2. Hodge structures and their classifying spaces

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Example X smooth proj. variety $/\mathbb{C}$ $\rightsquigarrow$ $X(\mathbb{C})$ complex manifold and $H^n_{sing}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} \cong H^n(X, \mathbb{C})$ can be computed via deRham cohomology, i.e. as the cohomology of

$$\left[\Omega^0(X) \longrightarrow \Omega^1(X) \longrightarrow \cdots \longrightarrow \Omega^{2\dim X}(X) \right]$$ (as a real mfd with smooth functions)

A differential form in $\Omega^n(X)$ can be written in coordinates $(z_i, \bar{z}_i)$ as a sum of forms $f \, dz_1 \wedge \dots \wedge dz_p \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_q$ (in local coord.) with $n = p+q$.

This decomp. passes to cohomology and we obtain the

Hodge decomposition $\quad H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$

with $H^{p,q} \cong H^q(X, \Omega^p_X)$ with symmetry $\overline{H^{p,q}} = H^{q,p}$.

Definition (Hodge structure) Let R be a ring $\subseteq \mathbb{R}$ (think of $R = \mathbb{Z}$ or $R = \mathbb{Q}$). Let V be a finite free R-module. Then $V_{\mathbb{C}} := V \otimes_R \mathbb{C}$ admits a complex conjugation $\overline{v \otimes \lambda} := v \otimes \bar{\lambda}$. A pure Hodge structure of weight n is a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \quad \text{w/} \quad V^{p,q} = \overline{V^{q,p}}$$ (as $\mathbb{C}$-vector spaces)

$h^{p,q}(V) := \dim_{\mathbb{C}} V^{p,q}$ are the Hodge numbers.

A morph. of pure Hodge structures is a R-module homomorphism $f: V \longrightarrow W$ s.t. $f_{\mathbb{C}}(V^{p,q}) \subseteq W^{p,q}$ $\forall\, p+q=n$.

[illegible]

s.t. $f_{\mathbb{C}}(V^{p,q}) \subseteq W^{p,q}$ $\forall p+q=n$.

<u>Cor.</u> X compact Kähler mfd, then $H^n(X,\mathbb{Z})$ has a pure Hodge structure of weight n. If $f: X \to Y$ is a holomorphic map between compact Kähler mfds, $f^*: H^n(Y) \longrightarrow H^n(X)$ is a morph. of pure Hodge structures.

Equivalently a pure Hodge structure is given by a (decreasing) Hodge filtration

$F^\bullet : \_ \supseteq F^p \supseteq F^{p+1} \supseteq \_$ on $V_{\mathbb{C}}$ s.t.

$\forall p+q = n+1$: $F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}} = 0$ and $F^p V_{\mathbb{C}} \oplus \overline{F^q V_{\mathbb{C}}} = V_{\mathbb{C}}$

Then $V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$ and $F^p V_{\mathbb{C}} = \bigoplus_{i \geq p} V^{i,n-i}$

<u>Example</u> (Hodge structures of Tate)

$$\mathbb{Z}(m) := (2\pi i)^m \mathbb{Z} \subseteq \mathbb{C} \quad \text{and} \quad \mathbb{Z}(m) \otimes_{\mathbb{Z}} \mathbb{C} = [\mathbb{Z}(m) \otimes \mathbb{C}]^{-m,-m}$$

is a pure integral Hodge structure of weight $-2m$.

In general one can make sense of a tensor product of Hodge structures and defines the r^{th} Tate twist of an R-Hodge structure V as $V(r) := V \otimes_R R(r)$ where $R(r) := R \otimes_{\mathbb{Z}} \mathbb{Z}(r)$. If V has weight n, $V(r)$ has weight $n-2r$.

<u>Definition</u> (Mixed Hodge structure) Let V be a finite free R-module, $R \subseteq \mathbb{Q}$. A <u>(mixed) Hodge structure</u> on V is a increasing weight filtration $W_\bullet$ on $V_{\mathbb{Q}}$ together with a decreasing Hodge filtration $F^\bullet$ on $V_{\mathbb{C}}$ s.t. the induced

[illegible]

together with a decreasing Hodge filtration $F^\bullet$ on $V_\mathbb{C}$ s.t. the induced filtration on $\mathrm{Gr}_k^{W_\bullet} V_\mathbb{Q} = W_k V_\mathbb{Q} / W_{k-1} V_\mathbb{Q}$ defines a pure rational Hodge structure of weight k.

