4.11.21
Talk 3
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Locally symmetric varieties

Reminder (from Simon's talk):

A Hermetian symmetric domain is a homogenous connected cpx. mfd. with a hermetian metric, an involution at every point having this point as isolated fixed point s.t the mfd has negative curvature.

Thm:
$$\underbrace{Is(M,g)^+}_{\text{connected comp. of id}} \cong Is(\underbrace{M^\infty}_{\text{underlying smooth manifold}},g)^+ \cong Hol(M)^+$$

$\forall p \in M$: $K_p = \{ f \in Hol(M)^+ \mid f(p) = p \}$ is compact &

$$Is(M,g)^+/K_p \xrightarrow{\cong} M$$

Notation for today: • D is hermetian symmetric domain.
• Γ is a subgroup (of $Hol(D)^+$, $G(\mathbb{Q})$, ...)

Aim: For "nice" (see later) $\Gamma \subseteq Hol(D)^+$ has $D(\Gamma) := \Gamma \backslash D$ the structure of algebraic variety$/\mathbb{C}$ & $\# Aut(\Gamma \backslash D) < \infty$.

§1 $\Gamma \backslash D$ as a complex manifold

Prop.: Let $\Gamma \subseteq Hol(D)^+$ be torsion free. ($\operatorname{ord}(\gamma) < \infty \Rightarrow \gamma = e$; free action $\hat{=}\ K_p = \{e\}\ \forall p$.)
Then Γ acts freely on D & $\exists!$ complex structure on $\Gamma / D =: D(\Gamma)$ s.t. $\pi : D \longrightarrow D(\Gamma)$ is a local isomorphism.

Moreover: $\varphi : D(\Gamma) \longrightarrow M$ holom. $\Leftrightarrow$ $\varphi \circ \pi : D \longrightarrow M$ holom.

Proof (sketch): The crucial step is to show that Γ acts freely:

$\forall p \in D$: $K_p := \{ g \in Hol(D)^+ \mid g(p) = p \} \subseteq Hol(D)^+$ compact (see Simon's talk)

$$\begin{array}{ccc} \cup\!\!\!\!\!\!\! & & \rotatebox{90}{$\subseteq$} \\ \Gamma_p := \{ g \in \Gamma \mid g(p) = p \} & \subseteq & \Gamma \end{array} \qquad K_p \text{ compact} \Downarrow \ \Gamma_p \text{ compact}$$

$\Rightarrow$ Γ_p discrete & compact $\Rightarrow$ Γ_p finite, Γ_p torsion-free $\Big\}$ $\Rightarrow$ $\Gamma_p = \{e\}$, i.e. Γ acts free. $\square$

Remark: $\pi : D \longrightarrow D(\Gamma)$ is universal cover &

$$\Gamma \longrightarrow \pi_1(D(\Gamma), \pi(p)) \qquad \gamma \longmapsto \pi\big(\overset{\lambda}{p \curvearrowright \gamma p}\big)$$

§2. Arithmetic subgroups

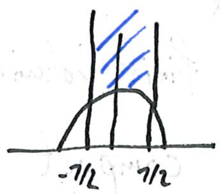
We want to consider quotients with "big" discrete subgroups.

Recall: D has a Riemannian metric g, hence a volume element Ω
$\hookrightarrow$ in local coord.: $\Omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \dots \wedge dx_n$

Def: $\Gamma \subseteq Hol(D)^+$ discrete has finite covolume, if $\int_{\Gamma \backslash D} \Omega < \infty$.

Rem: Similar def. possible for real Lie groups.

Example: $D = \mathcal{H}_1$.

1) $\Gamma = PSL_2(\mathbb{Z}) \rightsquigarrow$ fund. domain $F =$ 

metric (see Simon's talk): $y^{-2}\cdot(dx^2+dy^2)$

$$\int_{\Gamma\backslash D} \Omega = \int_F \frac{dx\,dy}{y^2} \leq \int_{\sqrt{3}/2}^{\infty} \frac{dy}{y^2} < \infty \implies \Gamma \text{ has finite covolume.}$$

2) $\Gamma = \mathbb{Z} : x \longmapsto x+n$ has infinite covolume.

Def: $S_1, S_2 \subseteq G$, G group, are commensurable, if $[S_i : S_1 \cap S_2] < \infty \ \forall i = 1,2$

Example: $\mathbb{Z}\cdot a, \mathbb{Z}\cdot b \subseteq \mathbb{R}$ are commensurable $\iff$ $a = 0 = b$ or $a \neq 0 \neq b$

Def: $G/\mathbb{Q}$ alg. group. $\Gamma \subseteq G(\mathbb{Q})$ is called arithmetic, if it is commensurable with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$ for some $G \hookrightarrow GL_n$.

