singular quasi-projective variety, and $\mathcal{F}^0_{\bar{h}}$ is embedded in it as a Zariski open subset. Hence, any connected component $\mathcal{F}^0_{\bar{h}}$ of $\mathcal{F}_{\bar{h}}$ has dimension $b_2 - 3$. The arithmetic group Γ preserves $\Omega^{\mathrm{a}}_{h^\perp}$, and the image of $\mathcal{F}^0_{\bar{h}}$ is contained in $\Gamma \backslash \Omega^{\mathrm{a}}_{h^\perp}$.

A.2. A special case

The rest of this Appendix is devoted to the proof of Theorem A.0.1. Let X be a hyper-Kähler manifold and assume that $b_2(X) > 6$. We fix a connected component $\mathfrak{M}$ of the moduli space of Λ-marked hyper-Kähler manifolds. We will show that given any (X_1, τ_1) and (X_2, τ_2) in $\mathfrak{M}$ with X_1 and X_2 projective, then $X_1 \sim X_2$, in the notation of the Theorem. We start by considering the following case.

A.2.1. Proposition. — *Let (X_1, τ_1) and (X_2, τ_2) be points of $\mathfrak{M}$. Assume that $\mathcal{P}(X_1, \tau_1)$ and $\mathcal{P}(X_2, \tau_2)$ are both in $\Omega^{\mathrm{a}}_{h^\perp}$, for some positive class $h \in \Lambda$. Then $X_1 \sim X_2$.*

Proof. — Let $(Y, \phi) \in \mathfrak{M}$ be such that $\mathcal{P}(Y, \phi) \in \Omega^{\mathrm{a}}_{h^\perp}$. Replacing Y with a birational hyper-Kähler variety if necessary, we may assume that $\phi^{-1}(h) = c_1(L)$ for an ample divisor L on Y. Following André [**3**, §3.3], there exists a local universal polarized deformation $\mathfrak{Y} \to S$ of (Y, L). This is a smooth and projective family of hyper-Kähler varieties over a non-singular and connected variety S, with a distinguished fibre $\mathfrak{Y}_s = Y$. Moreover, denoting by $\tilde{S} \to S$ the universal covering of S, we obtain a period map $\tilde{S} \to \Omega^{\mathrm{a}}_{h^\perp}$, which, upon replacing S with a finite étale cover, descends to a map $\Psi\colon S \to \Gamma\backslash\Omega^{\mathrm{a}}_{h^\perp}$. The period map Ψ is a generically finite, dominant morphism of schemes, by [**14**].

Let (X_1, τ_1) and (X_2, τ_2) be as in the statement of the proposition. Replacing X_i with a birational model if necessary, the class $\tau_i^{-1}(h)$ is the first Chern class of an ample line bundle L_i on X_i. Consider the respective polarized universal deformations $\mathfrak{X}_1 \to S_1$ and $\mathfrak{X}_2 \to S_2$ of (X_1, L_1) and (X_2, L_2) obtained as above, and let $\Psi_1\colon S_1 \to \Gamma\backslash\Omega^{\mathrm{a}}_{h^\perp}$ and $\Psi_2\colon S_2 \to \Gamma\backslash\Omega^{\mathrm{a}}_{h^\perp}$ be the corresponding period maps. Since $\Omega^{\mathrm{a}}_{h^\perp}$ is connected, $\Psi_1(S_1) \cap \Psi_2(S_2)$ is not empty; by the surjectivity of the period map there exists $(Y, \phi) \in \mathfrak{M}$ such that the image of $\mathcal{P}(Y, \phi)$ in $\Gamma\backslash\Omega^{\mathrm{a}}_{h^\perp}$ lies in $\Psi_1(S_1) \cap \Psi_2(S_2)$. We can then find hyper-Kähler varieties Y_1 and Y_2 which are both birational to Y and such that Y_1 (resp. Y_2) is a fibre of $\mathfrak{X}_1 \to S_1$ (resp. $\mathfrak{X}_2 \to S_2$). We therefore have $X_1 \sim Y_1 \sim Y_2 \sim X_2$. □

By the proposition, to prove Theorem A.0.1 it is sufficient to establish the following.

A.2.2. Claim. — *Let h_1 and h_2 be two positive classes in Λ. Then there exist (X_1, τ_1) and (X_2, τ_2) in $\mathfrak{M}$ with $\mathcal{P}(X_1, \tau_1) \in \Omega^{\mathrm{a}}_{h_1^\perp}$ and $\mathcal{P}(X_2, \tau_2) \in \Omega^{\mathrm{a}}_{h_2^\perp}$ such that $X_1 \sim X_2$.*

We will first deal with the following special case, which is the key step in the proof. We will complete the proof of Claim A.2.2 in the next section.

A.2.3. Proposition. — *Let h_1 and h_2 be positive classes in Λ such that the lattice $\langle h_1, h_2\rangle$ is of signature $(1, 1)$ and $(h_1, h_2) > 0$. Then the conclusion of Claim A.2.2 holds for h_1 and h_2.*

Let h_1 and h_2 be as above. We fix the constant N given by Theorem A.1.2 for our deformation type. To prove Proposition A.2.3, we will state and use several results whose proofs are collected in §A.2.8. First of all, we replace h_1 and h_2 with classes of which we can control the square.

A.2.4. Lemma. — *There exist a prime number $p > N$ congruent to 3 modulo 4, an odd integer $j \gg 0$ and positive classes l_1 and l_2 in Λ such that:*

– *$\Omega^{\mathrm{a}}_{h_1^\perp} \cap \Omega^{\mathrm{a}}_{l_1^\perp} \neq \emptyset$ and $\Omega^{\mathrm{a}}_{h_2^\perp} \cap \Omega^{\mathrm{a}}_{l_2^\perp} \neq \emptyset$;*
– *we have $(l_1, l_1) = p^j f_1$ and $(l_2, l_2) = p^j f_2$, for positive integers f_1 and f_2 not divisible by p and such that $f_1 f_2$ is not a square modulo p;*
– *the lattice generated by l_1 and l_2 has signature $(1, 1)$, and $(l_1, l_2) > 0$.*

Thanks to Proposition A.2.1, we may therefore assume that h_1 and h_2 have Beauville–Bogomolov square $p^j f_1$ and $p^j f_2$ respectively, with p, j, f_1 and f_2 as in Lemma A.2.4 above. These assumptions on h_1 and h_2 are now in force.