Alternatively choose $R \subseteq \mathbb{R}$ and define weight filt. $/R$.

A morph. of mixed Hodge structures is a morph. preserving both filtrations.

Equiv. $V_\mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$ with $V^{p,q} = \overline{V^{q,p}}$ so $\bigoplus_{p+q=n} V^{p,q} \subseteq V_\mathbb{C}$ is is a subspace stable under complex conj., so defined over $\mathbb{R}$, i.e. a complexification of an $\mathbb{R}$-subspace V_n.

A priori the weight filtr. for is only defined $/\mathbb{R}$.

$V_\mathbb{R} = \bigoplus_{n\in\mathbb{Z}} V_n$ is the weight decomp. of $V_\mathbb{R}$ and V_n admits a pure Hodge structure of weight n.

The set of pairs (p,q) with $V^{p,q} \neq 0$ is called the type of $(V, W_\bullet, F^\bullet)$

Hodge structures as torus representations

Let $\mathbb{G}_m$ be the torus $/\mathbb{C}$ and $\mathbb{S}$ it's restriction along $\mathbb{R} \hookrightarrow \mathbb{C}$

Then $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}_\mathbb{C} := \mathbb{S} \times_\mathbb{R} \mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$ ← $\mathbb{S}$ splits $/\mathbb{C}$

↪ repr. of $\mathbb{S}_\mathbb{C}$ are diagonalizable

Fix the second iso s.t. $\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ is induced by

$$z \longmapsto (z, \bar{z})$$

So complex conj. on $\mathbb{S}_\mathbb{C}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ sends $(z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)$

We have a weight homomorphism $w : \mathbb{G}_m \xrightarrow{\text{defined } /\mathbb{R}} \mathbb{S}$ s.t. $w(\mathbb{R}) : \underbrace{\mathbb{G}_m(\mathbb{R})}_{=\mathbb{R}^\times} \longrightarrow \underbrace{\mathbb{S}(\mathbb{R})}_{=\mathbb{C}^\times}$, $r \longmapsto r^{-1}$

The character group $X^*(\mathbb{S}_\mathbb{C}) = \{ \text{morph. } \mathbb{S}_\mathbb{C} \to \mathbb{G}_m \}$

$$= \{ (z_1, z_2) \mapsto z_1^r z_2^s \mid r, s \in \mathbb{Z} \} \cong \mathbb{Z} \times \mathbb{Z}$$

$\mapsto (r, s)$

admits a complex conjugation given by $\overline{(p,q)} = (q,p) \in \mathbb{Z} \times \mathbb{Z}$

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Thus giving a representation $\mathbb{S} \longrightarrow GL_k(\mathbb{R}) = \mathrm{Aut}(V)$ is the same as giving a $\mathbb{Z}\times\mathbb{Z}$ graduation on $V_{\mathbb{C}}$ s.t. $V^{p,q} = \overline{V^{q,p}}$

Galois action of complex conj.

This is the weight space decomp. as in the 1st talk, since $\mathbb{R} \hookrightarrow \mathbb{C}$ is Galois and a rep/$\mathbb{R}$ is fixed by Galois action

More precisely to a repr. $h : \mathbb{S} \longrightarrow \mathrm{Aut}(V)$, V an $\mathbb{R}$-vector space, splits $/\mathbb{C}$ into a decomposition $V_{\mathbb{C}} = \bigoplus V^{p,q}$ weight space decomp.

$$V^{p,q} = \{ v \in V_{\mathbb{C}} \mid h_{\mathbb{C}}(z_1,z_2)v = z_1^{-p} z_2^{-q} v \} = \{ v \in V_{\mathbb{C}} \mid h(z)v = z^{-p} \bar{z}^{-q} v \}$$

The weight decomposition is given by $V_n = \{ v \in V \mid w_h(r)v = r^n v \}$, $w_h = h \circ w$

If $V = V' \otimes_{\mathbb{Q}} \mathbb{R}$, the weight decomposition will be defined $/\mathbb{Q}$ iff $w_h : \mathbb{G}_m \to \mathrm{Aut}(V)$ is already defined over $\mathbb{Q}$.