Remark: i) It is the same as asking for all $G \hookrightarrow GL_n$.

ii) $\Gamma \subseteq G(\mathbb{Q})$ is discrete in $G(\mathbb{R})$, but does not need to have finite covolume:

$G_m(\mathbb{Z}) = \Gamma = \{\pm 1\} \subseteq G_m(\mathbb{Q})$ is arithmetic of infinite covolume.

Example: $G = GL_n$, $\Gamma(N) := \{ g \in GL_n(\mathbb{Z}) \mid g \equiv I_n \bmod N\}$ is arithmetic, since

$$0 \to \Gamma(N) \to GL_n(\mathbb{Z}) \longrightarrow \underbrace{GL_n(\mathbb{Z}/N\mathbb{Z})}_{\text{finite}} \to 0$$

Prop: $G/\mathbb{Q}$ alg. group, $\Gamma \subseteq G(\mathbb{Q})$ arithmetic. Then $\exists N > 0$ s.t. $\Gamma' = \Gamma(N) \cap \Gamma$ for some $G \hookrightarrow GL_n$ has finite index in Γ and $\forall \gamma \in \Gamma'$: $\langle$eigenvalues of $\gamma\rangle \subseteq \mathbb{C}^\times$ is torsion free.

"Γ' is a neat subgroup"

Thm: Let $G/_{\mathbb{Q}}$ be reductive, $\Gamma \subseteq G(\mathbb{Q})$ arithmetic.

a) $\Gamma \backslash G(\mathbb{R})$ has finite volume $\Leftrightarrow$ $\mathrm{Hom}(G, \mathbb{G}_m) = 0$ ($\Leftarrow$ G semisimple)

b) $\Gamma \backslash G(\mathbb{R})$ is compact $\Leftrightarrow$ $\underbrace{\mathrm{Hom}(G, \mathbb{G}_m) = 0 \text{ \& } G(\mathbb{Q}) \text{ contains no unipotent elements}}_{\text{does not depend on } \Gamma \text{ !}}$

Idea: (for $\mathcal{H}_1$) unipotent elements $\hat{=}$ cusps.

Example: Let $B/\mathbb{Q}$ be a quaternion alg., s.t. $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$

Let $G/\mathbb{Q}$ be linear alg. gp. s.t. $G(\mathbb{Q}) = \{ b \in B \mid \mathrm{nm}(b) = 1 \}$.

$\Rightarrow G(\mathbb{R}) \cong SL_2(\mathbb{R}) \curvearrowright \mathcal{H}_1$.

Let $\Gamma \subseteq G(\mathbb{Q})$ be arithmetic.

1. Case: $B \cong M_2(\mathbb{Q}) \rightsquigarrow G \cong SL_2$ semisimple, $\Rightarrow \Gamma \backslash G(\mathbb{R})$ has finite volume

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Q})$ unipotent $\Rightarrow \Gamma \backslash \mathcal{H}_1$ not compact

2. Case: $B \not\cong M_2(\mathbb{Q}) \Rightarrow B$ division alg. $\Rightarrow G(\mathbb{Q})$ has no unipotent element $\neq e$.

$\Rightarrow \Gamma \backslash G(\mathbb{R})$ compact, i.e. $\Gamma \backslash \mathcal{H}_1$ is compact.

Def: H connected real Lie gp, $\Gamma \subseteq H$.

Γ is called arithmetic, if $\exists\, G/\mathbb{Q}$ alg. gp, $G(\mathbb{R})^+ \twoheadrightarrow \mathrm{Hol}(D)^+$ with compact kernel & $\Gamma_0 \subseteq G(\mathbb{Q})$ arithmetic s.t. $\Gamma_0 \cap G(\mathbb{R})^+ \twoheadrightarrow \Gamma$.

Interesting case: H semisimple $\rightsquigarrow G$ semisimple.

Prop: H as above admitting faithful fin. dim. rep., H semisimple

$\Longrightarrow \forall\, \Gamma \subseteq H$ arithmetic: Γ is discrete of finite covolume & it contains a torsion-free subgp of finite index.

Proof: Γ gets these properties from Γ_0.

