# Hermitian Symmetric Domains

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Hermitian Symmetric Domains provide a higher-dimensional version of the upper half plane.

## Hermitian Symmetric Spaces

Recall that a *smooth manifold* is a locally ringed space that is locally isomorphic to spaces of the form $(U, \mathcal{O}_U^\infty)$, where $U$ is an open subset of $\mathbb{R}^n$ and $\mathcal{O}_U^\infty$ is the sheaf of smooth functions on $U$. Similarly, a *complex manifold* is a locally ringed space that is locally isomorphic to $(U, \mathcal{O}_U^\omega)$, where $U$ is an open subset of $\mathbb{C}^n$ and $\mathcal{O}_U^\omega$ is the sheaf of holomorphic functions on $U$.
Let's add some more structure: Given a smooth manifold $M$, a *Riemannian metric* on $M$ is a positive-definite smooth contravariant 2-tensor field $g$ on $M$. This means: $g$ consists of positive-definite bilinear forms $g_p\colon \mathrm{T}_p(M) \times \mathrm{T}_p(M) \to \mathbb{R}$ for each point $p \in M$ such that for smooth vector fields $X, Y$ on $M$, the function $g(X, Y)\colon p \mapsto g_p(X_p, Y_p)$ is smooth. In local coordinates $(x^i)_{1\le i\le n}$ a Riemannian metric has the form

$$g_p = \sum_{i,j} g_{ij}(p)\mathrm{d}x^i \otimes \mathrm{d}x^j$$

for some symmetric positive-definite matrix $g_{ij}(p)$ that depends smoothly on $p$. A diffeomorphism of $M$ is called *isometry* if it preserves the metric.
In differential geometry one uses Riemannian metrics to measure distances on a manifold $M$. Moreover, one can define geodesics on $M$, which are locally shortest paths between two points. Intuitively, geodesics are obtained by "walking in a straight line" along $M$.
Given a complex manifold $M$ of dimension $n$, there is an underlying smooth manifold $M^\infty$ of dimension $2n$. The complex structure on $M$ gives each tangent space of $M^\infty$ a complex structure $J_p\colon \mathrm{T}_p(M^\infty) \to \mathrm{T}_p(M^\infty)$, $J_p^2 = -1$. A *Hermitian metric* on $M$ is a Riemannian metric $g$ on $M^\infty$ such that $g(JX, JY) = g(X, Y)$ for all vector fields $X, Y$. A *Hermitian manifold* is a complex manifold together with a Hermitian metric. An *isometry* of a Hermitian manifold $(M, g)$ is a holomorphic diffeomorphism of $M$ that is also an isometry of $(M^\infty, g)$. The group of isometries is denoted $\mathrm{Is}(M, g)$. It is known, that $\mathrm{Is}(M, g)$ is a Lie group.
A manifold (smooth, complex, Hermitian, etc.) is *homogeneous*, if its automorphism group acts transitively. It "looks the same" everywhere. A *symmetry $s_p$ at $p$* for $p \in M$ is an automorphism of $M$ such that $s_p^2 = \mathrm{id}$ and in some neighbourhood, $p$ is the only fixed point of $s_p$. A *Hermitian symmetric space* is a Hermitian manifold that is connected, homogeneous, and has a symmetry at some point $p$. By homogeneity, there is a symmetry at each point.

**Example.** (a) Let $\Lambda$ be a discrete subgroup of $\mathbb{C}$. Then $\mathbb{C}/\Lambda$ is a Hermitian symmetric space. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the metric is given by the standard form $\mathrm{d}x^2 + \mathrm{d}y^2$. The translations act transitively on $\mathbb{C}/\Lambda$ and $x \mapsto -x$ is a symmetry at 0. Geodesics are images of straight lines in $\mathbb{C}$.

(b) The Riemann sphere $\mathbb{P}^1(\mathbb{C}) \cong \mathrm{S}^2$ is a Hermitian symmetric space. The metric is induced from the standard metric $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ of ambient space $\mathbb{R}^3$ of $\mathrm{S}^2$. Rotations act transitively on $\mathrm{S}^2$ and symmetries are given by rotations by $\pi$ around a diameter. Geodesics are great circles.