J mimics multiplication with i, $J^2 = -\mathrm{id}$

Example Let J be a complex structure on a vector space. $x+iy \mapsto x+yJ$

Then J can be identified as $\mathbb{R}$-algebra homomorphism $J : \mathbb{C} \longrightarrow \mathrm{End}_{\mathbb{R}}(V)$

J gives a morph. of groups $\mathbb{C}^\times \longrightarrow \mathrm{Aut}(V)$ which defines a Hodge structure of type $(-1,0), (0,-1)$

Here $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ is the eigenspace decomp. $\ker(J - i\,\mathrm{id}) \oplus \ker(J + i\,\mathrm{id})$

and $J_{\mathbb{C}}(v) = iv \Leftrightarrow \overline{J_{\mathbb{C}}(v)} = J_{\mathbb{C}}(\bar{v}) = -iv$, so

$\overline{V^{-1,0}} = V^{0,-1}$

Example The Hodge structures of Tate $\mathbb{Q}(k)$ correspond to the homomorphisms $h : \mathbb{S} \longrightarrow \mathbb{G}_m$ sending $z \mapsto (z\bar{z})^m$

defined $/\mathbb{R}$.

Every Hodge structure $(V, h: \mathbb{S} \longrightarrow \mathrm{Aut}(V))$ defines an $\mathbb{R}$-linear automorphism $C := h(i)$, the <u>Weil operator</u>, which acts on $V^{p,q}$ as i^{q-p} and $C^2 = h(-1)$ acts as $(-1)^n$ on V_n.

<u>Example</u> If V is of type $(-1,0), (0,-1)$, C coincides with the complex structure J and $(V, (V^{-1,0}, V^{0,-1})) \longmapsto (V, C)$ yields an equivalence of categories between $\{$cpx structures of type $(-1,0), (0,-1)\}$ and $\{$finite dim $\mathbb{C}$-vector spaces$\}$

Polarizations of Hodge structures

V a pure $\mathbb{R}$-Hodge structure of weight n. A <u>polarization</u> is a morph. of Hodge structures is a morph. $\psi : V \underset{\mathbb{R}}{\otimes} V \longrightarrow \mathbb{R}(-n)$ s.t. the real-valued bilinear form $\psi_C(u,v) := (2\pi i)^n \psi(u, C(v))$ is pos. definite.

we define Weil operator on $V \otimes \mathbb{R}$ as before $\longrightarrow$ on $V \underset{\mathbb{R}}{\otimes} \mathbb{R}$.

A mixed Hodge structure is polarizable if every pure Hodge structure of the graded pieces of it's weight filtration is polarizable.

<u>Example</u> X smooth proj. $/\mathbb{C}$. Then an embedding $X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ determines a polarization on the primitive part of the Hodge structure $H^n(X, \mathbb{Q})$ for each n (Hodge index thm)

↳ more generally the choice of a Kähler form defines a polarization on a compact Kähler mfd.

Variations of Hodge structures

Variations of Hodge structures

Idea: Like the period map in the last semester: $\pi : V \to S$ family of alg. varieties$/\mathbb{Q}$ with fibers V_s smooth proj. $\rightsquigarrow H^n(V_s, \mathbb{Q})$ form local system of $\mathbb{Q}$-vector spaces varying continuously in $s \in S(\mathbb{C})$. The Hodge filtration on $H^n(V_s, \mathbb{C})$ vary holomorphically in $s \in S(\mathbb{C})$.

S connected cpx mfd. V an $\mathbb{R}$-vector spaces. Fix weight n and assume $\forall s \in S$ h_s is a pure Hodge structure of weight n on $V_{\mathbb{C}}$. Denote $V_s^{p,q} = V_{h_s}^{p,q}$, $F_s^p = F_{h_s}^p V_{\mathbb{C}}$.

<u>Definition</u> The family of Hodge structures $(h_s)_{s \in S}$ is <u>continuous</u> if for fixed (p,q), $s \mapsto V_s^{p,q}$ is continuous, i.e. $\dim V_s^{p,q} =: d(p,q)$ is const. and the map $S \to Gr_{d(p,q)}(V_{\mathbb{C}})$ is continuous.

A continuous family of Hodge structures is <u>holomorphic</u> if the map

$$S \xrightarrow{\varphi} Flag_{(-,d(p),-)}(V_{\mathbb{C}}), \quad s \mapsto F_s^{\bullet} \quad \text{is holomorphic.}$$

here $d(p) = \dim F_s^p V_{\mathbb{C}} = \sum_{r \geq p} d(r, p-r)$

For a holomorphic family of Hodge structures we can differentiate φ at $s \in S$ to obtain $\underset{\mathbb{C}\text{-linear}}{d\varphi_s} : T_s S \longrightarrow T_{F_s^{\bullet}}(Flag_d(V_{\mathbb{C}})) \subseteq \bigoplus_p Hom(F_s^p, V_{\mathbb{C}}/F_s^p)$

The family satisfies <u>Griffiths transversality</u> if $image(d\varphi_s) \subseteq \bigoplus_p Hom(F_s^p, F_s^{p-1}/F_s^p)$ $\forall s$.