A.2.5. Lemma. — *There exist classes $v_1, v_2 \in \langle h_1, h_2\rangle^\perp \subset \Lambda$ such that:*

– *$(v_1, v_1) = p\epsilon_1$ and $(v_2, v_2) = p\epsilon_2$, for negative integers ϵ_1 and ϵ_2 not divisible by p;*
– *$\epsilon_1 f_1$, $\epsilon_2 f_2$ and $\epsilon_1\epsilon_2$ are not squares modulo p;*
– *(v_1, v_2) is divisible by p^2.*

Let v_1 and v_2 be as above. Consider the rank 2 sublattices of Λ:

$$L_1 = \langle h_1, v_1\rangle \quad \text{and} \quad L_2 = \langle h_2, v_2\rangle.$$

By construction, they have signature $(1, 1)$.

A.2.6. Lemma. — *Let $v \in L_1 \otimes \mathbb{Q}$ or $v \in L_2 \otimes \mathbb{Q}$. Then p divides (v, v).*

For $k > 0$, we define $w_{1,k} \in L_1$ and $w_{2,k} \in L_2$ as:

$$w_{1,k} = p^k h_1 + v_1 \qquad w_{2,k} = p^k h_2 + v_2.$$

We let $S_k \subset \Lambda$ be the lattice generated by $w_{1,k}$ and $w_{2,k}$. If $k \gg 0$, this lattice has signature $(1, 1)$, because this is the signature of the lattice generated by h_1 and h_2.

A.2.7. Lemma. — *For $k \gg 0$, given any $v \in S_k \otimes \mathbb{Q}$, the prime p divides (v, v).*

We can now complete the proof of Proposition A.2.3.

Proof of Proposition A.2.3. — Let L_1 and L_2 be the lattices of Lemma A.2.6. Since they are of signature $(1, 1)$, by the surjectivity of the period map we can find (Y_1, ϕ_1) and (Y_2, ϕ_2) in $\mathfrak{M}$ such that $\mathrm{NS}(Y_1) = \phi_1^{-1}(L_1)$ and $\mathrm{NS}(Y_2) = \phi_2^{-1}(L_2)$. Moreover, the class $\phi_1^{-1}(h_1)$ (resp. $\phi_2^{-1}(h_2)$) belongs to the positive cone $\mathrm{NS}^+(Y_1)$ (resp. $\mathrm{NS}^+(Y_2)$).

Since $p > N$, Lemma A.2.6 ensures that $\mathrm{NS}(Y_1)$ and $\mathrm{NS}(Y_2)$ contain no MBM classes; hence, by Theorem A.1.2,

$$\mathrm{Amp}(Y_1) = \mathrm{NS}^+(Y_1) \quad \text{and} \quad \mathrm{Amp}(Y_2) = \mathrm{NS}^+(Y_2).$$

We fix $k \gg 0$. There exists $(Z, \psi) \in \mathfrak{M}$ such that $\mathrm{NS}(Z) = \psi^{-1}(S_k)$ and both the classes $\psi^{-1}(w_{1,k})$ and $\psi^{-1}(w_{2,k})$ belong to the positive cone $\mathrm{NS}^+(Z)$; this is possible because $(w_{1,k}, w_{2,k}) > 0$. By Lemma A.2.7, there are no MBM classes in $\mathrm{NS}(Z)$; hence, we have $\mathrm{Amp}(Z) = \mathrm{NS}^+(Z)$, by Theorem A.1.2.

We therefore obtain:

$$\mathcal{P}(Y_1, \phi_1) \in \Omega^{\mathrm{a}}_{h_1^\perp} \cap \Omega^{\mathrm{a}}_{w_{1,k}^\perp}, \quad \mathcal{P}(Z, \psi) \in \Omega^{\mathrm{a}}_{w_{1,k}^\perp} \cap \Omega^{\mathrm{a}}_{w_{2,k}^\perp}, \quad \mathcal{P}(Y_2, \phi_2) \in \Omega^{\mathrm{a}}_{w_{2,k}^\perp} \cap \Omega^{\mathrm{a}}_{h_2^\perp}.$$

Applying Proposition A.2.1, we conclude that $Y_1 \sim Z \sim Y_2$. □

A.2.8. Technical proofs. — We give here the proof of the announced lemmata.

Proof of Lemma A.2.4. — Pick projective marked hyper-Kähler manifolds (X_1', ψ_1) and (X_2', ψ_2) in $\mathfrak{M}$ such that:

- $\mathcal{P}(X_i', \psi_i) \in \Omega^{\mathrm{a}}_{h_i^\perp}$, for $i = 1, 2$;
- $\mathrm{NS}(X_i')$ contains an isotropic class $\psi_i^{-1}(y_i)$, for $i = 1, 2$.

This is possible because $b_2 > 6$: the maximal Picard number is then $b_2 - 2 \geq 5$; by Meyer's theorem [**58**], any indefinite lattice of rank at least 5 contains an isotropic vector. Hence, the Néron-Severi lattice of any Y with maximal Picard number of the given deformation type will contain an isotropic vector. Since hyper-Kähler varieties of maximal Picard rank are dense in the moduli space, we find X_1' and X_2' as above.

