$x_1^2 + x_2^2 + \cdots + x_n^2 = x_1 x_2 \cdots x_n$

Helmut Postl, Torsten Sillke *

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1 Introduction

We are looking for integral solutions of the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = \alpha x_1 x_2 \cdots x_n$. This problem is called Hurwitz equation and it is known that only for $1 \le \alpha \le n$ there are integral solutions [1, section D12: Markoff numbers] [2]. And for any n there is a finite set of solutions which generate all others. The special case $n = \alpha = 3$ it is called Markoff equation. The special case $\alpha = 1$ was stated by Karl Scherer [3]. The related equation $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$ is partially analyzed in [4] [1, section D24]. In this paper we will effectively construct all solutions for a given n. We will show that the number of solutions with $2x_i^2 \le x_1 x_2 \cdots x_n$ is finite for every n. These solutions we call *basic* solutions [5]. All other solutions can be constructed from this ones.

2 Trivial cases

If one x_i is zero then the product is zero and therefore all x_i must be zero. This gives the zero solution. So assume in the following all x_i are positive integers.

Case n=1: The equation $x_1^2 = x_1$ that is $x_1(x_1 - 1) = 0$ has only one positive solution $x_1 = 1$.

Case n=2: The equation $x_1^2 + x_2^2 = x_1 x_2$ has no positive solution as $x_1^2 + x_2^2 \ge 2x_1 x_2$. This follows from $(x_1 - x_2)^2 \ge 0$.

In the following we assume $n \ge 3$.

3 Reduction to basic solution

Let $P_k = \prod_{i=k}^{n} x_i$ and $S_k = \sum_{i=k}^{n} x_i^2$ Then we have a quadratic equation $x_1 P_2 = x_1^2 + S_2$ in the variable x_1. Is x_1 a root of the equation so is $P_2 - x_1$ the other using Vieta.

A solution is called *basic* if $2x_k^2 \le P_1$ for all $1 \le k \le n$. The transformation

$$x_k \longleftarrow \frac{P_1}{x_k} - x_k \tag{1}$$

generates further solutions. Each solution contains at most one k for which we have $2x_k^2 > P_1$. Assume on the contrary there were a solution with $2x_1^2 > P_1$ and $2x_2^2 > P_1$. Then we had $2x_1 x_2 > P_1$ and this means $2 > P_3$ that is $P_3 = 1$ which is impossible. Therefore we have for each solution a unique reduction path to its *basic* solution. in other words – the graph of solutions is a forest. Note: If we have for a solution $d = \gcd(x_1, x_2, \ldots, x_n)$ then d is the gcd for all solutions of the same tree, as d is a factor of the transformation. We will see that $d = 3$ for $n = 3$ and $d = 2$ for $n = 4$. Otherwise $d = 1$ as the *basic* solutions contain the value one.

4 Determing the basic solutions

In the following let $x_1 \ge x_2 \ge \cdots \ge x_n$. For a *basic* solution $(x_1, x_2, \ldots, x_n)$ we have $x_1 \le \frac{P_2}{2}$. Then x_1 is determined by the quadratic equation $x_1 P_2 = x_1^2 + S_2$ as

$$x_1 = \frac{P_2}{2} - \sqrt{\frac{P_2^2}{4} - S_2}. \tag{2}$$

*e–mail: Torsten.Sillke@uni-bielefeld.de

From the inequality $2x_2 \leq 2x_1 = P_2 - \sqrt{P_2^2 - 4S_2}$ we derive $\sqrt{P_2^2 - 4S_2} \leq P_2 - 2x_2$. Dividing x_2 and squaring gives $P_3 \leq \frac{S_2}{x_2^2} + 1$ or the inequality

$$P_3 \leq \frac{S_3}{x_2^2} + 2. \tag{3}$$

Now we can bound S_3 as $S_3 = \sum_{i=3}^{n} x_i^2 \leq (n-2)x_2^2$ which gives us $P_3 \leq n$. As P_3 must be at least three we have