A family satisfying Griffiths transversality is called <u>variation of Hodge structures</u>.

↳ note that this can be rephrased in terms of connections preserving a filtration

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we omit this at this point as we did before discussing curvature

Tangent spaces of flag varieties: $\mathrm{Flag}_{\underline{d}}(V_{\mathbb{C}})$ can be seen as a quotient variety G/P, $G = GL(V)$, P the parabolic subgroup fixing the flag $F^\bullet$.

Then by some general theory about Lie algebras associated to alg. groups we have for the point $F^\bullet \in G/P$ that $T_{F^\bullet} G/P = \mathfrak{gl}(V) / \mathfrak{p}_{F^\bullet}(V)$ is the quotient of the Lie algebras. Here $\mathfrak{p}_{F^\bullet}(V) = \mathrm{End}_{F^\bullet}(V)$

$$\mathrm{End}_{F^\bullet}(V) = \{\phi : V \to V \mid \phi(V^i) \subseteq V^i\}$$

So the vector space $T_{F^\bullet} G/P$ can be identified with maps sequences $(\phi_i : F^i V \to V/F^i V \mid \phi_i|_{F^{i+1}V} \equiv \phi_{i+1} \bmod F^i V)_i$ which is a subspace of $\bigoplus_i \mathrm{Hom}(F^i V, V/F^i V)$

V an $\mathbb{R}$-vector space, $T = (t_i)_i$ family of tensors, i.e. multilinear maps $t_i : V \otimes \cdots \otimes V \longrightarrow \mathbb{R}$ with a distinguished $t_0 : V \otimes V \longrightarrow \mathbb{R}$ defining a a nondeg. bilinear form. Fix a number $n \in \mathbb{Z}$ and a function $d : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}_{\geq 0}$ a function with $d(p,q) = 0$ for almost all (p,q),

$$d(p,q) = d(q,p)$$
$$d(p,q) = 0 \quad \text{if } p+q \neq n$$

$$S(d,V) = \left\{ \begin{array}{l} \text{Hodge structures of weight } n \text{ } h \text{ on } V \text{ s.t. } \dim V_h^{pq} = d(p,q), \\ \text{each } t_i : \underbrace{V \otimes \cdots \otimes V}_{r\text{-copies}} \longrightarrow \mathbb{R}\left(\frac{-nr}{2}\right) \text{ is a morph. of} \\ \text{Hodge structures and } t_0 : V \otimes V \longrightarrow V(-n) \text{ defines} \\ \text{a polarization} \end{array} \right\}$$

| a "polarization |

$S(d,V)$ can be treated as a subset of $\prod_{d(p,q)\neq 0} \mathrm{Gr}_{d(p,q)}(V_{\mathbb{C}})$

send Hodge structure $\bigoplus_{p,q} V^{p,q}$ to $(V^{p,q})_{p,q}$

Grassmanian of $d(p,q)$-dim subspaces

This endows $S(d,V)$ with a topology.

with analytic topology

<u>Thm</u> (Deligne '79) S^+ a connected component of $S(d,T)$, then:

(a) S^+ admits a unique complex structure s.t. $(h_s)_{s\in S^+}$ is a holomorphic family of Hodge structures

(b) with above cpx structure, S^+ is a hermitian symmetric domain iff $(h_s)_s$ is a variation of Hodge structures

N.B. the converse is not true. Example is given in Milnes notes

(c) Every irreducible hermitian symmetric domain can be realized as an S^+ for a suitable choice of V, d and T.

<u>Proof (Sketch)</u>: (a) the complex structure is via the injective map $\varphi: S^+ \ni s \longmapsto F_s^\bullet \in \mathrm{Flag}_{\underline{d}}(V_{\mathbb{C}})$ sending a Hodge structure to it's Hodge filtration. (Inj. since Hodge structures are pure)

Set G = smallest alg. subgroup of GL_V with $h(\mathbb{S}) \subseteq G$ $\forall h\in S^+$

For a point $h_0 \in S^+$, $G(\mathbb{R})^+ \cdot h_0 = S^+$ let K_0 be the stabilizer of h_0

identity component

acting by conj.

then $S^+ = \left(G(\mathbb{R})^+ / K_0\right)\cdot h_0 \cong G(\mathbb{R})^+/K_0$

identity component — $\mathcal{G}$ … copy.

then $S^+ = \left(\mathcal{G}(\mathbb{R})^+ / K_0\right) \cdot h_0 \cong \mathcal{G}(\mathbb{R})^+ / K_0$

K_0 is a closed subgroup, thus $\mathcal{G}(\mathbb{R})^+/K_0$ is a smooth mfd. Via above bijection S^+ is endowed with a smooth mfd structure.