By assumption, up to replacing X_1' (resp. X_2') with a different birational model, $\psi_1^{-1}(h_1)$ (resp. $\psi_2^{-1}(h_2)$) is the first Chern class of an ample divisor on X_1' (resp. on X_2'). Hence, the class $\psi_1^{-1}(\lambda h_1 + y_1)$ (resp. $\psi_2^{-1}(\lambda h_2 + y_2)$) is also ample on X_1' (resp. on X_2') for $\lambda \gg 0$. We introduce the notation $e_i = (h_i, y_i)$ and $d_i = (h_i, h_i)$, for $i = 1, 2$. By the Hodge index theorem [**42**, §1.10], the orthogonal to h_i in $\mathrm{NS}(X_i')$ is negative definite; since y_i is isotropic, e_i must be not zero, for $i = 1, 2$.

We now choose a large prime number $p > N$ which does not divide neither $d_1 d_2$ nor $e_1 e_2$, and such that $p \equiv 3$ modulo 4. For a big enough odd integer j, the classes

$$l_1 = p^j h_1 + y_1 \quad \text{and} \quad l_2 = p^j h_2 + y_2,$$

are such that $\psi_1^{-1}(l_1)$ (resp. $\psi_2^{-1}(l_2)$) is ample on X_1' (resp. on X_2'). We have

$$(l_1, l_1) = p^j(p^j d_1 + 2e_1) \quad \text{and} \quad (l_2, l_2) = p^j(p^j d_2 + 2e_2).$$

We let $f_i = (p^j d_i + 2e_i)$; then f_i is not divisible by p. Moreover, for j sufficiently large, the lattice generated by l_1 and l_2 is of signature $(1,1)$ and $(l_1, l_2) > 0$, because (h_1, h_2) is positive by assumption.

Finally, we may assume that $f_1 f_2$ is not a square modulo p. Otherwise, we must have that $e_1 e_2$ is a square modulo p. We then choose an integer r which is not a square modulo p, and replace l_2 with $p^j h_2 + r y_2$; if j is an odd integer large enough, the class $\psi_2^{-1}(p^j h_2 + r y_2)$ is the first Chern class of an ample divisor on X_2', with Beauville–Bogomolov square $p^j(p^j d_2 + 2re_2)$. Note that re_1e_2 is not a square modulo p; if we now let $f_2 = p^j d_2 + 2re_2$, then $f_1 f_2$ is not a square modulo p. $\square$

Proof of Lemma A.2.5. — The orthogonal $\langle h_1, h_2\rangle^\perp \subset \Lambda$ to the sublattice generated by h_1 and h_2 is of signature $(2, b_2 - 4)$. Since $b_2 \geq 7$, we can find non-proportional isotropic vectors w_1 and w_2 in $\langle h_1, h_2\rangle^\perp$ such that $(w_1, w_2) = t < 0$. We may assume that p does not divide t.

Since $f_1 f_2$ is not a square modulo p, exactly one of them, say f_1, is a square modulo p. Assume that $2t$ is a square modulo p. Then we define:

$$\begin{aligned} v_1 &= pw_1 + (p-1)w_2, \\ v_2 &= pw_1 + w_2. \end{aligned}$$

We have

$$\begin{aligned} (v_1, v_1) &= 2(p-1)pt = p\epsilon_1, \\ (v_1, v_2) &= p(p-1)t + pt = p^2 t, \\ (v_2, v_2) &= 2pt = p\epsilon_2, \end{aligned}$$

where $\epsilon_1 = 2(p-1)t$ and $\epsilon_2 = 2t$. By construction, (v_1, v_2) is divisible by p^2 and ϵ_1 and ϵ_2 are negative integers not divisible by p. Moreover, ϵ_1 is not a square modulo p, while ϵ_2 is a square modulo p; note that $p-1$ is not a square modulo p since $p \equiv 3$ modulo 4. It follows that $\epsilon_1 f_1$, $\epsilon_2 f_2$ and $\epsilon_1\epsilon_2$ are not squares modulo p, as desired.

If instead $2t$ is not a square modulo p, we define

$$v_1 = pw_1 + w_2,$$
$$v_2 = pw_1 + (p-1)w_2,$$

and conclude similarly. □

Proof of Lemma A.2.6. — By construction, the intersection matrices of L_1 and L_2 are

$$\begin{pmatrix} p^j f_1 & 0 \\ 0 & p\epsilon_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p^j f_2 & 0 \\ 0 & p\epsilon_2 \end{pmatrix},$$

respectively, with $j \gg 0$ odd and $f_1, f_2, \epsilon_1, \epsilon_2$ not divisible by p. Moreover, $\epsilon_1 f_1$ and $\epsilon_2 f_2$ are not squares modulo p.

Let $v \in L_1 \otimes \mathbb{Q}$; the case of $v \in L_2 \otimes \mathbb{Q}$ is analogous. There exist integers γ, λ, δ such that $\gamma v = \lambda h_1 + \delta v_1$. We then have

$$\gamma^2(v,v) = p(\lambda^2 p^{j-1} f_1 + \delta^2 \epsilon_1).$$

Assume by contradiction that (v,v) is not divisible by p, and let m be the biggest integer such that p^m divides both λ and δ. We can then write

$$\gamma^2(v,v) = p^{2m+1}(\lambda_0^2 p^{j-1} f_1 + \delta_0^2 \epsilon_1),$$

where p does not divide both λ_0 and δ_0. The left hand-side is divisible by an even power of p. This forces δ_0 to be divisible by p, and hence $\delta_0 = p\delta_1$ for some integer δ_1. Therefore, λ_0 is not divisible by p. We obtain

$$\gamma^2(v,v) = p^{2m+3}(\lambda_0^2 p^{j-3} f_1 + \delta_1^2 \epsilon_1).$$

Again, p has to divide δ_1, so $\delta_0 = p^2 \delta_2$ and

$$\gamma^2(v,v) = p^{2m+5}(\lambda_0^2 p^{j-5} f_1 + \delta_2^2 \epsilon_1).$$

Proceeding in this way we find $\delta_{(j-1)/2}$ such that $\delta_0 = p^{(j-1)/2}\delta_{(j-1)/2}$ and

$$\gamma^2(v,v) = p^{2m+j}(\lambda_0^2 f_1 + \delta_{(j-1)/2}^2 \epsilon_1).$$

Now p has to divide $\lambda_0^2 f_1 + \delta_{(j-1)/2}^2 \epsilon_1$. But, since $f_1\epsilon_1$ is not a square modulo p, this implies that p divides λ_0, a contradiction. □