$$3 \leq P_3 \leq n \tag{4}$$

and therefore we have finitely many vectors $(x_3, x_4, \ldots, x_n)$ which can occur in a *basic* solution. With inquality 3 we derive

$$x_3^2 \leq x_2^2 \leq \frac{S_3}{P_3 - 2} \leq S_3. \tag{5}$$

This proves the theorem

Theorem 1 *For each n the number of* basic *solutions is finite.*

The lower bound for P_3 is sharp as there are infinitely many *basic* solutions with $P_3 = 3$. These are $\frac{3}{2}x_2 \geq x_1 \geq x_2 \geq 3 = x_3 > x_4 = \cdots = x_n = 1$. In this case n is determined by $n = x_1x_2 - (x_1 - x_2)^2 - 6$.

On the other hand the upper bound $P_3 \leq n$ is only sharp for $n = 4$. The *basic* solutions $x_1 = x_2 = \cdots = x_k = 2 > 1 = x_{k+1} = \cdots = x_n$ with $k \geq 4$ have $n = 2^k - 3k$ and $P_3 = 2^{k-2}$. So there are *basic* solutions having $P_3 = \frac{1}{4}n + O(\log(n))$.

5 A question of Karl Scherer

Now it is easy to answer a question of Karl Scherer, who wanted to know: Given an integer x what is the minimal n^* for which x is member of a solution? As we know every $x \geq 3$ is member of a $P_3 = 3$ *basic* solution. So the sequence of Scherer is well defined. In the following table this sequence is calculated upto $x = 120$ showing that the constuction method given in this note is very effective.

$n^*(x)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	4	3	5	5	3	13	10	5	7	13	5	10	8	3	14	13	7	25	10
+20	14	4	5	34	13	46	23	13	10	28	5	25	5	23	5	14	7	35	3	22
+40	25	23	47	5	38	19	10	10	25	86	39	8	27	39	37	58	5	7	13	5
+60	53	38	13	73	19	46	7	62	30	35	14	74	13	19	23	26	70	14	26	49
+80	5	4	13	130	53	23	3	49	23	67	43	79	13	14	10	65	59	70	35	37
+100	73	3	127	25	19	54	5	10	25	8	34	49	193	86	13	13	65	67	7	38

As an example we derive a solution with $x = 84$ for $n = 130$ from a *basic* solution. $(9, 8, 4, 1_{127}) \longrightarrow (23, 8, 4, 1_{127}) \longrightarrow (84, 23, 4, 1_{127})$.

6 Number of basic solutions

The number of *basic* solution of at most n terms is a slow growing function $s(n)$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	20	50	100	200	500	1000
$s(n)$	1	2	3	3	4	5	5	6	6	6	7	9	9	13	45	110	286	989	2544

It is a open question to determine an asymptotic formula for this function.