(c) The upper half-plane $\mathcal{H}_1$ is a Hermitian symmetric domain. After identifying $\mathbb{C}$ with $\mathbb{R}^2$, the metric is given by $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}_1$ by Möbius transforms (no, it's not fun to that they are isometries). A symmetry at i is given by $z \mapsto \bar{z}^{-1}$. Geodesics are vertical lines and half-circles with midpoint on the real line.

## Hermitian Symmetric Domains

Given a Riemannian manifold $M$, one can define the curvature at a point $p$. It is measured by taking two different geodesics through $p$ and determining, whether they converge or diverge. If they always converge, $M$ has positive curvature at $p$. If they always diverge, $M$ has negative curvature at $p$. If they never converge or diverge, $M$ has zero curvature at $p$. If $M$ is homogeneous, this notion is independent of $p$.

**Example.** Drawing some pictures shows:

(a) $\mathbb{C}/\Lambda$ has zero curvature.

(b) $\mathbb{P}^1(\mathbb{C})$ has positive curvature.

(c) $\mathcal{H}_1$ has negative curvature.

An important difference between these three cases is how the group $\mathrm{Is}(M,g)^+$ of holomorphic isometries behaves. In positive curvature, $\mathrm{Is}(M,g)^+$ is an adjoint compact Lie group. In negative curvature, it is an adjoint non-compact Lie group. If $(M,g)$ has zero curvature, not much can be said.
It is known that a Hermitian symmetric space $M$ can be decomposed as $M^+ \times M^0 \times M^-$, where $M^+$ has positive curvature, $M^0$ has zero curvature, and $M^-$ has negative curvature. We are only interested in the domains with negative curvature. A *Hermitian symmetric domain* is a Hermitian symmetric space with negative curvature.
One way to construct Hermitian Symmetric Domains is using Bergman metrics. Let $D$ be an open subset of $\mathbb{C}^n$ that is connected, bounded, homogeneous, and has a holomorphic symmetry at some point.

**Theorem** (Bergman)**.** There is a (up to a constant factor) canonical Hermitian metric on $D$. With this metric, $D$ has negative curvature.

**Example.** The *Siegel upper half space* $\mathcal{H}_g$ is given by the set of symmetric complex matrices $Z = X + \mathrm{i}Y$ such that $Y$ is positive definite. The group

$$\mathrm{Sp}_{2g}(\mathbb{R}) = \{M \in \mathrm{M}_{2g}(\mathbb{R}) : M^{\mathrm{T}} J M = J\} \qquad J = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

acts transitively on $\mathcal{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

$J$ is a matrix in $\mathrm{Sp}_{2g}(\mathbb{R})$ that has $\mathrm{i}\mathbb{1}$ as sole fixed point. The only thing missing to give $\mathcal{H}_g$ the structure of a Hermitian symmetric domain is a metric. The map $Z \mapsto (Z - \mathrm{i}\mathbb{1})(Z + \mathrm{i}\mathbb{1})$ identifies $\mathcal{H}_g$ with the set $\mathcal{D}_g$ of symmetric complex matrices $M$ such that $\mathbb{1} - M^*M$ is positive definite. The last condition can be reformulated as $\|M\|_2 < 1$. Therefore, $\mathcal{D}_g$ is bounded. Now $\mathcal{D}_g$ and $\mathcal{H}_g$ obtain a Bergman metric, so $\mathcal{H}_g$ is a Hermitian symmetric domain.

## Automorphisms of Hermitian Symmetric Domains

If $G$ is a topological group, $G^+$ denotes the connected component of the identity. Let $(M,g)$ be a Hermitian symmetric domain.

**Theorem.** The groups $\mathrm{Is}(M,g)^+$, $\mathrm{Is}(M^\infty,g)^+$, and $\mathrm{Hol}(M)^+$ are identical and act transitively on $M$. Let $p \in M$. The stabiliser $K_p$ of $p$ in $\mathrm{Is}(M,g)^+$ is compact. There is an isomorphism $\mathrm{Is}(M,g)^+/K_p \to M$ of smooth manifolds.