Proof of Lemma A.2.7. — We let $b = (h_1, h_2)$ and $(v_1, v_2) = p^2 e$. Recall that the classes v_1 and v_2 are orthogonal to $\langle h_1, h_2 \rangle$. The intersection matrix of S_k is

$$\begin{pmatrix} p^{2k+j} f_1 + p\epsilon_1 & p^{2k} b + p^2 e \\ p^{2k} b + p^2 e & p^{2k+j} f_2 + p\epsilon_2 \end{pmatrix},$$

where $f_1 f_2$, $\epsilon_1 f_1$, $\epsilon_2 f_2$ and $\epsilon_1 \epsilon_2$ are not squares modulo p (in particular, p does not divide f_1, f_2, ϵ_1, ϵ_2).

Given $v \in S_k \otimes \mathbb{Q}$, we find integers γ, λ, δ such that $\gamma v = \lambda w_{1,k} + \delta w_{2,k}$. Then:

$$\gamma^2 (v, v) = p(\lambda^2 (p^{2k+j-1} f_1 + \epsilon_1) + 2\lambda\delta(p^{2k-1} b + pe) + \delta^2 (p^{2k+j-1} f_2 + \epsilon_2)).$$

Let m be the biggest integer such that p^m divides both λ and δ; we have $\lambda = p^m \lambda_0$ and $\delta = p^m \delta_0$, with at least one among λ_0 and δ_0 not divisible by p. We can then write

$$\gamma^2 (v, v) = p^{2m+1} (\lambda_0^2 \epsilon_1 + \delta_0^2 \epsilon_2 + D),$$

where

$$D = \lambda_0^2 p^{2k+j-1} f_1 + 2\lambda_0 \delta_0 (p^{2k-1} b + pe) + \delta_0^2 p^{2k+j-1} f_2$$

is divisible by p, since $k \gg 0$.

Assume by contradiction that p does not divide (v, v). Then $\gamma^2 (v, v)$ is divisible by an even power of p, and hence p necessarily divides the term $(\lambda_0^2 \epsilon_1 + \delta_0^2 \epsilon_2 + D)$. But then we must have

$$\lambda_0^2 \epsilon_1 + \delta_0^2 \epsilon_2 \equiv 0 \text{ modulo } p.$$

Since $\epsilon_1 \epsilon_2$ is not a square modulo p, this implies $\lambda_0 \equiv \delta_0 \equiv 0$ modulo p, which is a contradiction. □

A.3. Conclusion of the proof

We now complete the proof of Claim A.2.2 and, hence, of Theorem A.0.1. By Proposition A.2.3, we are reduced to prove the following statement.

A.3.1. Claim. — *Let h_1, h_2 be positive classes in Λ. Then there exists finitely many vectors $v_1, v_2, v_3, \ldots, v_k \in \Lambda$ such that:*

- *$v_1 = h_1$ and $v_2 = h_2$;*
- *$(v_i, v_i) > 0$, for each $i = 1, \ldots, k$;*
- *$\langle v_i, v_{i+1} \rangle$ has signature $(1, 1)$ and $(v_i, v_{i+1}) > 0$, for each $i = 1, \ldots, k-1$.*

Proof. — We distinguish several cases. We follow an argument due to Soldatenkov [**79**, §6.2].

Case 0: $\langle h_1, h_2 \rangle$ is of signature $(1,1)$ and $(h_1, h_2) > 0$. In this case, there is nothing to do.

Case 1: $\langle h_1, h_2 \rangle$ is positive definite. We will then find a positive class $v \in \Lambda$ such that $(h_1, v) > 0$ and $(h_2, v) > 0$, and the lattices $\langle h_1, v \rangle$ and $\langle h_2, v \rangle$ are both of signature $(1,1)$, reducing to the previous case. We may assume that $(h_1, h_2) = 0$, for, otherwise, we pick a positive class $h_3 \in \langle h_1, h_2 \rangle^{\perp}$ and apply the argument below to h_1, h_3 and h_3, h_2 in place of h_1, h_2.

The subset V of $v \in \Lambda \otimes \mathbb{R}$ such that $\langle h_1, v \rangle$ and $\langle v, h_2 \rangle$ are both of signature $(1,1)$ with both (h_1, v) and (h_2, v) positive is an open cone in $\Lambda \otimes \mathbb{R}$; therefore, it suffices to show that V is not empty. We choose $w \in \langle h_1, h_2 \rangle^{\perp}$ such that $(w, w) < 0$. We let u_1, u_2, u_3 be the orthogonal basis of $\langle h_1, h_2, w \rangle \otimes \mathbb{R}$ such that

$$(u_1, u_1) = 1, \quad (u_2, u_2) = 1, \quad (u_3, u_3) = -1,$$

and $h_1 = \alpha u_1$, $h_2 = \beta u_2$ and $w = \gamma u_3$, for positive real numbers α, β and γ. A straightforward computation shows that the vector $v = e_1 + e_2 + \delta u_3$ is positive for $\delta^2 < 2$, and both the real vector spaces $\langle h_1, v \rangle$ and $\langle h_2, v \rangle$ are of signature $(1,1)$ for $\delta^2 > 1$. Moreover, $(h_1, v) = \alpha$ and $(h_2, v) = \beta$ are positive. Hence, if $1 < \delta^2 < 2$, the vector $v \in V$.