Consider $\mathcal{G} \subseteq gl(V)$ giving $\mathfrak{g} = \mathrm{Lie}(\mathcal{G}) \hookrightarrow \mathrm{End}(V)$

The adjoint representation yields a Hodge structure on $\mathfrak{g}$: $\mathbb{S} \xrightarrow{h_0} \mathcal{G} \xrightarrow{\mathrm{ad}} \mathrm{Aut}(\mathfrak{g})$

which makes $\mathfrak{g} \hookrightarrow \mathrm{End}(V)$ into an inclusion of Hodge structures

The identification $\mathfrak{g}^{00} = \mathrm{Lie}(K^0) = \mathfrak{g}^{0,0} \cap \mathfrak{g} = \ker\left(\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}/F^0\right)$ ← this construction works in general for weight 0 Hodge structures

↳ I think one needs to argue further to determine the 0-weight

N.B.: The repr. $h_0 : \mathbb{S} \to \mathrm{Aut}(V)$ is not the same as $h_0 : \mathbb{S} \to \mathrm{Aut}(V) \xrightarrow{\mathrm{ad}} \mathrm{End}(V) = \mathrm{Lie}(\mathrm{Aut}(V))$

yields a diagram

$$\begin{tikzcd}
T_{h_0}(S^+) = \mathfrak{g}/\mathfrak{g}^{00} \arrow[r, hook] \arrow[d, "s"] \arrow[dr, "(d\varphi)_{h_0}"] & \mathrm{End}(V)/\mathrm{End}(V)^{00} \arrow[d, "s"] \\
\mathfrak{g}_{\mathbb{C}}/F^0 \arrow[r, hook] & \mathrm{End}(V)/F^0 \cong T_{h_0}\mathrm{Flag}_{\underline{d}}(V_{\mathbb{C}})
\end{tikzcd}$$

identifying $T_{h_0}(S^+)$ with a complex subspace of $T_{h_0}\mathrm{Flag}_{\underline{d}}(V_{\mathbb{C}})$

This endows S^+ with an almost complex structure (note that the construction works for all $h_0 \in S^+$ smooth in h_0)

One can show that this structure is integrable, hence S^+ is a complex mfd s.t. φ is a holomorphic embedding.

(b) Here we use the converse direction of the thm in the 1st talk.

We have noted that any $h, h_0 \in S^+$ are conjugate, i.e. $\exists g \in \mathcal{G}(\mathbb{R}^+)$ s.t. $h = g h_0 g^{-1}$

$V_{\mathbb{R}}$ is of weight n $\Rightarrow$ all $h \in S^+$ act as $h(r)v = r^{-n}v$ $\forall v \in V, r \in \mathbb{R}$

$\Rightarrow g h_0(r) g^{-1} = h_0(r)$ $\forall g \in \mathcal{G}(\mathbb{R})^+$ and thus $h_0(r) \in Z(\mathcal{G})$

$\Rightarrow g\, h_0(r)\, g^{-1} = h_0(r) \;\; \forall g \in \mathcal{G}(\mathbb{R})^+$ and thus $h_0(r) \in Z(\mathcal{G})$

This allows to define $u_0 : U(1) \longrightarrow \mathfrak{g}^{ad}$, $z \longmapsto h_0(\sqrt{z})$.

two solutions differ by ± 1 so their difference lies in $Z(\mathcal{G}(\mathbb{R}))$. $\mathfrak{g}^{ad} = \text{image}(ad : \mathfrak{g} \to \mathfrak{g})$ so $Z(\mathcal{G}(\mathbb{R}))$ is send to zero. $\Rightarrow$ well-defined

One can show that the polarization t_0 ensures that $C = h_0(i) = u_0(-1)$ is a Cartan involution on $\mathfrak{g}$ (and on $\mathfrak{g}^{ad}$). That provides cond. (b) of Thm talk 1

Griffith's transversality shows (a) and (c).