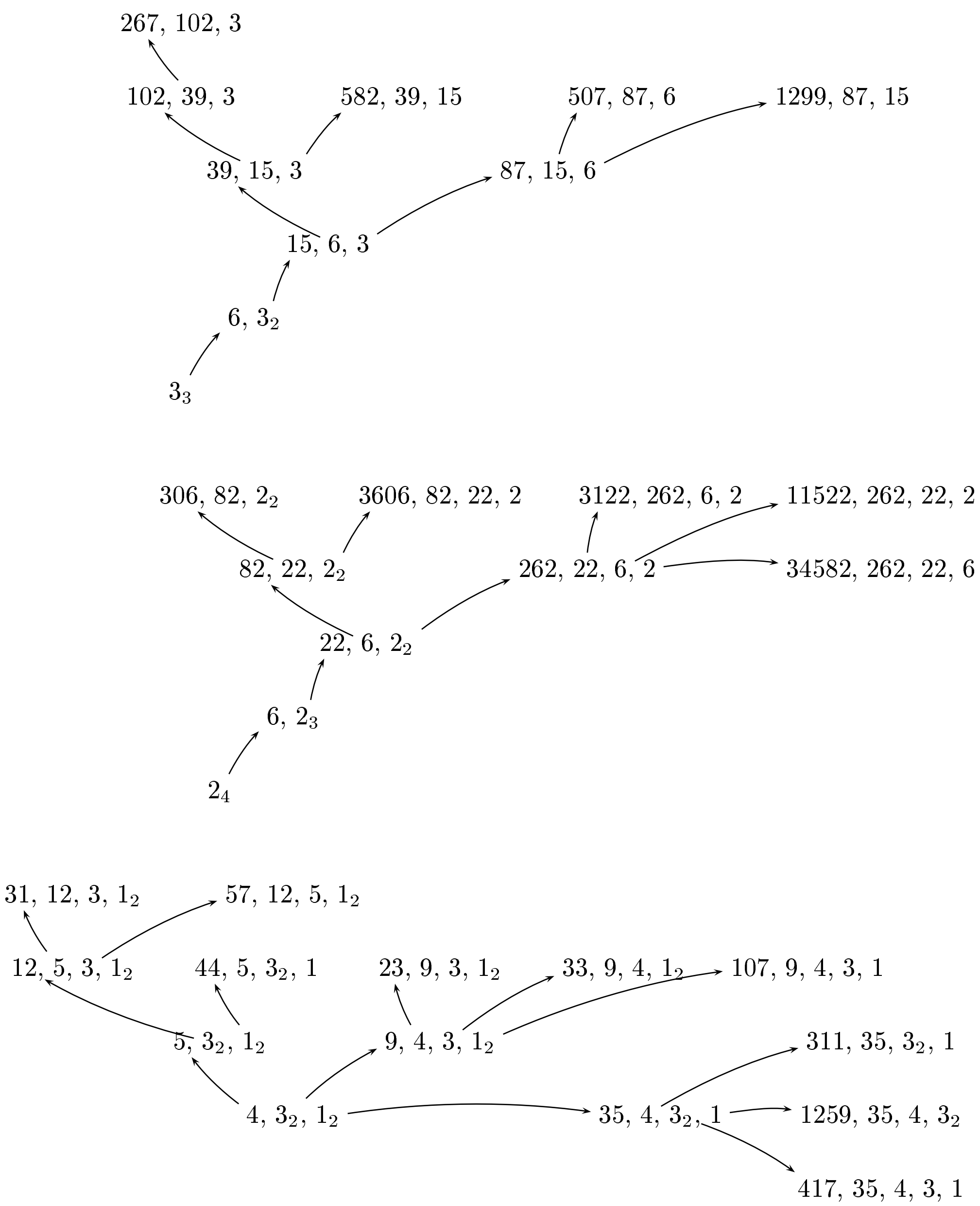
7 Table of basic solutions

To save space the solutions are given in run-length encoding. E. g. 3_3 reads as $3,3,3$.