Case 2: $\langle h_1, h_2 \rangle$ is of signature $(1,1)$ and $(h_1, h_2) < 0$. In this case we simply let v be a positive class in $\langle h_1, h_2 \rangle^{\perp}$. Then $\langle h_1, v \rangle$ and $\langle h_2, v \rangle$ are positive definite, and we conclude by Case 1.

Case 3: $\langle h_1, h_2 \rangle$ is degenerate. Then it suffices to find a positive class v such that $\langle h_1, v \rangle$ and $\langle v, h_2 \rangle$ are both non-degenerate to reduce to the previous cases. The subset V of $v \in \Lambda \otimes \mathbb{R}$ such that $\langle h_1, v \rangle$ and $\langle v, h_2 \rangle$ are both non-degenerate is an open cone in $\Lambda \otimes \mathbb{R}$; it suffices to show that V is not empty. But this is obvious, as otherwise the open cone of $z \in \Lambda \otimes \mathbb{R}$ such that $(z, z) > 0$ would be contained in a hypersurface, which is impossible. $\square$

BIBLIOGRAPHY

[1] E. Amerik & M. Verbitsky – "Teichmüller space for hyperkähler and symplectic structures", *Journal of Geometry and Physics* **97** (2015), p. 44–50.

[2] ———, "Collections of Orbits of Hyperplane Type in Homogeneous Spaces, Homogeneous Dynamics, and Hyperkähler Geometry", *International Mathematics Research Notices* **2020** (2020), no. 1, p. 25–38.

[3] Y. André – "On the Shafarevich and Tate conjectures for hyperkähler varieties", *Mathematische Annalen* **305** (1996), p. 205–248.

[4] ———, "Pour une théorie inconditionnelle des motifs", *Publications Mathématiques de l'IHÉS* **83** (1996), p. 5–49.

[5] ———, *Une introduction aux motifs (Motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses, vol. 17, Société Mathématique de France, Paris, 2004.

[6] D. Arapura – "Motivation for Hodge cycles", *Adv. Math.* **207** (2006), no. 2, p. 762–781.

[7] M. Artin, A. Grothendieck & J.-L. Verdier – *Séminaire de Géométrie Algébrique du Bois-Marie 4: Théorie de Topos et Cohomologie étale des Schemas I, II, III*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer, 1972.

[8] A. Beauville – "Some remarks on Kähler manifolds with $c_1 = 0$", in *Classification of Algebraic and Analytic Manifolds, Katata, 1982*, Progr. Math., vol. 39, Birkhäuser Boston, 1983, p. 1–26.

[9] ———, "Variétés Kählériennes dont la première classe de Chern est nulle", *J. Differential Geom.* **18** (1983), no. 4, p. 755–782.

[10] ———, "Holomorphic symplectic geometry: a problem list", in *Complex and differential geometry*, Springer Proc. Math., vol. 8, Springer, 2011, p. 49–63.

[11] F. A. Bogomolov – "Hamiltonian Kähler manifolds", *Dokl. Akad. Nauk SSSR* **243** (1978), no. 5, p. 1101–1104.

[12] ———, "On the cohomology ring of a simple hyperkähler manifold (on the results of Verbitsky)", *Geometric & Functional Analysis GAFA* **6** (1996), no. 4, p. 612–618.

[13] S. Boissière, M. Nieper-Wisskirchen & A. Sarti – "Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties", *Journal de mathématiques pures et appliquées* **95** (2011), no. 5, p. 553–563.

[14] A. Borel – "Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem", *Journal of Differential Geometry* **6** (1972), no. 4, p. 543–560.

[15] T.-H. Bülles – "Motives of moduli spaces on K3 surfaces and of special cubic fourfolds", *Manuscripta Math.* **161** (2020), p. 109–124.

[16] A. Cadoret & B. Moonen – "Integral and adelic aspects of the Mumford-Tate conjecture", *Journal of the Institute of Mathematics of Jussieu* **19** (2018), no. 3, p. 869–890.

[17] M. de Cataldo & L. Migliorini – "The Chow groups and the motive of the Hilbert scheme of points on a surface", *J. Algebra* **251** (2002), no. 2, p. 824–848.

[18] ———, "The Chow motive of semismall resolutions", *Math. Res. Lett.* **11** (2004), no. 2, p. 151–170.

[19] F. Charles – "Birational boundedness for holomorphic symplectic varieties, Zarhin's trick for K3 surfaces, and the Tate conjecture", *Annals of Mathematics* **184** (2016), no. 2, p. 487–526.

[20] J. Commelin – "The Mumford–Tate conjecture for products of abelian varieties", *Algebr. Geom.* **6** (2019), no. 6, p. 650–677.

[21] O. Debarre & C. Voisin – "Hyper-Kähler fourfolds and Grassmann geometry", *J. Reine Angew. Math* **649** (2010), p. 63–87.

[22] P. Deligne – "La conjecture de Weil pour les surfaces K3", *Inventiones mathematicae* **15** (1971), p. 206–226.

[23] ———, "Théorie de Hodge: II", *Publications Mathématiques de l'IHÉS* **40** (1971), p. 5–57.

[24] ———, "La conjecture de Weil: II", *Publications Mathématiques de l'IHÉS* **52** (1980), p. 137–252.

[25] ———, "Hodge cycles on abelian varieties", in *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900, Springer, 1982, p. 9–100.

[26] E. Fatighenti & G. Mongardi – "Fano varieties of K3 type and IHS manifolds", *International Mathematics Research Notices* **2021** (2021), no. 4, p. 3097–3142.

[27] S. Floccari – "On the Mumford-Tate conjecture for hyper-Kähler varieties", *arXiv preprint: 1904.06238* (2019).

[28] ———, "Galois representations on the cohomology of hyper-Kähler varieties", *arXiv preprint: 2007.01841* (2020).

[29] S. Floccari, L. Fu & Z. Zhang – "On the motive of O'Grady's ten-dimensional hyper-Kähler varieties", *Communications in Contemporary Mathematics* (2020), 2050034.