**Theorem.** Let $\mathfrak{h}$ be the Lie algebra of $\mathrm{Hol}(M)^+$. There is a unique connected algebraic subgroup $G$ of $\mathrm{GL}(\mathfrak{h})$, such that $G(\mathbb{R})^+ = \mathrm{Hol}(M)^+$. Moreover, $G$ is adjoint.

*Proof.* One can show that $\mathrm{Hol}(M)^+$ is an adjoint Lie group (= semisimple, trivial center). Therefore, the adjoint representation of $\mathrm{Hol}(M)^+$ on $\mathfrak{h}$ is faithful, so $\mathrm{Hol}(M)^+$ can be seen as a subgroup of $\mathrm{GL}(\mathfrak{h})$. By some theorem of Borel, there is an algebraic subgroup $G$ of $\mathrm{GL}(V)$ with Lie algebra $[\mathfrak{h}, \mathfrak{h}]$. As $\mathrm{Hol}(M)^+$ is adjoint, $\mathfrak{h}$ is semisimple, so $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. It follows $G(\mathbb{R})^+ = \mathrm{Hol}(M)^+$. Adjointness of $G$ follows from the fact that $\mathfrak{h}$ is semisimple. ☐

Let $U(1) = \{z \in \mathbb{C} : |z| = 1\}$. Using a lot of differential geometry, one proves:

**Theorem.** Let $p \in M$. There is a unique homomorphism $u_p : U(1) \to \mathrm{Hol}(M)$ such that $u_p(z)(p) = p$ for all $z$ and $u_p(z)$ acts on $\mathrm{T}_p(M)$ as multiplication by $z$.

# Representations of $U(1)$

Let $T$ be a torus over a field $k$. Our goal is to describe the representations of $T$ on finite-dimensional $k$-vector spaces. Firs suppose that $T$ is split. Given a representation $\rho : T \to \mathrm{GL}_V$, each $\rho(t)$ acts semisimply since $t$ is semisimple. Additionally, all $\rho(t)$ commute. Therefore, they can be diagonalised simultaneously. Now $V$ decompses as a sum of characters (= representations of dimension 1). We can write

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $X^*(T)$ is the set of characters of $T$ and $t \in T$ acts on $V_\chi$ by multiplication with $\chi(t)$.
Now suppose $T$ does not split over $k$ but over some Galois extension $K$ of $k$. Then $V \otimes_k K$ decomposes as above

$$V \otimes_k K = \bigoplus_{\chi \in X^*(T)} V_\chi.$$

Being a $k$-representation, $\rho$ is stable under the action of $\mathrm{Gal}(K/k)$. This is equivalent to $\sigma V_\chi = V_{\sigma\chi}$ for all $\chi \in X^*(T)$ and $\sigma \in \mathrm{Gal}(K/k)$.
Now consider the real torus $U(1)$. It splits over $\mathbb{C}$. The characters are given by $\chi_n : z \mapsto z^n$ for $n \in \mathbb{Z}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts via $\bar{\chi}_n = \chi_{-n}$. Thus, a representation of $U(1)$ on a real vector space $V$ is given by a grading

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{n \in \mathbb{Z}} V_n$$

such that $\bar{V}_n = V_{-n}$. By collecting spaces $V_n$ and $V_{-n}$ together and descending to $\mathbb{R}$ one obtains that a real representation of $U(1)$ decomposes as a sum of the representations of the following type:

- $V = \mathbb{R}$ with the trivial action, $V \otimes_{\mathbb{R}} \mathbb{C} = V_0$
- $V = \mathbb{R}^2$, $x + \mathrm{i}y \in U(1)$ acts via the matrix $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^n$ for some $n \in \mathbb{N}_+$, $V \otimes_{\mathbb{R}} \mathbb{C} = V_n \oplus V_{-n}$

# Classification via Real Groups

Let $G$ be a connected algebraic group over $\mathbb{R}$. An involution $\vartheta$ of $\mathbb{R}$ is a *Cartan involution* if the subgroup

$$G^{(\vartheta)}(\mathbb{R}) = \{g \in G(\mathbb{C}) : \vartheta(\bar{g}) = g\}$$

is compact.