n	$S_1 = P_1$	S_2	P_2	S_3	P_3	$x_1, x_2, \ldots$
3	27	18	9	9	3	3_3
4	16	12	8	8	4	2_4
5	36	20	9	11	3	$4, 3_2, 1_2$
7	24	15	8	11	4	$3, 2_3, 1_3$
8	32	16	8	12	4	$4, 2_3, 1_4$
10	48	32	12	16	3	$4_2, 3, 1_7$
13	60	35	12	19	3	$5, 4, 3, 1_{10}$
14	72	36	12	20	3	$6, 4, 3, 1_{11}$
14	36	27	12	18	4	$3_2, 2_2, 1_{10}$
17	32	28	16	24	8	$2_5, 1_{12}$
19	75	50	15	25	3	$5_2, 3, 1_{16}$
19	48	32	12	23	4	$4, 3, 2_2, 1_{15}$
19	64	48	16	32	4	$4_3, 1_{16}$
22	60	35	12	26	4	$5, 3, 2_2, 1_{18}$
23	90	54	15	29	3	$6, 5, 3, 1_{20}$
23	72	36	12	27	4	$6, 3, 2_2, 1_{19}$
25	105	56	15	31	3	$7, 5, 3, 1_{22}$
26	80	55	16	39	4	$5, 4_2, 1_{23}$
27	54	45	18	36	6	$3_3, 2, 1_{23}$
28	64	48	16	32	4	$4_2, 2_2, 1_{24}$
28	48	39	16	35	8	$3, 2_4, 1_{23}$
30	108	72	18	36	3	$6_2, 3, 1_{27}$
31	96	60	16	44	4	$6, 4_2, 1_{28}$
34	112	63	16	47	4	$7, 4_2, 1_{31}$
35	126	77	18	41	3	$7, 6, 3, 1_{32}$
35	80	55	16	39	4	$5, 4, 2_2, 1_{31}$
35	128	64	16	48	4	$8, 4_2, 1_{32}$
37	100	75	20	50	4	$5_2, 4, 1_{34}$
37	64	48	16	44	8	$4, 2_4, 1_{32}$
38	144	80	18	44	3	$8, 6, 3, 1_{35}$
38	72	56	18	47	6	$4, 3_2, 2, 1_{34}$
39	162	81	18	45	3	$9, 6, 3, 1_{36}$
40	96	60	16	44	4	$6, 4, 2_2, 1_{36}$
43	147	98	21	49	3	$7_2, 3, 1_{40}$
43	112	63	16	47	4	$7, 4, 2_2, 1_{39}$
44	128	64	16	48	4	$8, 4, 2_2, 1_{40}$
44	80	55	16	51	8	$5, 2_4, 1_{39}$
46	100	75	20	50	4	$5_2, 2_2, 1_{42}$
46	120	84	20	59	4	$6, 5, 4, 1_{43}$
46	64	60	32	56	16	$2_6, 1_{40}$
47	90	65	18	56	6	$5, 3_2, 2, 1_{43}$
47	72	63	24	54	8	$3_2, 2_3, 1_{42}$
49	168	104	21	55	3	$8, 7, 3, 1_{46}$
49	96	60	16	56	8	$6, 2_4, 1_{44}$
49	81	72	27	63	9	$3_4, 1_{45}$
52	112	63	16	59	8	$7, 2_4, 1_{47}$
53	189	108	21	59	3	$9, 7, 3, 1_{50}$
53	140	91	20	66	4	$7, 5, 4, 1_{50}$
53	125	100	25	75	5	$5_3, 1_{50}$
53	128	64	16	60	8	$8, 2_4, 1_{48}$
54	108	72	18	63	6	$6, 3_2, 2, 1_{50}$
55	210	110	21	61	3	$10, 7, 3, 1_{52}$
55	120	84	20	59	4	$6, 5, 2_2, 1_{51}$
55	96	80	24	64	6	$4_2, 3, 2, 1_{51}$
58	192	128	24	64	3	$8_2, 3, 1_{55}$