[30] S. Frei – "Moduli spaces of sheaves on K3 surfaces and Galois representations", *Selecta Mathematica* **26** (2020), no. 6, p. 1–16.

[31] L. Fu & Z. Li – "Supersingular irreducible symplectic varieties", in *Rationality of Algebraic Varieties, Schiermonnikoog, 2019*, to appear.

[32] L. Fu, Z. Li & H. Zou – "Supersingular O'Grady varieties of dimension six", *International Mathematics Research Notices* (2021), rnaa349.

[33] L. Fu, Z. Tian & C. Vial – "Motivic hyper-Kähler resolution conjecture I: Generalized Kummer varieties", *Geometry & Topology* **23** (2019), no. 1, p. 427–492.

[34] A. Fujiki – "On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold", in *Algebraic Geometry, Sendai, 1985*, Mathematical Society of Japan, 1987, p. 105–165.

[35] B. v. Geemen & C. Voisin – "On a conjecture of Matsushita", *International Mathematics Research Notices* **2016** (2016), no. 10, p. 3111–3123.

[36] M. Green, Y.-J. Kim, R. Laza & C. Robles – "The LLV decomposition of hyper-Kähler cohomology", *arXiv preprint: 1906.03432* (2019).

[37] P. A. Griffiths – "Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems", *Bull. Amer. Math. Soc.* **76** (1970), no. 2, p. 228–296.

[38] V. Gritsenko, K. Hulek & G. K. Sankaran – "Moduli spaces of irreducible symplectic manifolds", *Compositio Mathematica* **146** (2010), no. 2, p. 404–434.

[39] A. Grothendieck – "Standard conjectures on algebraic cycles", in *Algebraic geometry, Tata Institute, Bombay 1968*, Oxford Univ. Press, London, 1969, p. 193–199.

[40] N. P. Gushel – "On Fano varieties of genus 6", *Izv. Math.* **21** (1983), no. 3, p. 445–459.

[41] B. Hassett & Y. Tschinkel – "Hodge theory and Lagrangian planes on generalized Kummer fourfolds", *Moscow Mathematical Journal* **13** (2013), no. 1, p. 33–56.

[42] D. Huybrechts – "Compact hyperkähler manifolds: basic results", *Invent. Math.* **135** (1999), p. 63–113.

[43] ______, *Complex Geometry: An Introduction*, Universitext (Berlin. Print), Springer, 2005.

[44] ______, "A global Torelli Theorem for hyperkähler manifolds [after M. Verbitsky]", *Séminaire Bourbaki Juin 2011, Astérisque* (2012), no. 1040, p. 375–403.

[45] ______, *Lectures on K3 surfaces*, vol. 158, Cambridge University Press, 2016.

[46] D. Huybrechts & M. Lehn – *The Geometry of Moduli Spaces of Sheaves*, 2^{nd} ed., Cambridge University Press, 2010.

[47] D. Kaledin, M. Lehn & C. Sorger – "Singular symplectic moduli spaces", *Invent. Math.* **164** (2006), p. 591–614.

[48] N. Kurnosov, A. Soldatenkov & M. Verbitsky – "Kuga-Satake construction and cohomology of hyperkähler manifolds", *Adv. Math.* **351** (2019), p. 275–295.

[49] R. Laterveer – "A remark on the motive of the Fano variety of lines of a cubic", *Annales Mathématiques du Québec* **41** (2017), p. 141–154.

[50] M. Lehn & C. Sorger – "La singularité de O'Grady", *J. Algebraic Geom.* **15** (2006), p. 753–770.

[51] E. Looijenga & V. A. Lunts – "A Lie algebra attached to a projective variety", *Invent. Math.* **129** (1997), p. 361–412.

[52] Y. I. Manin – "Correspondences, motifs and monoidal transformations", *Mat. Sb. (N.S.)* **77** (1968), no. 119, p. 475–507.

[53] E. Markman – "Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces", *J. Reine Angew. Math.* **544** (2002), p. 61–82.

[54] ______, "Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces", *Advances in Mathematics* **208** (2007), no. 2, p. 622–646.

[55] ______, "On the monodromy of moduli spaces of sheaves on K3 surfaces", *J. Algebraic Geom.* **17** (2008), p. 29–99.

[56] ———, “A survey of Torelli and monodromy results for holomorphic-symplectic varieties”, in *Complex and differential geometry*, Springer Proc. Math., vol. 8, Springer, 2011, p. 257–322.

[57] ———, “The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians”, *arXiv preprint: 1805.11574* (2019).

[58] A. Meyer – “Mathematische Mittheilungen”, *Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich* **29** (1884), p. 209–222.

[59] G. Mongardi – “A note on the Kähler and Mori cones of hyper-Kähler manifolds”, *Asian Journal of Mathematics* **19** (2015), no. 4, p. 583–592.

[60] G. Mongardi, A. Rapagnetta & G. Saccà – “The Hodge diamond of O’Grady’s six-dimensional example”, *Compositio Mathematica* **154** (2018), no. 5, p. 984–1013.

[61] G. Mongardi & M. Wandel – “Automorphisms of O’Grady’s manifolds acting trivially on cohomology”, *Algebraic Geometry* **4** (2017), no. 1, p. 104–119.

[62] B. Moonen – “Families of Motives and the Mumford-Tate Conjecture”, *Milan J. of Math.* **85** (2017), p. 257–307.

[63] ———, “On the Tate and Mumford–Tate conjectures in codimension 1 for varieties with $h^{2,0} = 1$”, *Duke Math J.* **166** (2017), no. 4, p. 739–799.

[64] S. Mukai – “Symplectic structure of the moduli space of sheaves on an abelian or K3 surface”, *Invent. Math.* **77** (1984), p. 101–116.

[65] ———, “On the moduli space of bundles on K3 surfaces. I”, in *Vector bundles on algebraic varieties, Tata Institute, Bombay 1984)*, vol. 11, Fund. Res. Stud. Math, 1987, p. 341–413.