**Example.** (a) $G^{(\mathrm{id})}(\mathbb{R}) = \{g \in G(\mathbb{C}) : g = \bar{g}\} = G(\mathbb{R})$, so id is a Cartan involution iff $G(\mathbb{R})$ is compact.

(b) $\vartheta : M \mapsto M^{-\mathrm{T}}$ is an involution of $\mathrm{GL}_n$.

$$\mathrm{GL}_n^{(\vartheta)}(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{C}) : M = \bar{M}^{-\mathrm{T}} = M\} = \{M : M^* M = \mathbb{1}\} = \mathrm{SU}(n)$$

is compact, so $\vartheta$ is a Cartan involution.

Let $D$ be a Hermitian symmetric domain and let $G$ be the connected algebraic group with $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$. Given $p \in D$, we have a homomorphism $u_p : U(1) \to \mathrm{Hol}(D)^+ = G(\mathbb{R})^+$. One can show that it extends to a algebraic morphism $u_p : U(1) \to G$.

**Theorem.** The morphism $u_p$ constructed above has the following properties:

(a) The representation of $U(1)$ on $\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ induced by $u_p$ contains only the complex representations $\chi_0$, $\chi_1$, and $\chi_{-1}$.

(b) $\mathrm{ad}(u_p(-1))$ is a Cartan involution. (ad means "conjugation with")

(c) $u_p(1)$ does not project to 1 in any simple factor of $G$.

Conversely, suppose $G$ is a real adjoint algebraic group and $u : U(1) \to G$ satisfies (a), (b), and (c). Let $D$ be the set of conjugates of $u$ by elements of $G(\mathbb{R})^+$. Then $D$ is a Hermitian symmetric domain with $\mathrm{Hol}(D)^+ = G(\mathbb{R})^+$ and $u(-1)$ is a symmetry at $u \in D$.

*Proof.* Let $K_p$ be the stabiliser of $p$ inside $G(\mathbb{R})^+$. We have seen that $G(\mathbb{R})^+ / K_p \cong D$. The action of $U(1)$ on $D$ via $u_p$ corresponds to the action on the quotient by conjugation with $u_p(z)$. Considering tangent spaces, we get $\mathrm{Lie}(G)/\mathrm{Lie}(K_p) \cong \mathrm{T}_p(D)$. One can calculate that the action of $U(1)$ on $\mathrm{Lie}(K_p)$ is trivial. By definition of $u_p$, the action of $z \in U(1)$ on $\mathrm{T}_p(D)$ is given by multiplication with $z$. After complexifying, this representation therefore involves only $\chi_1$ and $\chi_{-1}$. For $\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ this means that only $\chi_0$, $\chi_1$, and $\chi_{-1}$ can occur. This proves (a).
$u_p(-1)$ is a symmetry at $p$. It is known that conjugation by a symmetry is a Cartan involution iff the space has negative curvature. This is the case, so (b) holds.
Suppose, $u_p(-1)$ maps to 1 in some factor $G'$ of $G$. Then the identity is a Cartan involution of $G'$, so $G'$ is compact. This contradicts negative curvature of $D$. Thus, (c) is proven.
Sketch for conversely: By (b), the centraliser $K_u$ of $u$ in $G(\mathbb{R})^+$ is compact. Then $D \cong G(\mathbb{R})^+ / K_u$, so $D$ is a real manifold. The tangent space $\mathrm{T}_p(D)$ can be identified with $\mathrm{Lie}(G)/\mathrm{Lie}(G)_0$. By (a), this space carries a complex structure, which can be used to give $D$ the structure of a complex manifold. Construct a $K_u$-invariant positive definite bilinear form on $\mathrm{T}_p(D)$ and bring it to every point using homogeneity. Then $D$ is a Hermitian symmetric space. Negative curvature follows from (b) and (c), since each factor of the automorphism group is non-compact. ☐

**Corollary.** There is a natural one-to-one correspondence between pointed Hermitian symmetric domains and pairs $(G, u)$ consisting of a real adjoint algebraic group and a homomorphism $u : U(1) \to G(\mathbb{R})$ such that (a), (b), and (c) are satisfied.

**Remark.** Note that there is another classification theorem for Hermitian symmetric domains that involves special nodes on connected Dynkin diagrams.