§3. Algebraic varieties vs. complex mfds (GAGA)

<u>Recall</u>: $\exists$ analytification functor

$$(\text{non-sing. varieties}/\mathbb{C}) \xrightarrow{(\,)^{an}} (\text{complex mfds})$$
$$\mathbb{A}^n \longmapsto \mathbb{C}^n$$
$$\text{étale maps} \longmapsto \text{local isom.}$$
$$\text{Zar. open is complex open.}$$

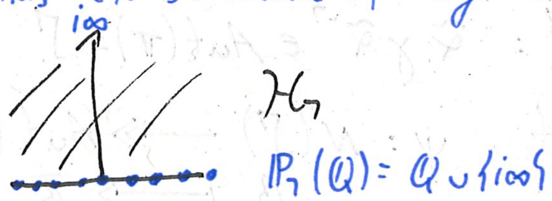
<u>Thm (Chow)</u>: $(\text{proj. alg. varieties}) \underset{\text{equiv. of cat.}}{\xrightarrow{(\,)^{an}}} (\text{proj. compl. analytic spaces})$

<u>Hironaka's thm</u> (on resolution of singularities):
$\forall$ non-sing. alg. variety V: $\exists V^*$ complete non-sing. variety s.t. $V \underset{\text{open}}{\hookrightarrow} V^*$, $V^* \setminus V$ is a divisor with normal crossings, i.e. complex locally it looks like $D_1^{\times r} \times D_1^s \hookrightarrow D_1^{r+s}$ with $D_1 = \{|z| \le 1\}$; $D_1^\times = D_1 \setminus \{0\}$.

§4 $D(\Gamma)$ as alg. variety

<u>Thm</u> (Baily & Borel, '66): Let $\Gamma \subseteq \mathrm{Hol}(D)^+$ be torsion free & arithmetic. Then $D(\Gamma)$ has a canonical realization as Zariski-open subset of proj. alg. variety $D(\Gamma)^*$. Hence $D(\Gamma)$ has the structure of alg. variety.

For $D = \mathcal{H}_1$: $\mathcal{H}_1^* = \mathcal{H}_1 \cup \mathbb{P}^1(\mathbb{Q})$



$\rightsquigarrow \Gamma\backslash\mathcal{H}_1^*$ is compact Riemann surface, modular forms give embedding into projective space + Chow.

$(\Gamma\backslash\mathcal{H}_1^*) \setminus (\Gamma\backslash\mathcal{H}_1)$ finite $\rightsquigarrow$ Zariski-open.

<u>Remark</u>: $D(\Gamma)^* = \mathrm{Proj}(\oplus A_n)$ with A_n = autom. forms for the n^{th} power of canonical automorphy form

<u>Def</u>: Such $D(\Gamma)$'s are called locally symmetric variety.

Thm (Borel '72): Let $D(\Gamma)$ be a locally symmetric variety, V quasi-proj, non-sing / $\mathbb{C}$.
Every holom. map $f: V^{an} \longrightarrow D(\Gamma)^{an}$ is regular.

This follows from Hironaka's thm &

Lemma: All holom. $D_1^{\times r} \times D_1^{s} \longrightarrow D(\Gamma)$ holom. can be ext. to $D_1^{r+s} \longrightarrow D(\Gamma)^*$.

Corollary: The structure of an alg. variety on $D(\Gamma)$ is unique.

Remark: Borel's thm also holds for singular varieties, but Torsion-free is necessary, i.e. e.g. $\Gamma(1) \backslash H_1 \cong \mathbb{A}^1$ & $\exp: \mathbb{C} \to \mathbb{C}$ is holom, not regular.

Thm: Let $D(\Gamma)$ be a locally symmetric variety. Then
$\# \underset{\text{cpx. mfd.}}{Aut}(D(\Gamma)) < \infty$

Idea: $\pi: D \longrightarrow D(\Gamma)$ universal cover

$$\rightsquigarrow \quad \Gamma = Aut(\pi) = \left\{ \begin{tikzcd} D \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & \circlearrowleft & D \arrow[dl, "\pi"] \\ & D(\Gamma) & \end{tikzcd} \right\}$$

$\alpha \in Aut(D(\Gamma))$ lifts to $\tilde{\alpha}: D \xrightarrow{\sim} D$.

$\forall \gamma \in \Gamma: \ \tilde{\alpha}\gamma\tilde{\alpha}^{-1} \in Aut(\pi) = \Gamma \ \Rightarrow \ \tilde{\alpha} \in N(\Gamma) = \{\beta \in Aut(D) : \beta\Gamma\beta^{-1} = \Gamma\}$

We get
$$\psi: N(\Gamma) \longrightarrow\!\!\!\!\rightarrow Aut(D(\Gamma))$$
$$\beta \longmapsto \bar{\beta}$$

$\ker(\psi) = \{\beta \in N(\Gamma) : \gamma\beta = \beta\gamma \ \forall \gamma \in \Gamma\} = C(\Gamma)$, i.e.

$$Aut(D(\Gamma)) \cong N(\Gamma)/C(\Gamma)$$

Magic with arithmetic subgroups: $N(\Gamma)/C(\Gamma)$ is finite $\square$.