n	$S_1 = P_1$	S_2	P_2	S_3	P_3	$x_1, x_2, \ldots$
58	160	96	20	71	4	$8, 5, 4, 1_{55}$
59	144	108	24	72	4	$6_2, 4, 1_{56}$
59	126	77	18	68	6	$7, 3_2, 2, 1_{55}$
61	180	99	20	74	4	$9, 5, 4, 1_{58}$
62	140	91	20	66	4	$7, 5, 2_2, 1_{58}$
62	200	100	20	75	4	$10, 5, 4, 1_{59}$
62	144	80	18	71	6	$8, 3_2, 2, 1_{58}$
63	162	81	18	72	6	$9, 3_2, 2, 1_{59}$
64	96	80	24	71	8	$4, 3, 2_3, 1_{59}$
65	216	135	24	71	3	$9, 8, 3, 1_{62}$
67	160	96	20	71	4	$8, 5, 2_2, 1_{63}$
67	150	114	25	89	5	$6, 5_2, 1_{64}$
68	144	108	24	72	4	$6_2, 2_2, 1_{64}$
69	108	92	27	83	9	$4, 3_3, 1_{65}$
70	240	140	24	76	3	$10, 8, 3, 1_{67}$
70	180	99	20	74	4	$9, 5, 2_2, 1_{66}$
70	168	119	24	83	4	$7, 6, 4, 1_{67}$
70	120	95	24	79	6	$5, 4, 3, 2, 1_{66}$
71	200	100	20	75	4	$10, 5, 2_2, 1_{67}$
73	264	143	24	79	3	$11, 8, 3, 1_{70}$
73	96	87	32	83	16	$3, 2_5, 1_{67}$
74	288	144	24	80	3	$12, 8, 3, 1_{71}$
75	243	162	27	81	3	$9_2, 3, 1_{72}$
78	108	99	36	90	12	$3_3, 2_2, 1_{73}$
79	168	119	24	83	4	$7, 6, 2_2, 1_{75}$
79	192	128	24	92	4	$8, 6, 4, 1_{76}$
79	175	126	25	101	5	$7, 5_2, 1_{76}$
79	120	95	24	86	8	$5, 3, 2_3, 1_{74}$
80	128	112	32	96	8	$4_3, 2, 1_{76}$
83	270	170	27	89	3	$10, 9, 3, 1_{80}$
83	144	108	24	92	6	$6, 4, 3, 2, 1_{79}$
85	196	147	28	98	4	$7_2, 4, 1_{82}$
86	216	135	24	99	4	$9, 6, 4, 1_{83}$
86	180	144	30	108	5	$6_2, 5, 1_{83}$
87	135	110	27	101	9	$5, 3_3, 1_{83}$
88	192	128	24	92	4	$8, 6, 2_2, 1_{84}$
89	297	176	27	95	3	$11, 9, 3, 1_{86}$
89	200	136	25	111	5	$8, 5_2, 1_{86}$
89	128	112	32	96	8	$4_2, 2_3, 1_{84}$
91	240	140	24	104	4	$10, 6, 4, 1_{88}$
91	150	125	30	100	6	$5_2, 3, 2, 1_{87}$
92	144	108	24	99	8	$6, 3, 2_3, 1_{87}$
93	324	180	27	99	3	$12, 9, 3, 1_{90}$
94	300	200	30	100	3	$10_2, 3, 1_{91}$
94	196	147	28	98	4	$7_2, 2_2, 1_{90}$
94	264	143	24	107	4	$11, 6, 4, 1_{91}$
94	168	119	24	103	6	$7, 4, 3, 2, 1_{90}$
95	351	182	27	101	3	$13, 9, 3, 1_{92}$
95	216	135	24	99	4	$9, 6, 2_2, 1_{91}$
95	288	144	24	108	4	$12, 6, 4, 1_{92}$
97	225	144	25	119	5	$9, 5_2, 1_{94}$
98	224	160	28	111	4	$8, 7, 4, 1_{95}$
98	144	128	36	112	9	$4_2, 3_2, 1_{94}$
98	128	112	32	108	16	$4, 2_5, 1_{92}$
100	240	140	24	104	4	$10, 6, 2_2, 1_{96}$

8 The forest of solutions



References

[1] GUY (Richard K.) — *Unsolved Probems in Number Theory* 2nd ed., Springer Verlag, New York/Berlin, 1994

[2] HERZBERG (N. P.) — *On a problem of Hurwitz*,
Pacific Journal of Mathematics 50 (1974) 485-493. MR 50#233, Zbl 0247.10010

[3] SCHERER (Karl) — *Sums of Squares and Products*,
Journal of Recreational Mathematics 26:4 302 Question 2190 by Karl Scherer
Journal of Recreational Mathematics 27:4 (1995) 315-318 partial solution to Q 2190

[4] ECKER (Michael) and CONN (Herb) — $\sum_{i=1}^n x_i = \prod_{i=1}^n x_i$,
REC 11:5 (Mar.-Apr. 1997) issue 79, 8-10

[5] POSTL (Helmut) — $\sum_{i=1}^n x_i^2 = \prod_{i=1}^n x_i$, Letter to T. Sillke dated 26 Jan. 1998.