[66] ———, “Biregular classification of Fano 3-folds and Fano manifolds of coindex 3”, *Proc. Nat. Acad. Sci. U.S.A.* **86** (1989), no. 9, p. 3000–3002.

[67] K. O’Grady – “The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface”, *J. Algebraic Geom.* **6** (1997), no. 4, p. 599–644.

[68] ———, "Desingularized moduli spaces of sheaves on a K3", *J. Reine Angew. Math.* **512** (1999), p. 49–117.

[69] ———, "A new six-dimensional irreducible symplectic variety", *J. Algebraic Geom.* **12** (2003), no. 3, p. 435–505.

[70] K. Oguiso – "No cohomologically trivial nontrivial automorphism of generalized Kummer manifolds", *Nagoya Mathematical Journal* **239** (2020), p. 110–122.

[71] C. Pedrini – "On the finite dimensionality of a K3 surface", *Manuscripta Math.* **138** (2012), no. 1-2, p. 59–72.

[72] A. Perego & A. Rapagnetta – "Deformation of the O'Grady moduli spaces", *J. Reine Angew. Math.* **678** (2013), p. 1–34.

[73] R. Pink – "ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture", *J. Reine Angew. Math.* **495** (1998), p. 187–237.

[74] A. Rapagnetta – "Topological invariants of O'Grady's six dimensional irreducible symplectic variety", *Mathematische Zeitschrift* **256** (2007), no. 1, p. 1–34.

[75] ———, "On the Beauville form of the known irreducible symplectic varieties", *Mathematische Annalen* **340** (2008), no. 1, p. 77–95.

[76] U. Riess – "On the Chow ring of birational irreducible symplectic varieties", *Manuscripta Math.* **145** (2014), no. 3-4, p. 473–501.

[77] U. Schlickewei – "On the André motive of certain irreducible symplectic varieties", *Geom. Dedicata* **156** (2012), no. 1, p. 141–149.

[78] J.-P. Serre – *Local Fields*, Graduate Texts in Mathematics, no. 67, Springer, 1979.

[79] A. Soldatenkov – "Deformation principle and André motives of projective hyper-Kähler manifolds", *arXiv preprint: 1904.11320* (2019).

[80] ———, "On the Hodge structure of compact hyper-Kähler manifolds", *arXiv preprint: 1905.07793* (2019).

[81] S. G. Tankeev – "Cycles of abelian varieties of prime dimension over finite and number fields", *Izv. Akad. Nauk SSSR Ser. Mat.* **47** (1983), no. 2, p. 356–365.

[82] M. Verbitsky – "Action of the Lie algebra SO(5) on the cohomology of a hyper-Kähler manifold", *Funktsional'nyi Analiz i ego Prilozheniya* **24** (1990), no. 3, p. 70–71.

[83] ______, "Cohomology of compact hyper-Kähler manifolds and its applications", *Geometric & Functional Analysis GAFA* **6** (1996), no. 4, p. 601–611.

[84] ______, "Mapping class group and a global Torelli theorem for hyper-Kähler manifolds", *Duke Math. J.* **162** (2013), no. 15, p. 2929–2986.

[85] ______, "Errata for "Mapping class group and a global Torelli theorem for hyper-Kähler manifolds" by misha verbitsky", *Duke Math. J.* **169** (2020), no. 5, p. 1037–1038.

[86] C. Vial – "Algebraic cycles and fibrations", *Doc. Math.* **18** (2013), p. 1521–1553.

[87] ______, "Remarks on motives of abelian type", *Tohoku Math. J.* **69** (2017), no. 2, p. 195–220.

[88] E. Viehweg – *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, vol. 30, Springer, 1995.

[89] C. Voisin – *Hodge Theory and Complex Algebraic Geometry I*, Cambridge Stud. Adv. Math., no. 76, Cambridge University Press, 2002.

[90] Z. Xu – "Algebraic cycles on a generalized Kummer variety", *Int. Math. Res. Not.* **3** (2018), p. 932–948.

[91] S.-T. Yau – "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation, I", *Comm. Pure Appl. Math.* **31** (1978), no. 3, p. 339–411.

[92] K. Yoshioka – "Moduli spaces of stable sheaves on abelian surfaces", *Math. Ann.* **321** (2001), no. 4, p. 817–884.

SAMENVATTIG

Hyper-Kähler variëteiten vormen een belangrijke en interessante klasse van hoger-dimensionale algebraïsche variëteiten. In dimensie 2 zijn het K3-oppervlakken, en dankzij het werk van vele auteurs hebben we nu een rijke theorie van hyper-Kähler variëteiten die in veel opzichten gelijkenis vertoont met die van deze oppervlakken. Desondanks zijn er nog veel fundamentele vragen onbeantwoord; zo ontbreekt er bijvoorbeeld een topologische classificatie van hyper-Kähler variëteiten.

Veel van de bekende constructiemethoden van hyper-Kähler variëteiten zijn gebaseerd op het nemen van moduliruimten van schoven op een abels oppervlak of een K3-oppervlak. In zulke gevallen moeten de oppervlakken op de een of andere manier de meetkunde van de verkregen hyper-Kähler variëteit bepalen. Deformaties van zo'n moduliruimte geven hyper-Kähler variëteiten die niet meer van deze vorm zijn en die, a priori, niet direct gerelateerd zijn aan een oppervlak.

In dit proefschrift wordt onderzocht hoe de motieven van hyper-Kähler variëteiten worden beheerst door kleinere, "oppervlakte-achtige" motieven. Meer precies formuleren we de verwachting dat het André motief $\mathcal{H}^\bullet(X)$ van een hyper-Kähler variëteit X gereconstrueerd kan worden uit zijn Künneth component in graad 2 door middel van tensorconstructies.

Om dit probleem aan te pakken, koppelen we aan elke hyper-Kähler variëteit X (met $b_2(X) > 3$) een algebraïsche groep $P(X)$, die het falen van onze verwachting meet. We noemen $P(X)$ de *defectgroep* van X; deze groep is triviaal dan en slechts dan als het motief $\mathcal{H}^\bullet(X)$ behoort tot de tensorcategorie die wordt voortgebracht door $\mathcal{H}^2(X)$. Door diepgaande ideeën over families van motieven, die teruggaan tot

Deligne, André en Moonen, te combineren met een representatietheoretische constructie van Verbitsky en Looijenga–Lunts, kunnen we aantonen dat $P(X)$ een aantal mooie eigenschappen heeft. Dit leidt ertoe dat we onze voorspelling over motieven van hyper-Kähler variëteiten kunnen bewijzen voor alle momenteel bekende deformatietypen van hyper-Kähler variëteiten.

Stelling. — *De defectgroep van alle thans bekende voorbeelden van hyper-Kähler variëteiten is triviaal.*

De vraag blijft open voor mogelijk nog te ontdekken deformatietypen van hyper-Kähler variëteiten. Niettemin kunnen we aantonen dat hun motieven bepaald worden door de Künneth component in graad 2, in de volgende zin.

Stelling. — *Stel X en Y zijn deformatie-equivalente hyper-Kähler variëteiten met $b_2 > 6$ en triviale oneven cohomologie, en neem aan dat er een Hodge isometrie $H^2(X) \simeq H^2(Y)$ bestaat. Dan zijn de André motieven van X en Y isomorf.*

Voor hyper-Kähler variëteiten waarvan de cohomologie in oneven graad niet triviaal is bewijzen we een vergelijkbaar resultaat onder een extra technische aanname.

De belangrijkste toepassing van deze resultaten is het Mumford–Tate vermoeden voor hyper-Kähler variëteiten. Voor een gladde en projectieve variëteit X voorspelt het Mumford–Tate vermoeden dat er een direct verband is tussen de Hodgestructuur en de ℓ-adische Galoisrepresentatie op de cohomologie van X. Dit moeilijke vermoeden is zeer opmerkelijk doordat de structuren die in deze relatie een rol spelen heel verschillend van aard zijn. We bewijzen dat het Mumford–Tate vermoeden waar is voor alle thans bekende hyper-Kähler variëteiten.

Stelling. — *Het Mumford–Tate vermoeden is waar voor alle thans bekende hyper-Kähler variëteiten en alle producten van zulke variëteiten.*

Een direct gevolg van deze stelling is dat voor elk product van thans bekende hyper-Kähler variëteiten het Hodge-vermoeden equivalent is met het Tate-vermoeden.

SUMMARY

Hyper-Kähler varieties are an important class of higher dimensional algebraic varieties. In dimension 2 they are K3 surfaces, and, thanks to the work of many authors, we now have a rich theory of hyper-Kähler varieties which parallels in many respects that of these surfaces. Despite this, many fundamental questions remain out of reach, for instance, the topological classification of hyper-Kähler varieties is still unknown.

Many of the known construction methods of hyper-Kähler varieties involve taking moduli spaces of sheaves on an abelian or K3 surface. In such cases, the surface should somehow govern the geometry of the hyper-Kähler variety obtained. Deformations of such a moduli space give hyper-Kähler varieties which are not anymore of this form, and a priori, not related to any surface.

This thesis investigates how the motives of hyper-Kähler varieties are controlled by smaller, "surface-like" motives. More precisely, we formulate the expectation that the André motive $\mathcal{H}^\bullet(X)$ of a hyper-Kähler variety X can be reconstructed from its Künneth component in degree 2 by means of tensor constructions.

To tackle this problem, to any hyper-Kähler variety X (with $b_2(X) > 3$) we attach an algebraic group $P(X)$, which measures the failure of our expectation. We call $P(X)$ the *defect group* of X; it is trivial if and only if the motive $\mathcal{H}^\bullet(X)$ belongs to the tensor category generated by $\mathcal{H}^2(X)$. Combining deep ideas on families of motives going back to Deligne, André and Moonen with a representation theoretic construction by Verbitsky and Looijenga–Lunts, we are able to show that $P(X)$ enjoys several nice property, and confirm our prediction for all hyper-Kähler varieties of known deformation type.

Theorem. — *The defect group of any known hyper-Kähler variety is trivial.*

The question remains open for potentially yet to be discovered deformation types of hyper-Kähler varieties. Nevertheless, we are still able to show that their motives are determined by the Künneth component in degree 2, in the following sense.

Theorem. — *Let X and Y be deformation equivalent hyper-Kähler varieties with $b_2 > 6$ and trivial odd cohomology, and assume that there exists a Hodge isometry $H^2(X) \simeq H^2(Y)$. Then the André motives of X and Y are isomorphic.*

In presence of non-trivial odd cohomology we prove a similar result under an additional technical assumption.

Our main application is to the Mumford–Tate conjecture for hyper-Kähler varieties. For any smooth and projective variety X, the Mumford–Tate conjecture predicts a comparison between the Hodge structure and the ℓ-adic Galois representation attached to X. This challenging conjecture is most remarkable due to the very different nature of the objects involved in the comparison. We obtain new evidence towards the validity of the Mumford–Tate conjecture.

Theorem. — *The Mumford–Tate conjecture holds for any known hyper-Kähler variety and any product of such varieties.*

A direct consequence of this theorem is that the conjectures of Hodge and Tate are equivalent for any product of known hyper-Kähler varieties.

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CURRICULUM VITAE

Salvatore Floccari was born in Padova, Italy, on October 7th 1992. First of two twins, during his happy childhood he tried several different sports and played the piano. Since a young age he showed a keen interest in science and mathematics.

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In February 2017, Salvatore moved to the Netherlands where he started working at Radboud Universiteit Nijmegen as a PhD student of Ben Moonen and Arne Smeets. The outcome of the research carried out during these years is the subject of this booklet.

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